The Landweber Exact Functor Theorem (Lecture 16)

April 27, 2010

Our goal in this lecture is to prove the following result:

Theorem 1. *Let M be a module over the Lazard ring. Then M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the elements $v_0 = p, v_1, v_2, \ldots \in L$ form a regular sequence for M.*

We first note that M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the localization $M_{(p)} = M \otimes \mathbf{Z}_{(p)}$ is flat over $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. We therefore fix a prime number p and work locally at p.

Lemma 2. *Let $q : \operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \to \mathcal{M}_{\mathrm{FG}}$ be the flat map considered in the previous lecture. Let M be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. Then M is flat over $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ if and only if $q^* M$ is a flat $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ module.*

Proof. The "only if" direction is immediate from the definitions. Conversely, suppose that $q^* M$ is flat. Fix any map $f : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$; we wish to prove that $f^* M$ is a flat R-module. Form a pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "f"] \\
\operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \arrow[r, "q"] & \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}.
\end{tikzcd}$$

We saw in the last lecture that q is faithfully flat, so $R \to B$ is a faithfully flat map of commutative rings. Consequently, it will suffice to show that $f^* M \otimes_R B$ is a flat B-module. But $f^* M \otimes_R B = q^* M \otimes_{\mathbf{Z}_{(p)}[v_1, v_2, \ldots]} B$, which if flat over B since $q^* M$ is flat over $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$. ∎

Let us now return to the proof of Theorem 1. Let M be a module over the localized Lazard ring $L_{(p)}$ such that $v_0 = p, v_1, v_2, \ldots$ is a regular sequence on M. We wish to prove that the pushforward of M along the map $\operatorname{Spec} L_{(p)} \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ is flat. Form a pullback square

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} L_{(p)} \arrow[d] \\
\operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \arrow[r] & \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}.
\end{tikzcd}$$

By the Lemma, it will suffice to show that $M_B = M \otimes_{L_{(p)}} B$ is flat as a module over the ring $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$. In other words, we wish to prove that for every R-module N, the groups $\operatorname{Tor}_i^{\mathbf{Z}_{(p)}[v_1, \ldots]}(M_B, N)$ vanish for $i > 0$.

Since the functor $N \mapsto \operatorname{Tor}_i^{\mathbf{Z}_{(p)}[v_1, v_2, \ldots]}(M_B, N)$ commutes with filtered colimits, it will suffice to show that the groups $\operatorname{Tor}_i^R(M_B, N)$ vanish when $i > 0$ and N is a finitely presented $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$-module (every module is a filtered colimit of finitely presented modules). Note that a finite presentation of an $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$-module can reference only finitely many of the polynomial generators $v_1, v_2, \ldots$. In other words, we may assume that there exists an integer $n \geq 1$ such that $N \simeq N_0[v_{n+1}, v_{n+2}, v_{n+3}, \ldots]$, where N_0 is a module over the ring $\mathbf{Z}_{(p)}[v_1,\ldots,v_n]$. In this case, we have $\mathrm{Tor}_i^{\mathbf{Z}_{(p)}[v_1,v_2,\ldots]}(M_B,N) \simeq \mathrm{Tor}_i^{\mathbf{Z}_{(p)}[v_1,\ldots,v_n]}(M_B,N_0)$. In other words, we are reduced to proving that M_B is a flat module over $\mathbf{Z}_{(p)}[v_1,\ldots,v_n]$ for all n.

Let us now address a potentially confusing point. By construction, the ring B is equipped with homomorphisms $\phi' : \mathbf{Z}_{(p)}[v_1,v_2,\ldots] \to B$ and $\phi'' : L_{(p)} \to B$. Consequently, we obtain two *different* sequences of elements

$$v_0', v_1', v_2', \ldots \qquad v_0'', v_1'', v_2'', \ldots$$

in B, given by $v_i' = \phi'(v_i)$ and $v_i'' = \phi''(v_i)$. It follows that, for each $m \geq 0$, the finite sequences $(v_0', v_1', v_2', \ldots, v_{m-1}')$ and $(v_0'', v_1'', \ldots, v_{m-1}'')$ generate the same ideal $I_m \subseteq B$.

