The Nilpotence Theorem (Lecture 25)

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In the last lecture, we defined a ring spectrum E to be a *field* if $\pi_* E$ is a graded field. Every Morava K-theory is a field. Conversely, if E is any field, then we claim that E has the structure of a $K(n)$-module for some $0 \leq n \leq \infty$ (and some prime number p, if $n > 0$). Equivalently, we claim that $E \otimes K(n)$ is nonzero for some n.

Remark 1. The integer n is uniquely determined: the cohomology theory E is complex oriented and n can be characterized as the height of the associated formal group. (Similarly, the prime number p is uniquely determined: it is the characteristic of the field $\pi_0 E$).

For the remainder of this lecture, we will fix a prime number p.

Proposition 2. *Let $\{E^\alpha\}$ be a collection of ring spectra. The following conditions are equivalent:*

(1) *Let R be a p-local ring spectrum. If $x \in \pi_m R$ is a homotopy class whose image in $E^\alpha_0(R)$ is zero for all α, then x is nilpotent in $\pi_* R$.*

(2) *Let R be a p-local ring spectrum. If $x \in \pi_0 R$ is a homotopy class whose image in $E^\alpha_0(R)$ is zero for all α, then x is nilpotent in $\pi_0 R$.*

(3) *Let X be an arbitrary p-local spectrum. If $x \in \pi_0 X$ has trivial image under the Hurewicz map $\pi_0 X \to E^\alpha_0(X)$ for each α, then the induced class $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for $n \gg 0$.*

(4) *Let X be an arbitrary p-local spectrum, and let F be a finite spectrum. If $f : F \to X$ is such that each composite map $F \to X \to X \otimes E^\alpha_0$ is nullhomotopic, then $f^{\otimes n} : F^{\otimes n} \to X^{\otimes n}$ is nullhomotopic for $n \gg 0$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows by taking R to be the ring spectum $\bigoplus_n X^{\otimes n}_{(p)}$. The implication (3) $\Rightarrow$ (4) follows by replacing X by the function spectrum X^F. Suppose now that (4) is satisfied, and let $x \in \pi_m R$ be a class whose image vanishes in $E^\alpha_n(R)$ for all α. Let us identify x with a map $S^m \to R$. Then x^n can be identified with the composition

$$S^{mn} \stackrel{x^{\otimes n}}{\to} R^{\otimes n} \to R,$$

where the second map is given by the multiplication on n. Since $x^{\otimes n}$ is nullhomotopic for $n \gg 0$ by (4), we conclude that x is nilpotent. $\square$

We say that a collection of ring spectra $\{E^\alpha\}$ *detects nilpotence* if the equivalent conditions of Proposition 2 are satisfied.

The following fundamental result was proven by Devinatz, Hopkins, and Smith:

Theorem 3 (Nilpotence Theorem)**.** *For any ring spectrum R, the kernel of the map $\pi_* R \to \mathrm{MU}_*(R)$ consists of nilpotent elements. In particular, the single cohomology theory* MU *detects nilpotence.*

Corollary 4 (Nishida)**.** *For $n > 0$, every element of $\pi_n S$ is nilpotent.*

Proof. Let $x \in \pi_n S$. Then x is torsion, so the image of x in $\mathrm{MU}_*(S) = \pi_* \mathrm{MU} \simeq L$ is torsion. Since L is torsion free, we conclude that the image of x is zero so that x is nilpotent by Theorem 3. $\square$

We will use Theorem 3 to deduce the following:

Theorem 5. *The spectra $\{K(n)\}_{0 \leq n \leq \infty}$ detect nilpotence.*

We will prove that the spectra $\{K(n)\}_{0 \leq n \leq \infty}$ satisfy condition (3) of Proposition 2. Let T denote the homotopy colimit of the spectra

$$S \xrightarrow{x} X \xrightarrow{x} X^{\otimes 2} \xrightarrow{x} X^{\otimes 3} \to \cdots .$$

Lemma 6. *Let $x \in \pi_0 X$ and T be defined as above, and let E be any ring spectrum. The following conditions are equivalent:*

(1) *The spectrum T is E-acyclic.*

(2) *The image of $x^{\otimes n}$ in $E_0(X^{\otimes n})$ vanishes for $n \gg 0$.*

Proof. If (1) is satisfied, then the canonical map $S \to T \to T \otimes E$ is nullhomotopic. It follows that the map $S \xrightarrow{x^{\otimes n}} X^{\otimes n} \to X^{\otimes n} \otimes E$ is nullhomotopic for $n \gg 0$, so that (2) is satisfied. For the converse, we note that $T \otimes E$ can be identified with the homotopy colimit of the sequence

$$E \to X^{\otimes n} \otimes E \to X^{\otimes 2n} \otimes E \to \cdots$$

If (2) is satisfied, then each of the maps in this system is nullhomotopic, so the colimit is trivial. $\square$

We now turn to the proof of Theorem 5. Fix $x \in \pi_0 X$ whose image in each $K(n)_0(X)$ is zero. We wish to prove that some smash power $x^{\otimes n}$ is trivial. By the nilpotence theorem, it will suffice to show that the image of x in $\mathrm{MU}_0(X)$ is nilpotent. By the Lemma, this is equivalent to showing that $\mathrm{MU}_*(T) \simeq 0$: that is, the quasi-coherent sheaf $\mathcal{F}_{\Sigma^k T}$ on $\mathcal{M}_{\mathrm{FG}}$ vanishes for $k \in \mathbf{Z}$.

