The Stratification of $\mathcal{M}_{\mathrm{FG}}$ (Lecture 13)

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Let p be a prime number, fixed throughout this lecture. Our goal is to describe the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ of formal groups over p-local rings.

We begin by recalling a few definitions from the previous lecture. If $f(x,y) \in R[[x,y]]$ is a formal group law over a $\mathbf{Z}_{(p)}$-algebra R, we let v_n denote the coefficient on t^{p^n} in the p-series $[p](t)$. We obtain a sequence of elements $v_0 = p, v_1, \ldots \in R$. We say that f has *height* $\geq n$ if the elements v_i vanish for $i < n$, and *height exactly* n if it has height $\geq n$ and v_n is invertible.

Restricting our attention to the universal case, we can regard $v_0, v_1, \ldots$ as elements of the Lazard ring L. We now describe the relationship between these elements and our presentation of L as a polynomial ring $\mathbf{Z}[t_1, t_2, \ldots]$. In our earlier discussion, the coordinates t_i were not canonically determined. What is canonically determined is the isomorphism $(I/I^2)_{2n} \simeq \mathbf{Z} t_n$, where I is the ideal generated by elements of positive degree. We can regard each v_i as an element of $L_{2(p^i-1)}$, so that v_i has a canonically defined image in $(I/I^2)_{2(p^i-1)} \simeq \mathbf{Z} t_{p^i-1}$.

Proposition 1. *The image of $v_n \in (I/I^2)_{2(p^n-1)} \simeq \mathbf{Z}$ is $p^{p^n-1} - 1$. That is, we can write $v_n = -t_{p^n-1} + p^{p^n-1} t_{p^n-1} +$ decomposables.*

Proof. Let $k = p^n - 1$. The homomorphism $L \to \mathbf{Z} \oplus (I/I^2)_{2k} \simeq \mathbf{Z} \oplus \mathbf{Z} t_k$ classifies the formal group law

$$f(x,y) = x + y + \sum_{0 < i < p^n} \frac{1}{p} \binom{p^n}{i} t_k x^i y^{p^n - i}.$$

We obtain formally $f(x,y) = x + y + \frac{t_k}{p}((x+y)^{p^n} - x^{p^n} - y^{p^n})$. It follows by induction on a that the series $[a]$ is given by $[a](t) = at + \frac{t_k}{p}((at)^{p^n} - at^{p^n})$. In particular, the coefficient of t^{p^n} in $[p](t)$ is $\frac{t_k}{p}(p^{p^n} - p) = (p^{p^n-1} - 1) t_k$. □

It follows that after localizing at the prime p, we can choose another isomorphism $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$, where each t_{p^n-1} is given by v_n. In other words, the elements v_n in L can be regarded as the "interesting" generators of L (under the isomorphism $L \simeq \pi_* \mathrm{MU}$ of Quillen's theorem, these are the generators of Adams filtration 1).

Corollary 2. *Let k be a field of characteristic p. Then, for every integer $1 \leq n \leq \infty$, there exists a formal group law of height n over k.*

Proof. If $n = \infty$, we can take $f(x,y) \in k[[x,y]]$ to be the additive formal group law $f(x,y) = x + y$. If $n < \infty$, we take f to be any formal group law classified by a map $L \simeq \mathbf{Z}[t_1, t_2, \ldots] \to k$ such that $t_i \mapsto 0$ for $i < p^n - 1$, but $t_{p^n-1} \mapsto 1$. □

Recall that the condition that a formal group $f(x,y) \in R[[x,y]]$ have height $\geq n$ depends only on the isomorphism class of f. Moreover, it is a *local* condition: that is, if we are given a collection of elements $a_1, \ldots, a_k \in R$ with $a_1 + \cdots + a_k = 1$, then f has height $\geq n$ over R if and only if f has height $\geq n$ over $R[\frac{1}{a_i}]$ for all i. Consequently, if $\mathcal{F} : \mathrm{Alg}_R \to \mathrm{Ab}$ is a formal group over R which is not necessarily coordinatizable, it makes sense to demand that $\mathcal{F}$ has height $\geq n$: this is the requirement that $\mathcal{F} \mid \mathrm{Alg}_{R'}$ have height $\geq n$, whenever R' is an R-algebra such that $\mathcal{F} \mid \mathrm{Alg}_{R'}$ is coordinatizable.

Remark 3. Here is another interpretation of the height of a formal group. Let $\mathcal{F} : \mathrm{Alg}_R \to \mathrm{Ab}$ be a formal group of height exactly n. Then $\mathcal{F}[p] = \ker(\mathcal{F} \xrightarrow{p} \mathcal{F})$ is representable by a finite flat group scheme over R, of rank p^n. To see this, it suffices to work locally: we may therefore assume that $\mathcal{F}$ is defined by a formal group law $f(x,y) \in R[[x,y]]$ with p-series $[p](t) = \lambda t^{p^n} + \cdots$ where λ is invertible. Then $\mathcal{F}[p] = \mathrm{Spec}\, R[[t]]/(\lambda t^{p^n} + \cdots)$.

