Phantom Maps (Lecture 17)

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We begin by recalling Adams' variant of the Brown representability theorem:

Theorem 1 (Adams). *Let E be a spectrum and let h_* be a homology theory. Suppose we are given a map of homology theories $\alpha : E_* \to h_*$ (that is, a collection of maps $E_*(X, Y) \to h_*(X, Y)$, depending functorially on a pair of spaces $(Y \subseteq X)$ and compatible with boundary maps). Then there is a map of spectra $\beta : E \to E'$ and an isomorphism of homology theories $E'_* \simeq h'_*$ such that α is given by the composition $E_* \to E'_* \simeq h_*$.*

Corollary 2 (Adams). *Let E and E' be spectra, and let $\alpha : E_* \to E'_*$ be a map between the corresponding homology theories. Then α is induced by a map of spectra $\overline{\alpha} : E \to E'$.*

Proof. Let $h_* = E'_*$. Applying Theorem 1 the evident map $\alpha : E_* \oplus E'_* \to h_*$, we get a spectrum F and a map $E \oplus E' \to F$ inducing α. This comes from a pair of spectrum maps $f : E \to F$ and $g : E' \to F$. The map g induces an isomorphism $\pi_* E' = h_*(*) = \pi_* F$ and is therefore a homotopy equivalence. Then $\overline{\alpha} : g^{-1} \circ f$ is the desired map of spectra from E to E'. □

Corollary 3 (Adams). *Every homology theory h_* is represented by a spectrum E, which is uniquely defined up to (nonunique) homotopy equivalence.*

Proof. The existence of E follows from Theorem 1. For the uniqueness, we note that if E and E' are two spectra with $E_* \simeq h_* \simeq E'_*$, then the isomorphism $E_* \simeq E'_*$ is induced by a map of spectra $E \to E'$ (Corollary 2), which is automatically a homotopy equivalence. □

In the situation of Corollary 2, the map $\overline{\alpha}$ is generally not determined by α, even up to homotopy. This is due to the existence of *phantom maps*:

Definition 4. Let $f : E \to E'$ be a map of spectra. We say that f is a *phantom* if the underlying map of homology theories $E_* \to E'_*$ is zero: that is, for every space X, the map $E_*(X) \to E'_*(X)$ is identically zero.

Lemma 5. *Let $f : E \to E'$ be a map of spectra. The following conditions are equivalent:*

(1) *The map f is a phantom.*

(2) *For every spectrum X, the map $E_*(X) \to E'_*(X)$ is zero.*

(3) *For every finite spectrum X, the map $E_*(X) \to E'_*(X)$ is zero.*

(4) *For every finite spectrum X, the map $E^*(X) \to E'^*(X)$ is zero.*

(5) *For every finite spectrum X and every map $g : X \to E$, the composition $f \circ g : X \to E'$ is nullhomotopic.*

Proof. The implication (2) ⇒ (1) is obvious, and the converse follows from the fact that every spectrum X can be written as a filtered colimit $\varinjlim \Sigma^{\infty - n} \Omega^{\infty - n} X$. The implication (2) ⇒ (3) is obvious, and the converse follows from the fact that every spectrum is a filtered colimit of finite spectra. The equivalence of (4) and (5) follows by Spanier-Whitehead duality, and the equivalence of (4) and (5) is a tautology. □

Let us now return to the setting of the previous lectures. Let $L \simeq \mathbf{Z}[t_1, \ldots]$ denote the Lazard ring, and let M be a graded L-module. Assume that the grading on M is *even*: that is, $M_k \simeq 0$ for every odd number k. In the last lecture, we saw that if M satisfies Landweber's criterion: that is, if the sequence $v_0 = p, v_1, v_2, \ldots \in L$ is M-regular for every prime number p, then the construction

$$X \mapsto \mathrm{MU}_*(X) \otimes_L M$$

is a homology theory. It follows from Corollary 3 that this homology theory is represented by a spectrum E, which is unique up to homotopy equivalence. We will say that a spectrum E is *Landweber-exact* if it arises from this construction. Our goal in this lecture is to show that, as an object of the homotopy category of spectra, E is functorially determined by M. This is a consequence of the following assertion:

