Lubin-Tate Theory (Lecture 21)

April 27, 2010

We have seen that the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups admits a stratification. The open strata are locally closed substacks $\mathcal{M}^n_{\mathrm{FG}} \subseteq \mathcal{M}^n_{\mathrm{FG}}$ classifying formal groups of height exactly n (at some fixed prime p). These strata are relatively well understood: for $0 < n < \infty$, the stratum $\mathcal{M}^n_{\mathrm{FG}}$ can be identified with a quotient $\mathrm{Spec}\,\overline{\mathbb{F}}_{\mathrm{p}_p}/\mathbb{G}$, where $\mathbb{G}$ is a certain profinite group (the Morava stabilizer group). To understand the moduli stack $\mathcal{M}_{\mathrm{FG}}$ itself, we want to know how these strata fit together. In other words, we would like to understand what $\mathcal{M}_{\mathrm{FG}}$ looks like in a small neighborhood of some point of $\mathcal{M}^n_{\mathrm{FG}}$. This is the subject of *Lubin-Tate theory*.

Let us fix a perfect field k of characteristic p and a formal group law $f(x, y) \in k[[x, y]]$ of height n over k. We would like to understand formal group which are, in some sense, "close" to f.

Definition 1. An *infinitesimal thickening* of k is a commutative ring A with a surjective map $\phi : A \to k$ whose kernel $\mathfrak{m}_A = \ker(\phi)$ has the following properties:

(1) The ideal $\mathfrak{m}^a_A = 0$ for $a \gg 0$.

(2) Each quotient $\mathfrak{m}^a_A/\mathfrak{m}^{a+1}_A$ is a finite-dimensional vector space over k.

In other words, A is a local Artin ring having residue field k.

Definition 2. Let A be an infinitesimal thickening of k. A *deformation of f over A* is a formal group law f_A over A, whose image under the map $\mathrm{FGL}(A) \to \mathrm{FGL}(k)$ is f. We say that two deformations of f are *isomorphic* if they differ by an invertible power series $g(t) \in A[[t]]$ such that $g(t) \equiv t \mod \mathfrak{m}_A$. We will denote the collection of isomorphism classes of deformations of f over A by $\mathrm{Def}(A)$.

Remark 3. A priori, we expect that deformations of a formal group law f over A should form a groupoid. However, this groupoid is actually discrete. In other words, if f_A is a deformation of f over A, then any automorphism of f_A which is the identity modulo $\mathfrak{m}_A$ is automatically trivial. To prove this, we can replace f by the image of f_A in $\mathrm{FGL}(k)$ and thereby reduce to the case $\eta = \mathrm{id}$. Let $g(x) = b_0x + b_1x^2 + \cdots$ be an automorphism of the formal group law f_A. We will prove by induction on a that $g(x) \equiv x \mod \mathfrak{m}^a_A$. When $a = 1$, this is true by hypothesis; for a sufficiently large, we have $\mathfrak{m}^a_A = 0$ so that we will have proven $g(x) = x$. To complete the proof, we carry out the inductive step. Let A' be the quotient of $A[b_0^{\pm 1}, b_1, \ldots]$ which classifies automorphisms of f_A. The map g is classified by a ring homomorphism $\psi : A' \to A$, while the identity automorphism is classified by $\psi_0 : A' \to A$. Assume that the composite maps

$$\psi, \psi' : A' \to A \to A/\mathfrak{m}^a_A$$

agree. Then, modulo $\mathfrak{m}^{a+1}_A$, the difference $\psi - \psi'$ is a map $d : A' \to V$, where V is the k-vector space $\mathfrak{m}^a_A/\mathfrak{m}^{a+1}_A$. The map d is an A-linear derivation, and factors as a composition

$$A' \to A' \otimes_A k \xrightarrow{d'} V$$

where d' is a k-linear derivation. But $A' \otimes_A k$ is the ring classifying automorphisms of the formal group f of height n, and is therefore etale over k: it follows that $d' = 0$ so that $\psi \equiv \psi' \mod \mathfrak{m}^{a+1}_A$.

Remark 4. The set $\mathrm{Def}(A)$ can be identified with the set of isomorphism classes of formal groups $\mathcal{F}$ over A lifting the formal group $\mathcal{G}_f$ associated to f. To see this, we note that since A is local the formal group $\mathcal{F}$ automatically has the form $\mathcal{F}_{f'}$ for some $f' \in \mathrm{FGL}(A)$. By assumption, the image f'_k of f' in $\mathrm{FGL}(k)$ is isomorphic to f, via some invertible power series $g(t) \in k[[t]]$. Lifting the coefficients of g arbitrarily, we can assume that g is the image of a power series $\overline{g}(t) \in A[[t]]$ (automatically invertible). Conjugating f' by g, we obtain the desired deformation of f.