For example, if $\mathcal{F}$ is the formal multiplicative group, then $\mathcal{F}[p]$ is the group scheme μ_p, defined by $\mu_p(A) = \{a \in A : a^p = 1\}$. We have $\mu_p = \mathrm{Spec}\, R[a]/(a^p - 1)$, which has rank p.

We can define a closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n}$ of $\mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{Z}_{(p)}$ as follows: for every commutative $\mathbf{Z}_{(p)}$-algebra R, $\mathcal{M}_{\mathrm{FG}}^{\geq n}(R)$ is the category of formal groups of height $\geq n$ over R (with morphisms given by isomorphisms). We have $\mathcal{M}_{\mathrm{FG}}^{\geq n} = \mathrm{Spec}(L_{(p)}/(v_0, \ldots, v_{n-1}))/G^+$, where G^+ is the group scheme of coordinate transformations defined in the previous lecture. This makes sense because the ideal $(v_0, \ldots, v_{n-1})$ is G^+-invariant: this is a translation of the statement that the condition of having height $\geq n$ is an isomorphism invariant condition.

Remark 4. The elements $v_i \in L$ are not themselves G-invariant: that is, if f and f' are isomorphic formal group laws over a commutative ring R, then the p-series $[p]_f(t)$ and $[p]_{f'}(t)$ are generally different. However, if we assume that f and f' have height $\geq n$ and $g(t) = b_0 t + b_1 t^2 + \ldots$ is an invertible power series such that $gf(x,y) = f'(g(x), g(y))$, then $g \circ [p]_f \simeq [p]_{f'} \circ g'$. If $[p]_f(t) = v_n t^{p^n} + \cdots$ and $[p]_{f'} = v_n' t^{p^n} + \cdots$, then examining leading terms gives $b_0 v_n = b_0^{p^n} v_n'$. In other words, as an element in the quotient ring $L/(v_0, \ldots, v_{n-1})$, v_n is invariant under the subgroup $G \subseteq G^+$, and is acted on by the quotient $G^+/G \simeq \mathbb{G}_m$ by the character $\mathbb{G}_m \xrightarrow{p^n - 1} \mathbb{G}_m$. In more invariant terms, this means that we can descend v_n to a section of the line bundle $\omega^{p^n - 1}$ on the moduli stack $\mathcal{M}_{\mathrm{FG}}^{\geq n}$.

For $0 \leq n < \infty$, we let $\mathcal{M}_{\mathrm{FG}}^n$ denote the locally closed substack

$$\mathcal{M}_{\mathrm{FG}}^{\geq n} - \mathcal{M}_{\mathrm{FG}}^{\geq n+1} = (\mathrm{Spec}\, L_{(p)}[v_n^{-1}]/(v_0, \ldots, v_{n-1}))/G^+$$

of $\mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{Z}_{(p)}$. Also let $\mathcal{M}_{\mathrm{FG}}^{\infty} = \mathcal{M}_{\mathrm{FG}}^{\geq \infty} = (\mathrm{Spec}\, L/(v_0, v_1, \ldots))/G^+$ denote the moduli stack of formal groups having infinite height. Thus $\mathcal{M}_{\mathrm{FG}}^n$ are the open strata for a stratification of the moduli stack $\mathcal{M}_{\mathrm{FG}}$. We will see that each stratum as a relatively simple structure.

Example 5. The moduli stack $\mathcal{M}_{\mathrm{FG}}^0$ of formal groups of height 0 can be identified with $\mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{Q} \simeq B\mathbb{G}_m$.

Note that $\mathcal{M}_{\mathrm{FG}}^{\geq 1} = \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{F}_p$. For the remainder of the discussion, we will work with commutative rings R which have characteristic p: that is, we will assume that $p = 0$ in R.

The following characterization of height is convenient:

Proposition 6. *Let R be a commutative ring such that $p = 0$ in R, and let $f(x,y) \in R[[x,y]]$ be a formal group law over R. For $1 \leq n \leq \infty$, the following conditions are equivalent:*

(1) *The formal group law f has height $\geq n$.*

(2) *There exists a formal group law f' which is isomorphic to f such that $f'(x,y) \equiv x + y \mod (x,y)^{p^n}$ is congruent to $x + y$ modulo $(x,y)^{p^n}$.*

Lemma 7. *Let R be a commutative ring and let $f, f' \in R[[x,y]]$ be formal group laws. Suppose that $f(x,y)$ is congruent to $f'(x,y)$ modulo the ideal $(x,y)^m$. Let*

$$d = \begin{cases} p & \text{if } m = p^n \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a unique constant $\lambda \in R$ such that $f(x,y)$ is congruent to $f'(x,y) + \sum_{0<i<m} \frac{\lambda}{d}\binom{m}{i} x^i y^{m-i}$. modulo $(x,y)^{m+1}$.

Proof. There exist constants $\{\lambda_{i,j}\}_{i+j=m}$ such that $f(x,y)$ is congruent to $f'(x,y)+\sum_{0\leq i\leq m}\lambda_{i,m-i}x^iy^{m-i}$ modulo $(x,y)^{m+1}$. Since $f(x,0)=f'(x,0)=x$, we conclude that $\lambda_{m,0}=0$; similarly $\lambda_{0,m}=0$.