We will prove the following:

Claim 3. *For $m \leq n+1$, the quotient $M_B/I_m M_B$ is a flat module over the ring $\mathbf{Z}_{(p)}[v_1,v_2,\ldots,v_n]/(p,v_1,\ldots,v_{m-1})$.*

When $m=0$, Claim 3 reduces to what we need to know. We will prove Claim 3 by descending induction on m. Note that if $m=n+1$, then $\mathbf{Z}_{(p)}[v_1,\ldots,v_n]/(p,v_1,\ldots,v_n) \simeq \mathbf{F}_p$ is a field and there is nothing to prove. To carry out the inductive step, we need the following algebraic lemma:

Lemma 4. *Let R be a commutative ring containing a non zero-divisor x, and let M be an R-module. Then M is flat over R if and only if the following conditions are satisfied:*

(1) *The element x is a non zero-divisor on M.*

(2) *The quotient M/xM is a flat $R/(x)$-module.*

(3) *The module $M[x^{-1}]$ is flat over $R[x^{-1}]$.*

Proof. The necessity of conditions (1) through (3) is easy (and not needed for our application). Let us assume that conditions (1), (2), and (3) are satisfied. We wish to prove that M is flat over R: that is, for any R-module N, the groups $\mathrm{Tor}_i^R(M,N)$ vanish for $i>0$. We carry out the proof in several steps:

(a) Suppose that N is annihilated by x: that is, N is a module over $R/(x)$. Assumption (1) gives $\mathrm{Tor}_i^R(M,N) \simeq \mathrm{Tor}_i^{R/(x)}(M/xM,N)$, which vanishes for $i>0$ by assumption (2).

(b) Suppose that N is annihilated by x^k for some k. We prove by induction on k that $\mathrm{Tor}_i^R(M,N) \simeq 0$ for $i>0$. We have an exact sequence

$$0 \to K \to N \to xN \to 0,$$

where K is the kernel of the map $N \to xN$. Since $\mathrm{Tor}_i^R(M,K) \simeq 0$ by (a) and $\mathrm{Tor}_i^R(M,xN) \simeq 0$ by the inductive hypothesis, we deduce from the exact sequence

$$\mathrm{Tor}_i^R(M,K) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M,xN)$$

that $\mathrm{Tor}_i^R(M,N) \simeq 0$.

(c) Suppose that N consists of x-power torsion: that is, every element $n \in N$ satisfies $x^k n = 0$ for $k \gg 0$. Then N is a filtered colimit of submodules annihilated by x^k, so that $\mathrm{Tor}_i^R(M,N) \simeq 0$ for $i>0$ by part (b).

(d) Let N be arbitrary, and let K be the kernel of the map $N \to N[x^{-1}]$. Then K satisfies the hypothesis of (c), so that $\mathrm{Tor}_i^R(M,K) \simeq 0$ for $i>0$. Consequently, to prove that $\mathrm{Tor}_i^R(M,N) \simeq 0$, it suffices to show that $\mathrm{Tor}_i^R(M,N/K) \simeq 0$; that is, we may replace N by N/K and thereby assume that the map $N \to N[x^{-1}]$ is injective.

(e) Let N be as in (d), and let K' be the cokernel of the injection $N \to N[x^{-1}]$. Then K' satisfies the condition of (c), so that $\mathrm{Tor}_i^R(M,K') \simeq 0$ for $i>0$. Consequently, to prove that $\mathrm{Tor}_i^R(M,N) \simeq 0$, it will suffice to show that $\mathrm{Tor}_i^R(M,N[x^{-1}]) \simeq 0$.