We would like to understand the deformation functor $A \mapsto \mathrm{Def}(A)$. We begin by writing down a specific deformation of f. Let $W(k)$ denote the ring of Witt vectors of k, and let $R = W(k)[[v_1, \ldots, v_{n-1}]]$. There is a canonical map $R \to k$, whose kernel is the maximal ideal $\mathfrak{m}_R = (p, v_1, \ldots, v_{n-1})$. The formal group f over k is classified by a map $\phi_0 : L_{(p)} \to k$, where $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$. We may assume without loss of generality that $t_{p^i-1} = v_i$ for $1 \leq i \leq n-1$. Since f has height n, we conclude that $t_{p^i-1} \mapsto 0 \in k$ for $1 \leq i \leq n-1$. Let $\phi : L_{(p)} \to R$ be *any* homomorphism which lifts ϕ_0, and carries t_{p^i-1} to v_i for $0 < i < m$. This homomorphism determines a formal group law $\overline{f} \in \mathrm{FGL}(R)$ whose image in $\mathrm{FGL}(k)$ is f.

Theorem 5 (Lubin-Tate)**.** *The formal group law $\overline{f}$ over $R = W(k)[[v_1, \ldots, v_{n-1}]]$ is a universal deformation of f in the following sense: for every infinitesimal thickening A of k, $\overline{f}$ gives a bijection*

$$\mathrm{Hom}_{/k}(R, A) \to \mathrm{Def}(A).$$

The proof rests on the following pair of observations:

(1) The functor $A \mapsto \mathrm{Def}(A)$ is formally smooth: that is, if $A \to A'$ is a surjective map between infinitesimal thickenings of k, then the induced map $\mathrm{Def}(A) \to \mathrm{Def}(A')$ is surjective (this is because any formal group law over A' extends to a formal group law over A, since the Lazard ring L is polynomial).

(2) Given a pair of surjective maps $A \to B \leftarrow C$ between infinitesimal thickenings of k, the canonical map $\mathrm{Def}(A \times_B C) \to \mathrm{Def}(A) \times_{\mathrm{Def}(B)} \mathrm{Def}(C)$ is a bijection. To see this, it is best to think in terms of formal groups (Remark 4): $\mathrm{Spec}(A \times_B C)$ is obtained by gluing $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, C$ along the common closed subscheme $\mathrm{Spec}\, B$, so giving a formal group over $\mathrm{Spec}(A \times_B C)$ is equivalent to giving a formal groups over $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, C$, together with an isomorphism between their restrictions to $\mathrm{Spec}\, B$.

To prove Theorem 5 we work by induction on the length of the Artinian ring A. If A has length 1, then $A \simeq k$ and both $\mathrm{Hom}_{/k}(R, A)$ and $\mathrm{Def}(A)$ consist of a single element. If A has length > 1, then we can choose an element $x \in A$ which is annihilated by $\mathfrak{m}_A$. Let us study the relationship between $\mathrm{Def}(A)$ and $\mathrm{Def}(A/x)$. Using (2), we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Def}(A \times_{A/x} A) & \longrightarrow & \mathrm{Def}(A) \\ \downarrow & & \downarrow p \\ \mathrm{Def}(A) & \longrightarrow & \mathrm{Def}(A/x). \end{array}$$

Note that $A \times_{A/x} A \simeq k[x]/(x^2) \times_k A$. There is an addition map

$$k[x]/(x^2) \times_k k[x]/(x^2) \to k[x]/(x^2)$$

which, by (2), determines a group structure on $\mathrm{Def}(k[x]/(x^2))$. The multiplication

$$k[x]/(x^2) \times_k A \to A$$

determines an action of $\mathrm{Def}(k[x]/(x^2))$ on $\mathrm{Def}(A)$, and the pullback square above shows that p determines an embedding $\mathrm{Def}(A)/\mathrm{Def}(k[x]/(x^2)) \hookrightarrow \mathrm{Def}(A/x)$. It follows from (1) that this map is surjective: that is, $\mathrm{Def}(A)$ is a principal homogeneous space for $\mathrm{Def}(k[x]/(x^2))$ over $\mathrm{Def}(A/x)$. The same reasoning shows that $\mathrm{Hom}_{/k}(R, A)$ is a torsor for $\mathrm{Hom}_{/k}(R, k[x]/(x^2))$ over $\mathrm{Hom}_{/k}(R, A/x)$. Since $\mathrm{Hom}_{/k}(R, A/x) \simeq \mathrm{Def}(A/x)$ by the inductive hypothesis, we are reduced to proving the following special case of Theorem 5:

