Even Periodic Cohomology Theories (Lecture 18)

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Definition 1. Let R be a commutative ring and let $\mathcal{L}$ be an invertible R-module. An *$\mathcal{L}$-twisted formal group law* is a formal series

$$f(x,y) = \sum a_{i,j} x^i y^j$$

where $a_{i,j} \in \mathcal{L}^{\otimes(i+j-1)}$ which satisfies the identities

$$f(x,y) = f(y,x) \qquad f(x,0) = x \qquad f(x, f(y,z)) = f(f(x,y), z).$$

When $\mathcal{L} = R$, an $\mathcal{L}$-twisted formal group law is the same thing as a formal group law over R. Every $\mathcal{L}$-twisted formal group law $f(x,y)$ determines a formal group $\mathcal{G}_f$. More precisely, f defines a group structure on the functor $\operatorname{Spf} R[[\mathcal{L}]] = \operatorname{Spf}(\prod_n \mathcal{L}^{\otimes n})$ given by $A \mapsto \operatorname{Hom}_R(\mathcal{L}, \sqrt{A})$, where $\sqrt{A}$ denotes the ideal consisting of nilpotent elements of A. Note that the fiber of the map

$$(\operatorname{Spf} R[[\mathcal{L}]])(R[\epsilon/\epsilon^2]) \to (\operatorname{Spf} R[[\mathcal{L}]])(R)$$

is the collection of R-linear maps $\mathcal{L} \to \epsilon R/\epsilon^2 R$: that is, it is the R-module $\mathcal{L}^{-1}$. In other words, if f is any $\mathcal{L}$-twisted formal group law, there is a canonical isomorphism $\eta_f : \mathfrak{g}_{\mathcal{G}_f} \simeq \mathcal{L}^{-1}$, where $\mathfrak{g}_{\mathcal{G}_f}$ denotes the Lie algebra over $\mathcal{G}_f$. Conversely, we have the following:

Lemma 2. *Let R be a commutative ring and let $\mathcal{G}$ be a formal group over R with Lie algebra $\mathfrak{g}$. Then there exists a $\mathfrak{g}^{-1}$-twisted formal group law f and an isomorphism $\mathcal{G}_f \simeq \mathcal{G}$ lifting the isomorphism $\eta_f : \mathfrak{g}_{\mathcal{G}_f} \simeq \mathfrak{g}$.*

Proof. We first suppose that $\mathcal{G}$ is coordinatizable. In particular, we can choose an isomorphism $\alpha : \mathfrak{g} \simeq R$. We also have an isomorphism $\beta : \mathcal{G} \simeq \mathcal{G}_f$ for some formal group law $f(x,y) \in R[[x,y]]$. Replacing f by $\lambda^{-1} f(\lambda x, \lambda y)$ for some invertible constant λ, we can ensure that the composite map

$$R \overset{\alpha}{\simeq} \mathfrak{g} \overset{\beta}{\simeq} \mathcal{G}_{\mathfrak{f}} \overset{\eta_f}{\simeq} R$$

is the identity.

Let G denote the affine R-scheme which carries every R-algebra A to the group of power series of the form

$$g(t) = t + b_1 t^2 + b_2 t^3 + \cdots$$

where $b_n \in \mathcal{L}^{\otimes n}$, and let P be the affine R-scheme which carries every R-algebra A to the collection of all pairs (f, β), where f is an $(\mathcal{L} \otimes_R A)$-twisted formal group law and β is an isomorphism of $\mathcal{G}_f \simeq \mathcal{G}$ over $\operatorname{Spec} A$ which lifts the isomorphism η_f. There is an obvious action of G on P, and the above argument shows that P is a locally trivial G-torsor with respect to the Zariski topology. To prove the Lemma, we wish to show that $P(R)$ is trivial.