We have

$$f(f(x,y),z)-f'(f'(x,y),z)=f(x,f(y,z))-f'(x,f'(y,z)).$$

Extracting the coefficient of $x^iy^jz^k$ when $i+j+k=m$ and $i,j,k>0$, we conclude that

$$\binom{i+j}{j}\lambda_{i+j,k}=\binom{j+k}{j}\lambda_{i,j+k}.$$

In Lecture 3, we saw that all solutions to these equations are as stated in the Lemma. □

Proof of Proposition 6. First assume that n is finite. We prove by induction on $m<p^n$ that, after a change of variable, we can assume that $f(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. By the inductive hypothesis, we may assume that the congruence holds modulo $(x,y)^{m-1}$. Let d be defined as in Lemma 7, so that $f(x,y)$ is congruent to $x+y+\sum_{0<i<m}\frac{\lambda}{d}\binom{m}{i}x^iy^{m-i}$ for some $\lambda\in R$. If m is not a power of p, then we define $f'(x,y)=g^{-1}f(g(x),g(y))$ where $g(t)=t+\frac{\lambda t^m}{d}$; a simple calculation shows that $f'(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. If $m=p^{n'}$ then we necessarily have $n'<n$. We claim that $f(x,y)$ is automatically congruent to $x+y$ modulo $(x,y)^m$. This follows from the calculation of the previous lecture: f is classified by a homomorphism $L\simeq\mathbf{Z}[t_1,\ldots,]\rightarrow R$, and we wish to show that the image of each $t_{m'}$ is equal to zero for $m'<m$. By the inductive hypothesis, this holds for $m'<m-1$. Then the image of t_{m-1} is given by $-v_{n'}$ (here $v_{n'}$ is the coefficient of $t^{p^{n'}}$ in the p-series $[p](t)$), and therefore vanishes since we have assumed that f has height $\geq n$.

Suppose now that n is infinite. Using the above construction, we define a sequence of formal group laws $f_m(x,y)$ which are isomorphic to f such that $f_m(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. We have $f_m(x,y)=g_m^{-1}f(g_m(x),g_m(y))$. By construction, the power series $g_m(t)$ converge in the t-adic topology to an invertible power series $g(t)$; then $g^{-1}f(g(x),g(y))=x+y$ is the additive formal group. □

Corollary 8. *Let f be a formal group law of infinite height over a commutative ring R (necessarily with $p=0$ in R). Then f is isomorphic to the additive formal group law $f'(x,y)=x+y$.*

Remark 9. It follows that we can identify $\mathcal{M}_{\mathrm{FG}}^{\infty}$ with the classifying stack for the group of automorphisms of the additive formal group $f(x,y)=x+y\in\mathbf{F}_p[[x,y]]$. This is the group scheme whose R-points are given by power series of the form

$$g(t)=a_0t+a_1t^p+a_2t^{p^2}+\ldots\in R[[t]],$$

where a_0 is invertible. This group scheme is closely related to the structure of the (mod p) Steenrod algebra.

We now study formal groups of height n where $0<n<\infty$. The basic result is the following:

Theorem 10 (Lazard)**.** *Let k be an algebraically closed field of characteristic p. Then two formal group laws $f(x,y),f'(x,y)\in k[[x,y]]$ are isomorphic if and only if they have the same height.*

Here the condition that k be algebraically closed can be weakened, but not completely removed. To prove Theorem 10 we need to write down an isomorphism between f and f': that is, we need to find an invertible power series $g(t)=b_0t+b_1t^2+\ldots$ such that $gf(x,y)=f'(g(x),g(y))$. This identity amounts to a system of equations that the coefficients b_i must satisfy. Theorem 10 asserts that these equations can be solved with values in an algebraically closed field. In fact, we can be much more precise. Let $f(x,y),f'(x,y)\in R[[x,y]]$ be formal group laws of height exactly $n>0$ over a commutative ring R. Then we can define a ring $R'=R[b_0^{\pm1},b_1,\ldots]/I$ which parametrizes isomorphisms between f and f': take I to be the ideal generated by the coefficients on x^iy^j in the expression $gf(x,y)-f'(g(x),g(y))$. A more precise version of Theorem 10 can be formulated as follows:

Theorem 11. *Let $f(x,y), f'(x,y) \in R[[x,y]]$ be formal group laws of height exactly $n > 0$ and let R' be defined as above. Then R' isomorphic to the direct limit of a system of (injective) finite etale maps*

$$R = R(0) \hookrightarrow R(1) \hookrightarrow R(2) \hookrightarrow \cdots$$

When R is an algebraically closed field k, each $R(i)$ is a product of copies of k. It follows that we can choose a compatible system of ring homomorphisms $R(i) \to k$, which together define a map $R' \to k$ giving rise to the desired isomorphism of f with f'. In fact, we need not assume that k is algebraically closed: it is enough to suppose that k is separably closed or, more generally, that k is a strictly Henselian ring.

We will prove Theorem 11 in the next lecture.