Lemma 6. *The canonical map* $\theta : \mathrm{Hom}_{/k}(R, k[x]/(x^2)) \to \mathrm{Def}(k[x]/(x^2))$ *is bijective.*

To prove this, we construct a map $\theta' : \mathrm{Def}(k[x]/(x^2)) \to k^{n-1}$ as follows. Every deformation f' is classified by a map ϕ from the Lazard ring L into $k[x]/(x^2)$ and we have $\phi(v_i) = c_i x$ for $0 < i < n$. Set $\theta'(\phi) = (c_1, c_2, \ldots, c_{n-1})$.

Claim 7. *The sequence* $(c_1, c_2, \ldots, c_{n-1})$ *depends only on the isomorphism class of the deformation* ϕ.

To see this, let us suppose that f' and f'' are deformations of the formal group law f over $k[x]/(x^2)$ which differ by an autommorphism $g(t) = (1 + b_0 x)t + b_1 x t^2 + b_2 x t^3 + \cdots$. These formal group laws have p-series which we will denote by $[p]'(t)$ and $[p]''(t)$, which are related by the formula

$$g([p]'(g^{-1}(t))) = [p]''(t).$$

Since f has height $\geq n$, the power series $[p]'(t)$ and $[p]''(t)$ are divisible by x modulo t^{p^n}. Since $x^2 = 0$ and $g(t) \equiv t \mod (x)$, we deduce that $[p]'(t) \equiv [p]''(t) \mod (t^{p^n})$, thereby proving the claim.

It is not hard to see that θ' is a group homomorphism. Moreover, the composition

$$\mathrm{Hom}_{/k}(R, k[x]/(x^2)) \xrightarrow{\theta} \mathrm{Def}(k[x]/(x^2)) \xrightarrow{\theta'} k^{n-1}$$

is an isomorphism by construction. This proves that θ is injective. To prove that θ is surjective, it will suffice to show that θ' is injective. A deformation f' of f belongs to the kernel of θ' if and only if θ' has height exactly n. Let f'' be the trivial deformation of f; we wish to show that there is an isomorphism of f' with f'' which reduces to the identity modulo x.

Since f' and f'' are formal groups of height exactly n over $k[x]/(x^2)$, the collection of isomorphisms of f' and f'' is classified by a $k[x]/(x^2)$-algebra R which is an inductive limit of finite etale extensions of $k[x]/(x^2)$. It follows that $k[x]/(x^2)$-algebra homomorphism $R \to k$ lifts uniquely to a $k[x]/(x^2)$-algebra homomorphism $R \to k[x]/(x^2)$: in particular, the identity automorphism f extends uniquely to an isomorphism of f with f'. This completes the proof of Theorem 5.

Remark 8. Let A be a complete Noetherian local ring with residue field k and maximal ideal $\mathfrak{m}_A$. Then each $A/\mathfrak{m}_A^a$ is an infinitesimal thickening of k, and $A \simeq \varprojlim A/\mathfrak{m}_A^a$. It follows that Theorem 5 is also true for A: giving a deformation of the formal group f over A is equivalent to giving a ring homomorphism $W(k)[[v_1, \ldots, v_{n-1}]] \to A$ which is the identity on the common residue field k.

In particular, we see that $W(k)[[v_1, \ldots, v_{n-1}]]$ is characterized uniquely by Theorem 5. As such, it depends functorially on the residue field k together with the choice of formal group of height n over k.

In particular, if we take $k = \overline{\mathbb{F}}_{\mathrm{p}}$, then the Morava stabilizer group $\mathbb{G}$ acts on $W(\overline{\mathbb{F}}_{\mathrm{p}})[[v_1, \ldots, v_{n-1}]]$.

Remark 9. Let k and $R = W(k)[[v_1, \ldots, v_{n-1}]]$ be as above. Then the formal group law over R is Landweber exact: the sequence $v_0 = p, v_1, \ldots, v_{n-1}$ is regular by construction, and v_n has invertible image in $R/(v_0, v_1, \ldots, v_{n-1}) \simeq k$ by virtue of our assumption that the original formal group law f has height n.

Using results of previous lectures, we can construct an even periodic spectrum $E(n)$ with $\pi_* E(n) \simeq W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$, where β has degree 2. The cohomology theory $E(n)$ (which really depends not only on n, but on a choice of field k and a formal group of height n over k) is called *Morava E-theory*. It is also sometimes called *Lubin-Tate theory* or *completed Johnson-Wilson theory*.