Theorem 6. *Let E be a Landweber-exact spectrum, and let E' be a spectrum such that $\pi_k E' \simeq 0$ for k odd. Then every phantom map $f : E \to E'$ is nullhomotopic.*

Corollary 7. *Let E and E' be Landweber exact spectra. Then every phantom map $f : E \to E'$ is null-homotopic. In particular, every nontrivial endomorphism of E acts nontrivially on the homology theory E_*.*

To prove Theorem 6, we introduce two new notions:

Definition 8. We will say that a finite spectrum X is *even* if the homology groups $\mathrm{H}_k(X; \mathbf{Z})$ are free abelian groups, which vanish when k is odd. Equivalently, a finite spectrum X is *even* if it admits a finite cell decomposition using only even-dimensional cells.

We say that a spectrum E is *evenly generated* if, for every map $X \to E$ where X is a finite spectrum, there exists a factorization $X \to X' \to E$ where X' is a finite even spectrum.

Theorem 6 is a consequence of the following two assertions:

Proposition 9. *Every Landweber exact spectrum E is evenly generated.*

Proposition 10. *Let E be an evenly generated spectrum and let E' be a spectrum whose homotopy groups are concentrated in even degrees. Then every phantom map $f : E \to E'$ is null.*

We begin by proving Proposition 9. Let E be a Landweber-exact spectrum, associated to a graded L-module M, and let $f : X \to E$ be a map where X is a finite spectrum. We can associate to f an element of $E^0(X) = E_0(DX) = \mathrm{MU}_0(DX) \otimes_L M = \mathrm{MU}^0(X) \otimes_L M$, which can be written as $\sum c_i m_i$ where $c_i \in \mathrm{MU}^{d_i}(X)$ and $m_i \in M_{d_i}$. Then f factors as a composition

$$X \stackrel{\{c_i\}}{\to} \bigoplus \Sigma^{d_i} \mathrm{MU} \stackrel{m_i}{\to} E.$$

We may therefore replace E by $\bigoplus \Sigma^{d_i} \mathrm{MU}$: that is, it suffices to prove that $\bigoplus \Sigma^{d_i} \mathrm{MU}$ is evenly generated. Since M is evenly graded, each of the integers d_i is even. We can therefore reduce to showing that MU itself is evenly generated.

Since $\mathrm{MU} \simeq \varinjlim \mathrm{MU}(n)$, it suffices to show that each $\mathrm{MU}(n)$ is evenly generated. Recall that $\mathrm{MU}(n)$ is the Thom complex of the virtual bundle $\zeta - \mathbf{C}^n$, where ζ is the tautological vector bundle on $BU(n)$. We can write $BU(n) \simeq \varinjlim_m \mathrm{Grass}(n, n+m)$, wehre $\mathrm{Grass}(n, n+m)$ denotes the Grassmannian of n-dimensional subspaces of $\mathbf{C}^{n+m}$. It follows that $\mathrm{MU}(n)$ is a direct limit of Thom spectra associated to the finite-dimensional Grassmannians $\mathrm{Grass}(n, n+m)$. It therefore suffices to show that each of these Thom complexes is an even finite spectrum. We now note that the space $\mathrm{Grass}(n, n+m)$ admits a finite cell decomposition with cells of even dimension: for example, we can take the Bruhat decomposition. This proves Proposition 9.

We now prove Proposition 10. Let E be an evenly generated spectrum. We begin by describing the structure of phantom maps from E to other spectra. Let A be a set of representatives for all homotopy equivalence classes of maps $X_\alpha \to E$, where X_α is an even finite spectrum, and form a fiber sequence

$$K \to \bigoplus_{\alpha} X_\alpha \xrightarrow{u} E.$$

This sequence is classified by a map $u' : E \to \Sigma(K)$. Since E is evenly generated, every map from a finite spectrum X into E factors through u, so the composite map $X \to E \to \Sigma(K)$ is null: in other words, u' is a phantom map. Conversely, if $f : E \to E'$ is *any* phantom map, then $f \circ u$ is nullhomotopic, so that f factors as a composition $E \to \Sigma(K) \to E'$. Consequently, to prove Proposition 10, it will suffice to prove that every map $\Sigma(K) \to E'$ is nullhomotopic: that is, that the group $E'^{-1}(K)$ is zero.