(f) We are now reduced to the case where $N \simeq N[x^{-1}]$: that is, N is a module over $R[x^{-1}]$. We then have $\mathrm{Tor}_i^R(M,N) \simeq \mathrm{Tor}_i^{R[x^{-1}]}(M[x^{-1}],N)$, which vanishes for $i>0$ by assumption (3).

□

Let us now return to the proof of Claim 3. Let $m \leq n$; we wish to prove that $M_B/I_m M_B$ is flat over $R = \mathbf{Z}_{(p)}[v_1,\ldots,v_n]/(p,v_1,\ldots,v_{m-1})$. Note that $M_B/I_m M_B$ can be identified with the tensor product

$$B \otimes_{L_{(p)}} (M/(v_0,\ldots,v_{m-1})).$$

By assumption, v_m is a non zero-divisor on the quotient $M/(v_0,\ldots,v_{m_1})$. Since B is flat over $L_{(p)}$, we have an exact sequence

$$0 \to M_B/I_m M_B \xrightarrow{v''_m} M_B/I_m M_B \to M_B/I_{m+1}M_B \to 0.$$

Since v''_m is congruent to an invertible multiple of v'_m moduli I_m, we deduce that $v_m \in R$ is a non zero-divisor on $M_B/I_m M_B$. Moreover, the quotient $M_B/(I_m,v_m)M_B \simeq M_B/I_{m+1}M_B$ is flat over $R/(v_m)$ by the inductive hypothesis. By the Lemma, we are reduced to proving that $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $R[v_m]^{-1}$. We will prove the following stronger statement:

Claim 5. *For every integer $m \geq 0$, the module $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $(\mathbf{Z}_p[v_1,v_2,\ldots]/(p,v_1,\ldots,v_{m-1}))[v_m^{-1}]$.*

We have a pullback diagram of stacks

$$\begin{tikzcd}
\operatorname{Spec} B/(p,v_1,\ldots,v_{m-1})[v_m^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} L_{(p)}/(v_0,\ldots,v_{m-1})[v_m^{-1}] \arrow[d] \\
\operatorname{Spec}(\mathbf{Z}_{(p)}[v_1,v_2,\ldots]/(p,v_1,\ldots,v_{m-1}))[v_m^{-1}] \arrow[r] & \mathcal{M}_{\mathrm{FG}}^m .
\end{tikzcd}$$

Claim 5 is a special case of the assertion that the $L_{(p)}/(v_0,\ldots,v_{m-1})[v_m^{-1}]$-module $(M/(v_0,\ldots,v_{m-1})M)[v_m^{-1}]$ is flat over $\mathcal{M}_{\mathrm{FG}}^m$. This in turn follows from:

Claim 6. *Every quasi-coherent sheaf on the stack $\mathcal{M}_{\mathrm{FG}}^m$ is flat.*

We will prove this claim when $m>0$; the proof when $m=0$ is similar. Let X be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}^m$. We wish to prove that q^*X is a flat A-module, for any map $\operatorname{Spec} A \to \mathcal{M}_{\mathrm{FG}}^m$ classifying a formal group height exactly m on A. Working locally on $\operatorname{Spec} A$, we may assume that the formal group is coordinatizable. Choose a formal group law of height m over $\mathbf{F}_p$ classified by a map $f : \operatorname{Spec} \mathbf{F}_p \to \mathcal{M}_{\mathrm{FG}}^m$, and form a pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} A' \arrow[r] \arrow[d] & \operatorname{Spec} \mathbf{F}_p \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] & \mathcal{M}_{\mathrm{FG}}^m
\end{tikzcd}$$

In Lecture 14, we proved that A' is a direct limit of a sequence of injective finite etale ring extensions; in particular, A' is faithfully flat over A. Consequently, it will suffice to prove that $q^*X \otimes_A A'$ is flat over A'. But $q^*X \otimes_A A' \simeq f^*X \otimes_{\mathbf{F}_p} A'$. We are therefore reduced to proving that f^*X is flat over $\mathbf{F}_p$, which is obvious since $\mathbf{F}_p$ is a field.