For each $n \geq 1$, we let G_n denote the subgroup scheme of G consisting of those power series such that $b_i = 0$ for $i \leq n$. Then $P \simeq \varprojlim P/G_n$, and $P/G_0 \simeq *$. To prove that $P(R)$ is nonempty, it will suffice to show that each of the maps $P/G_n(R) \to P/G_{n-1}(R)$ is surjective. The obstruction to surjectivity lies in the group

$$\mathrm{H}^1(\operatorname{Spec} R; G_n/G_{n-1}) \simeq \mathrm{H}^1(\operatorname{Spec} R; \mathcal{L}^{\otimes n})$$

. This group is trivial, since $\mathcal{L}^{\otimes n}$ is a quasi-coherent sheaf on $\operatorname{Spec} R$. □

Remark 3. Let R be a commutative ring and let $\mathcal{L}$ be an invertible R-module. The data of an $\mathcal{L}$-twisted formal group law over R is equivalent to the data of a graded formal group law over the ring $\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$, where $\mathcal{L}^{\otimes n}$ has degree $2n$. That is, it is equivalent to giving a map of graded rings $L \to \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$.

Remark 4. Let f be an $\mathcal{L}$-twisted formal group law over a commutative ring R. The following conditions are equivalent:

(1) The associated formal group $\mathcal{G}_f$ is classified by a flat map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$.

(2) The graded L-module $\bigoplus \mathcal{L}^{\otimes n}$ is Landweber-exact.

By Landweber's theorem, condition (2) is equivalent to the sum $\bigoplus \mathcal{L}^{\otimes n}$ being flat over $\mathcal{M}_{\mathrm{FG}}$. In particular, this implies that $\mathcal{L}^{\otimes 0} \simeq R$ is flat over $\mathcal{M}_{\mathrm{FG}}$, so that (2) $\Rightarrow$ (1). The converse follows from the observation that $\bigoplus \mathcal{L}^{\otimes n}$ is flat over R.

In the situation of Remark 4, we can apply Landweber's theorem to obtain a spectrum E_R, whose underlying homology theory is given by $(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L (\bigoplus_n \mathcal{L}^{\otimes n})$.

Example 5. Let R be the Lazard ring L and let $\mathcal{L} = R$ be trivial, so that $\bigoplus_n \mathcal{L}^{\otimes n}$ can be identified with $L[\beta^{\pm 1}]$. Then the above construction applies to produce a spectrum E_L whose homology theory is given by

$$(E_L)_*(X) = \mathrm{MU}_*(X) \otimes_L L[\beta^{\pm 1}] \simeq \mathrm{MU}_*(X)[\beta^{\pm 1}].$$

This spectrum is called the *periodic complex bordism spectra*, and will be denoted by MP. Just as MU can be realized as the Thom spectrum of the universal virtual complex bundle of rank 0 over BU, MP can be realized as the Thom spectrum of the universal virtual complex bundle of arbitrary rank over the space $BU \times \mathbf{Z}$. We have $\mathrm{MP}_0(X) = \mathrm{MU}_{\mathrm{even}}(X)$.

Now suppose more generally, we are given a $\mathcal{L}$-twisted formal group law f over a commutative ring R satisfying the conditions of Remark 4. If we choose an isomorphism $\mathcal{L} \simeq R$, then we can identify f with a formal group law classified by a map $L \to R$, and $\bigoplus_n \mathcal{L}^{\otimes n}$ with the ring $R[\beta^{\pm 1}]$. Then the homology theory E_R is given by

$$(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L R[\beta^{\pm 1}] \simeq \mathrm{MP}_*(X) \otimes_L R.$$

In particular, we have $(E_R)_0(X) = \mathrm{MP}_0(X) \otimes_L R = \mathrm{MU}_{\mathrm{even}}(X) \otimes_L R$.