Since the homotopy groups of E' are concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence shows that $E'^{-1}(X) \simeq 0$ whenever X is a finite even spectrum. It will therefore suffice to prove the following:

($*$) The spectrum K is a retract of a direct sum of even finite spectra.

To prove ($*$), we will compare the cofiber sequence

$$K \to \bigoplus_{\alpha \in A} X_\alpha \to E$$

with another cofiber sequence of spectra. Let B be the collection of triples (α, α', f), where $\alpha, \alpha' \in A$ and f ranges over all homotopy classes of maps fitting into a commutative diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f} & X_{\alpha'} \\ & \searrow \quad \swarrow & \\ & E. & \end{array}$$

For each $\beta = (\alpha, \alpha', f) \in B$, we let $Y_\beta = X_\alpha$. We have a canonical map $\phi : \bigoplus_{\beta \in B} Y_\beta \to \bigoplus_{\alpha \in A} X_\alpha$, whose restriction to Y_β for $\beta = (\alpha, \alpha', f)$ given by the difference of the maps $Y_\beta = X_\alpha \to \bigoplus_{\alpha \in A} X_\alpha$ and

$$Y_\beta = X_\alpha \xrightarrow{f} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha.$$

Let F be the cofiber of the map ϕ. By construction, we have a map of fiber sequences

$$\begin{array}{ccccc} \bigoplus_{\beta \in B} Y_\beta & \longrightarrow & \bigoplus_{\alpha \in A} X_\alpha & \xrightarrow{u} & F \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & \bigoplus_{\alpha \in A} X_\alpha & \longrightarrow & E. \end{array}$$

We now construct a map of spectra $q : E \to F$. By Corollary 2, it will suffice to define a map of homology theories $E_* \to F_*$. We will give a map $E_*(X) \to F_*(X)$ defined for every spectrum X. Since homology theories commute with filtered colimits, it will suffice to consider the case where X is a finite spectrum. Replacing X by its Spanier-Whitehead dual, we are reduced to the problem of producing a map $q(f) : X \to F$ for every map of spectra $f : X \to E$ for X finite.

Here is our construction. Since E is evenly generated, every map $f : X \to E$ factors through some map $X \xrightarrow{f'} X_{\alpha'} \to E$ for $\alpha' \in A$. We define $q(f)$ to be the composite map $X \xrightarrow{f'} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha \to F$. We must show that this construction is well-defined; that is, it does not depend on the choice of f'. To this end, suppose we are given another factorization of f $X \xrightarrow{f''} X_{\alpha''} \to E$, where $\alpha'' \in A$. Let Y denote the pushout $X_{\alpha'} \coprod_X X_{\alpha''}$. Then Y is a finite spectrum, and our data gives a canonical map $Y \to E$. Since E is evenly generated, this map factors as a composition

$$Y \xrightarrow{g} X_\alpha \to E$$

for some $\alpha \in A$. Let h' denote the composite map $X_{\alpha'} \to X' \xrightarrow{g} X_\alpha$ and let h' be defined similarly. Then (α', α, h) and (α'', α, h) can be identified with elements of B. It follows that the composite maps

$$X \to X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha \to F$$

$$X \to X_{\alpha''} \to \bigoplus_{\alpha \in A} X_\alpha \to F$$

both coincide with the map

$$X \to Y \xrightarrow{g} X_\alpha \to \bigoplus_{\alpha \in A} X_\alpha \to F,$$

which proves that q is well-defined.

We now have a larger commutative diagram of fiber sequences

$$\begin{array}{ccccc}
K & \longrightarrow & \bigoplus_{\alpha \in A} X_\alpha & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{\beta \in B} Y_\beta & \longrightarrow & \bigoplus_{\alpha \in A} X_\alpha & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & \bigoplus_{\alpha \in A} & \longrightarrow & E.
\end{array}$$

The right vertical composition induces the identity map on the underlying homology theory E_*: that is, it differs from id_E by a phantom map. In particular, it is an equivalence, so that the left vertical composition is an equivalence of K with itself. It follows that K is a retract of $\bigoplus_{\beta \in B} Y_\beta$, which proves $(*)$.