Uniqueness of Morava K-Theory (Lecture 24)

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Fix a prime number p and an integer $0 < n < \infty$. In Lecture 22, we introduced the Morava K-theory spectrum $K(n)$: a homotopy associative (and commutative if $p > 2$) ring spectrum with $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}]$, $v_n \in \pi_{2(p^n-1)} K(n)$. However, our construction involved a number of arbitrary choices. Our goal in this lecture is to show that the underlying spectrum of $K(n)$ is independent of these choices (though its ring structure is not).

We begin with the following:

Definition 1. Let R be a commutative evenly graded ring. We will say that R is a *graded field* if every nonzero homogeneous element of R is invertible. Equivalently, R is a graded field if either R is either a field k concentrated in degree zero, or has the form $k[\beta^{\pm 1}]$ for some element β of positive even degree.

Remark 2. If R is a graded field, then every graded R-module M admits a free basis of homogeneous elements. This is clear if R is a field. If $R \simeq k[\beta^{\pm 1}]$ where β has degree $d > 0$, then any k-basis for $\bigoplus_{0 \leq i < d} M_i$ is an R-basis for M.

Definition 3. We will say that a homotopy associative ring spectrum E is a *field* if $\pi_* E$ is a graded field.

If E is as in Definition 3, then every E-module spectrum M is free: that is, it has the form $\bigoplus_\alpha \Sigma^{k_\alpha} E$ for some integers k_α. To see this, choose a homogeneous basis of $\pi_* M$ as a $\pi_* E$-module. Such a basis determines a map of E-module spectra $\alpha : \bigoplus \Sigma_{k_\alpha} E \to M$, which is obviously a homotopy equivalence.

Example 4. For every prime number p and every integer n, the Morava K-theory spectrum $K(n)$ is a field.

Example 5. For every field k (in the usual algebraic sense), the Eilenberg-MacLane spectrum Hk is a field in the sense of Definition 3. In particular, $H\,\mathbf{Q}$ and $H\mathbf{F}_p$ are fields. It is convenient to view these as special cases of the above: note that the definition of $K(n)$ makes sense when $n = 0$ and yields the Eilenberg-MacLane spectrum $K(0) \simeq H\,\mathbf{Q}$, and we agree to the convention that $K(\infty) = H\mathbf{F}_p$.

Remark 6. Let E be a field and let X and Y be spectra. Since E is a field, we can write $E \otimes X = \bigoplus \Sigma^{k_\alpha} E$; in particular, $E_* X = \pi_*(E \otimes X)$ is a free $\pi_* E$-module on generators of degree k_α. We have

$$E_*(X \otimes Y) \simeq \pi_*(E \otimes X \otimes Y) \simeq \pi_*(\bigoplus \Sigma^{k_\alpha} E \otimes Y) \simeq \bigoplus E_{*-k_\alpha}(Y) \simeq E_*(X) \otimes_{\pi_* E} E_*(Y).$$

In other words, for every field there is a Kunneth formula for computing the homology of a smash product of spectra (and therefore a Kunneth formulat for computing the homology of a product of spaces).

Lemma 7. *Let $f : X \to Y$ be a map of spectra. Suppose that X and Y each admit the structure of a $K(n)$-module, so that $\pi_* X$ and $\pi_* Y$ are modules over $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm}]$. Then the pushforward map $f_* : \pi_* X \to \pi_* Y$ is $\mathbf{F}_p[v_n^{\pm 1}]$-linear.*

Proof. We can factor f as a composition

$$X \to K(n) \otimes X \to K(n) \otimes Y \to Y.$$

Here we regard $K(n) \otimes X$ and $K(n) \otimes Y$ as $K(n)$-module spectra via the left action of $K(n)$ on itself. Each of the maps in this diagram is a $K(n)$-module map except for the first. It therefore suffices to treat the case $Y = K(n) \otimes X$. Since $K(n)$ is a field, the module X is free, so we may reduce to the case $X = K(n)$. In this case, we are required to prove that the two evident maps

$$f, g : K(n) \to K(n) \otimes K(n)$$

induce the same map on homotopy groups. In other words, we must show that $f_*(v_n) = g_*(v_n)$.

Note that $K(n) \otimes K(n)$ comes equipped with two complex orientations, determining two formal group laws over $R = \pi_{\text{even}}(K(n) \otimes K(n))$. These formal group laws have p-series $[p](t) \equiv f_*(v_n)t^{p^n} \mod (t)^{p^n+1}$ and $[p]'(t) \equiv g_*(v_n)t^{p^n} \mod (t)^{p^n+1}$. Since these formal group laws differ by a coordinate change of the form $t \mapsto t + b_1 t^2 + b_2 t^3 + \ldots$, we conclude that $f_* v_n = g_* v_n$. □

Proposition 8. *Let X be a spectrum which admits the structure of a $K(n)$-module, and let Y be a retract of X. Then Y admits the structure of a $K(n)$-module.*

Proof. We have maps

$$Y \xrightarrow{i} X \xrightarrow{r} Y$$

whose composition is homotopic to id_Y. The composition $f = i \circ r$ is a map from X to itself. It follows from the Lemma that f_* is an $\mathbf{F}_p[v_n^{\pm 1}]$-module map from $\pi_* X$ to itself. In particular, the image of f_* is a graded $\mathbf{F}_p[v_n^{\pm 1}]$-submodule of $\pi_* X$. This image is automatically free, and so has a basis of classes $\eta_\alpha \in \pi_{k_\alpha} X$. This basis determines a map $\alpha : \bigoplus \Sigma^{k_\alpha} K(n) \to X$. By construction, the map $r \circ \alpha$ induces an isomorphism on homotopy groups and therefore determines an equivalence $Y \simeq \bigoplus \Sigma^{k_\alpha} K(n)$. □