The above calculation can be expressed in a more invariant way. Recall that to any spectrum X we can associate a quasi-coherent sheaf $\mathcal{F}_X$ on $\mathcal{M}_{\mathrm{FG}}$, whose restriction to $\operatorname{Spec} L$ is given by $\mathrm{MU}_{\mathrm{even}}\, X$. Then $(\mathrm{MU}_{\mathrm{even}}\, X) \otimes_L R$ is the pullback of $\mathcal{F}_X$ along the map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$. From this description, it is clear that the homology theory $(E_R)_*$ depends only on the formal group $\mathcal{G}_f$ (or equivalently, the map q), and not on the particular choice of formal group law f. This calculation globalizes as follows:

Proposition 6. *Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a flat map. Then there exists a spectrum E_R which is determined up to canonical isomorphism (in the homotopy category of spectra) by its underlying homology theory, which is given by $(E_R)_0(X) = q^* \mathcal{F}_X$ (so that, more generally, $(E_R)_n(X) = (E_R)_0(\Sigma^{-n} X) = q^* \mathcal{F}_{\Sigma^{-n} X}$).*

Remark 7. Suppose we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \\ & {\scriptstyle q'} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathcal{M}_{\mathrm{FG}} & \end{array}$$

where q and q' are flat: that is, we have a Landweber-exact formal group over R whose restriction along a map of commutative rings $R \to R'$ is also Landweber-exact. Then we get an evident map $E_R \to E_{R'}$ (which is unique up to homotopy, by the results of the previous lecture).

Proposition 8. *Suppose we are given flat maps* $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ *and* $q' : \operatorname{Spec} R' \to \mathcal{M}_{\mathrm{FG}}$*. Then the smash product* $E_R \otimes E_{R'}$ *is homotopy equivalent to* E_B*, where* B *fits into a pullback diagram*

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \mathcal{M}_{\mathrm{FG}}\,. \end{array}$$

Proof. It is clear that $\operatorname{Spec} B$ is flat over $\mathcal{M}_{\mathrm{FG}}$. For simplicity, we will suppose that q and q' classify formal groups which admit coordinates, given by maps $L \to R$ and $L \to R'$. Note that

$$\mathrm{MP}_0(\mathrm{MP}) \simeq \mathrm{MU}_{\mathrm{even}}(\mathrm{MP}) \simeq \mathrm{MU}_*(\mathrm{MU})[b_0^{\pm 1}] \simeq \mathrm{MU}_*[b_0^{\pm 1}, b_1, \ldots].$$

Using this calculation, one sees that the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathrm{MP}_0\,\mathrm{MU} & \longrightarrow & \operatorname{Spec} L \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \longrightarrow & \mathcal{M}_{\mathrm{FG}} \end{array}$$

is a pullback square, so that $B \simeq R \otimes_L \mathrm{MP}_0\,\mathrm{MP} \otimes_L R'$.

Now let X be any spectrum. We have

$$\begin{aligned} (E_R \otimes E_{R'})_0(X) &\simeq (E_R)_0(E_{R'} \otimes X) & (1)\\ &\simeq R \otimes_L \mathrm{MP}_0(E_{R'} \otimes X) & (2)\\ &\simeq R \otimes_L (E_{R'})_0(\mathrm{MP} \otimes X) & (3)\\ &\simeq R \otimes_L \mathrm{MP}_0(\mathrm{MP} \otimes X) \otimes_L R' & (4)\\ &\simeq R \otimes_L (\mathrm{MP} \otimes \mathrm{MP})_0 X \otimes_L R'. & (5) \end{aligned}$$

where $(\mathrm{MP} \otimes \mathrm{MP})_0 X$ is the pullback of $\mathcal{F}_X$ to $\operatorname{Spec} \mathrm{MP}_0\,\mathrm{MP} \simeq \operatorname{Spec} L \times_{\mathcal{M}_{\mathrm{FG}}} \operatorname{Spec} L$. It follows that $(E_R \otimes E_{R'})_0 X$ is the pullback of $\mathcal{F}_X$ to $\operatorname{Spec} B$, thus giving a canonical homotopy equivalence $E_R \otimes E_{R'} \simeq E_B$. ☐