Choose cofiber sequences

$$\Sigma^{2k} \mathrm{MU}_{(p)} \xrightarrow{t_k} \mathrm{MU}_{(p)} \to M(k)$$

as in the previous lectures. For $n \geq 0$, let $P(n)$ denote the smash product (taken over $\mathrm{MU}_{(p)}$) of the spectra $\{M(k)\}_{k \neq p^m - 1}$ and $\{M(p^m - 1)\}_{m < n}$, so that $P(n)$ is a ring spectrum with

$$\pi_* P(n) \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]/(v_0, v_1, \ldots, v_{n-1}).$$

In particular, $P(0)$ is the ring spectrum BP; we have seen that $P(0)$ is Landweber exact and that the map $\pi_* P(0) \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ is faithfully flat. Then $P(0)_*(X)$ is the pullback of the quasi-coherent sheaf $\mathcal{F}_{\Sigma^{-*} X}$ on $\mathcal{M}_{\mathrm{FG}}$. It therefore suffices to show that $P(0)_*(T) \simeq 0$.

Let $P(\infty) \simeq \varinjlim P(n)$, so that $P(\infty) \simeq H\mathbf{F}_p$. By assumption, the image of x in $P(\infty)_0(X) \simeq \varinjlim P(n)_0(X)$ is zero. It follows that the image of x in $P(n)_*(X)$ vanishes for some $n < \infty$. By the lemma, we deduce that $P(n)_*(T) \simeq 0$.

We now prove that $P(m)_*(T) \simeq 0$ for all m, using descending induction on m. Assume that $P(m+1)_*(T) \simeq 0$. We have a cofiber sequence

$$\Sigma^{2(p^m - 1)} P(m) \xrightarrow{v_m} P(m) \to P(m+1).$$

It follows that multiplication by v_m is invertible on $P(m)_*(T)$, so that $P(m)_*(T) \simeq P(m)[v_m^{-1}]_* T$. Since $P(m)[v_m^{-1}]$ is a module over $\mathrm{MU}_{(p)}[v_m^{-1}]$, it will suffice to prove that T is $\mathrm{MU}_{(p)}[v_m^{-1}]$-acyclic. Note that $\mathrm{MU}_{(p)}[v_m^{-1}]$ is a Landweber-exact theory whose associated formal group has height $\leq m$ everywhere; it therefore suffices to show that T is $E(m)$-acyclic.

We now prove using ascending induction on $k \leq m$ that T is $E(k)$-acyclic. By the main result of Lecture 23, the inductive step is equivalent to showing that T is $K(k)$-acyclic. This follows from our lemma, since the image of x in $K(k)_0(X)$ vanishes by assumption.

Remark 7. Since $K(m)$ is a field, for each $n \geq 0$ the homology $K(m)_*(X^{\otimes n})$ is the nth (algebraic) tensor power of $K(m)_*(X)$ over $\pi_* K(m) \simeq \mathbf{F}_p[v_m^{\pm 1}]$. It follows that $x^{\otimes n}$ has trivial image in $K(m)_*(X^{\otimes n})$ if and only if x has trivial image in $K(m)_*(X)$ Consequently, we have the following slightly more precise result for a homotopy class $x \in \pi_0 X$ for a p-local spectrum X:

($*$) The class $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for $n \gg 0$ if and only if the image of x in $K(m)_0(X)$ vanishes for all m.

Remark 8. We can drop the requirement that X is p-local if we impose the same condition at all Morava K-theories (for all primes).

Corollary 9. *Let E be a nonzero p-local ring spectrum. Then $E \otimes K(n)$ is nonzero for some $0 \leq n \leq \infty$.*

Proof. If $K(n)_* E \simeq 0$ for all n, then Theorem 5 shows that every element of $\pi_0 E$ is nilpotent. In particular, the unit element $1 \in \pi_0 E$ is nilpotent, so that $E \simeq 0$. □

Combining this with the results of the previous lecture, we deduce:

Corollary 10. *Let E be a ring spectrum such that $\pi_* E$ is a graded field. Then E has the structure of a $K(n)$-module for some n (and some prime number p).*