Proposition 9. *Let E be any field and suppose that $E \otimes K(n)$ is nonzero. Then E admits the structure of a $K(n)$-module.*

Proof. If $E \otimes K(n)$ is nonzero, then the unit map $E \to E \otimes K(n)$ is nonzero. Using the assumption that E is a field, we deduce that E is a direct summand of $E \otimes K(n)$, and so admits a $K(n)$-module structure by Proposition 8. □

Proposition 10. *Let E be a complex-oriented ring spectrum whose associated formal group has height exactly n. If $E \neq 0$, then $E \otimes K(n) \neq 0$.*

Note that $E \otimes E(n-1)$ is a complex oriented ring spectrum whose formal group is both of height $\leq n-1$ and exactly n; it follows that $E \otimes E(n-1) \simeq 0$. If $E \otimes K(n) \simeq 0$, then it follows from the last lecture that $E \otimes E(n) \simeq 0$. But $\pi_*(E \otimes E(n))$ is the pullback of the quasi-coherent sheaf $\mathcal{F}_E$ on $\mathcal{M}_{\text{FG}}$ along the flat map $\operatorname{Spec} W(k)[[v_1, \ldots, v_{n-1}]] \to \mathcal{M}_{\text{FG}}$. It follows that $\mathcal{F}_E$ vanishes when restricted to the open substack $\mathcal{M}_{\text{FG}}^{\leq n}$. Since the formal group of E has height $\leq n$, we conclude that $E \simeq 0$.

Proposition 11. *Let E be any complex-oriented ring spectrum whose formal group has height exactly n, and whose homotopy groups are given by $\pi_* E \simeq \mathbf{F}_p[v_n^{\pm 1}]$. Then there is a homotopy equivalence of spectra $E \simeq K(n)$.*

Proof. Since $E \neq 0$ and the formal group of E has height n, we conclude that $E \otimes K(n) \neq 0$ (Proposition 10). It follows from Proposition 9 that E admits the structure of a $K(n)$-module. This module is automatically free (since $K(n)$ is a field); it follows by inspecting homotopy groups that E must be free of rank 1: that is, $E \simeq K(n)$. □

It follows from Proposition 11 that when $n = 1$, the Morava K-theory $K(n)$ reduces to something familiar. Let K denote the complex K-theory spectrum. Then K has a canonical complex orientation, whose associated formal group law is given by $f(x, y) = x + y + \beta xy$, where $\beta \in \pi_2 K$ denotes the Bott element. Fix a prime number p, and let $\widehat{K}$ denote the p-adic completion of K. Then $\pi_0 \widehat{K} \simeq \mathbf{Z}_p$, and the formal group law over $\mathbf{Z}_p$ deforms the multiplicative formal group law (of height 1) over $\mathbf{F}_p$. We deduce:

Proposition 12. *Let f be the multiplicative formal group law over the field $\mathbf{F}_p$. Then the associated Morava E-theory is given by $E(1) = \widehat{K}$.*

We have seen that, as a homology theory, $E(1)$ is functorial with respect to automorphisms of the underlying formal group. In particular, the automorphism group of the formal multiplicative group acts on $\widehat{K}$. We have seen that this group can be identified with the group of units $\mathbf{Z}_p^\times$. For every p-adic unit λ, we let ψ^λ denote the corresponding map from $\widehat{K}$ to itself (these are given by the classical *Adams operations*).

The group $\mathbf{Z}_p^\times$ contains a finite subgroup μ_{p-1}, consisting of $(p-1)$st roots of unity. This finite group acts on the associated homology theory $\widehat{K}_*$. We let $\widehat{K}_*^{Ad}$ denote the μ_{p-1}-invariants in $\widehat{K}_*$. Since $\widehat{K}_*$ takes values in $\mathbf{Z}_{(p)}$-modules, passage to invariants is an exact functor so that $\widehat{K}_*^{Ad}$ is a homology theory, represented by a spectrum $\widehat{K}^{Ad}$. This is called the *Adams summand* of $\widehat{K}$; we have

$$\pi_* \widehat{K}^{Ad} \simeq \mathbf{Z}_p[\beta^{\pm(p-1)}] \subseteq \mathbf{Z}_p[\beta^{\pm 1}] \simeq \pi_* \widehat{K}.$$

With a bit more effort, one can show that $\widehat{K}^{Ad}$ has the structure of a ring spectrum, and that the cofiber of the multiplication by p map $p : \widehat{K}^{Ad} \to \widehat{K}^{Ad}$ inherits the structure of a ring spectrum. We denote this cofiber by $\widehat{K}^{Ad}/p$. A simple calculation gives $\pi_*(\widehat{K}^{Ad}/p) \simeq \mathbf{F}_p[\beta^{\pm(p-1)}]$. Moreover, $\widehat{K}^{Ad}/p$ has a canonical complex orientation, so we get a class $v_1 \in \pi_{2p-2}(\widehat{K}^{Ad}/p)$ which coincides with β^{p-1} (in fact, the p-series for $f(x, y) = x + y + \beta xy$ is given by $[p](t) = \frac{(1+\beta t)^p - 1}{\beta} \equiv \beta^{p-1} \mod p$). It follows from Proposition 11 that $K(1) \simeq \widehat{K}^{Ad}/p$: that is the 1st Morava K-group is given by the Adams summand of p-adic K-theory, reduced modulo p.

Remark 13. We will later see that every field satisfies the hypotheses of Proposition 9 for some $0 \leq n \leq \infty$. In other words, the Morava K-theories $K(n)$ are essentially the only examples of fields in the stable homotopy category (provided that we allow the cases $n = 0$ and $n = \infty$).