Corollary 9. *For any flat map* $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$*, there is a canonical multiplication* $E_R \otimes E_R \to E_R$*, making* E_R *into a commutative and associative algebra in the homotopy category of spectra.*

Proof. Form a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\mathrm{FG}}\,. \end{array}$$

There is an evident diagonal map $\operatorname{Spec} R \to \operatorname{Spec} B$. By Remark 7, this induces a map

$$E_R \otimes E_R \simeq E_B \to E_R.$$

The commutativity and associativity properties of this construction are evident. ☐

Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a flat map classifying a formal group with Lie algebra $\mathfrak{g}$, and let E_R the associated ring spectrum. By construction, we have

$$\pi_n E_R \simeq \begin{cases} \mathfrak{g}^k & \text{if } n = -2k \\ 0 & \text{if } n = -2k+1. \end{cases}$$

Let us now axiomatize this structural phenomenon:

Definition 10. Let E be a ring spectrum. We will say that E is *even periodic* if the following conditions are satisfied:

(1) The homotopy groups $\pi_i E$ vanish when i is odd.

(2) The map $\pi_2 E \otimes_{\pi_0 E} \pi_{-2} E \to \pi_0 E$ is an isomorphism (so that, in particular, $\pi_2 E$ is an invertible E-module $\mathcal{L}$, and we have $\pi_{2n} E \simeq \mathcal{L}^{\otimes n}$ for all n).

If E is an even periodic ring spectrum, then E is automatically complex-orientable, so we obtain a formal group $\mathcal{G}$ over $\pi_* E$. However, in the periodic case we can do better: since $E^*(\mathbf{CP}^\infty) \simeq E^0(\mathbf{CP}^\infty) \otimes_{\pi_0 E} \pi_* E$, we get a formal group $\operatorname{Spf} E^0(\mathbf{CP}^\infty)$ over the commutative ring $R = \pi_0 E$, whose restriction to $\pi_* E$ is the formal group we have been discussing earlier in this course. This formal group is classified by a map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$.

We can summarize the situation as follows:

Proposition 11. *Let $\mathcal{C}$ be the category of pairs (R, η), where R is a commutative ring and $\eta : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ is a flat map (that is, η corresponds to a Landweber-exact formal group over $\operatorname{Spec} R$). Then the construction $R \mapsto E_R$ determines a fully faithful embedding Φ of $\mathcal{C}$ into the category of commutative algebras in the homotopy category of spectra. A ring spectrum E belongs to the essential image of this embedding if and only if E is even periodic, and the induced map $\pi_0 E \to \mathcal{M}_{\mathrm{FG}}$ is flat.*

To prove Proposition 11, we note that the construction $E \mapsto (\pi_0 E, \operatorname{Spf} E^0(\mathbf{CP}^\infty))$ provides a left inverse to Φ. What is not entirely clear is that this construction is also right-inverse to Φ: that is, if E is an even periodic ring spectrum which determines a map $q : \operatorname{Spec} \pi_0 E = \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$, can we identify E with the ring spectrum E_R? Choose a complex orientation on E, given by a map of ring spectra $\mathrm{MU} \to E$ which induces a map of graded rings $\phi : L \to \pi_* E$. Then the homology theory E_R is given by

$$(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L (\pi_* E).$$

We get an evident map of homology theories $(E_R)_*(X) \to E_*(X)$. This map is an isomorphism by construction when X is a point. Since E is even and E_R is Landweber exact, the results of the previous lecture show that we get a map of spectra $E_R \to E$ which is well-defined up to homotopy equivalence. This map induces an isomorphism $\pi_* E_R \to \pi_* E$ by construction, and is therefore an equivalence of spectra; it is easy to see that this equivalence is compatible with the ring structures on E_R and E.