Formal Groups (Lecture 11)

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We begin by recalling our discussion of the Adams-Novikov spectral sequence:

Claim 1. *Let X be any spectrum. Then $\mathrm{MU}_*(X)$ is a module over the commutative ring $L = \pi_* \mathrm{MU}$, and can therefore be understood as a quasi-coherent sheaf on the affine scheme $\operatorname{Spec} L$ which parametrizes formal group laws (here L denotes the Lazard ring). This quasi-coherent sheaf admits an action of the affine group scheme $G = \operatorname{Spec} \mathbf{Z}[b_1, b_2, \ldots]$ which assigns to each commutative ring R the group $\{g \in R[[t]] : g(t) = t + b_1 t^2 + b_2 t^3 + \cdots\}$, compatible with the action of G on $\operatorname{Spec} L$ by the construction*

$$(g \in G(R), f(x,y) \in \mathrm{FGL}(R) \subseteq R[[x,y]]) \mapsto gf(g^{-1}(x), g^{-1}(y)) \in \mathrm{FGL}(R) \subseteq R[[x,y]]$$

There is a spectral sequence $\{E_r^{p,q}, d_r\}$, called the Adams-Novikov spectral sequence, with the following properties. If X is connective, then $\{E_r^{p,q}, d_r\}$ converges to a finite filtration of $\pi_{p-q}X$. Moreover, the groups $E_2^{*,q}$ are given by the cohomology groups $\mathrm{H}^q(G; \mathrm{MU}_* X)$.

Equivalently, we can think of $E_2^{*,q}$ as the cohomology of the stack $\mathcal{M}_{\mathrm{FG}}^s = \operatorname{Spec} L/G$ with coefficients in the sheaf $\mathcal{F}_X$ determined by $\mathrm{MU}_*(X)$ with its G-action.

To be more precise, we should observe that the ring L, and the ring $\mathbf{Z}[b_1, \ldots,]$ are all equipped a canonical grading. In geometric terms, this grading corresponds to an action of the multiplicative group $\mathbb{G}_m$. This group acts on L by the formula

$$(\lambda \in R^\times, f(x,y) \in \mathrm{FGL}(R)) \mapsto \lambda f(\lambda^{-1}x, \lambda^{-1}y).$$

In fact, we can identify both $\mathbb{G}_m$ and G with subgroups of a larger group G^+, with $G^+(R) = \{g \in R[[x]] : g(t) = b_0 t + b_1 t^2 + \cdots, b_0 \in R^\times\}$. This group can be identified with a semidirect product of the subgroup $\mathbb{G}_m$ (consisting of those power series with $b_i = 0$ for $i > 0$) and G (consisting of those power series with $b_0 = 1$), and this semidirect product acts on $\operatorname{Spec} L$ by substitution.

For any spectrum X, $\mathrm{MU}_*(X)$ is a graded L-module, and the action of G on $\mathrm{MU}_*(X)$ is compatible with the grading. In the language of algebraic geometry, this means that $\mathrm{MU}_{\mathrm{even}}(X) = \oplus_n \mathrm{MU}_{2n}(X)$ can be regarded as a representation of the group G^+, compatible with the action of G^+ on $\operatorname{Spec} L$. In the language of stacks, this means that $\mathrm{MU}_{\mathrm{even}}(X)$ can be regarded as a quasi-coherent sheaf on the quotient stack $\operatorname{Spec} L/G^+$.

Definition 2. The quotient stack $\operatorname{Spec} L/G^+$ is called the *moduli stack of formal groups* and will be denoted by $\mathcal{M}_{\mathrm{FG}}$.

To understand $\mathcal{M}_{\mathrm{FG}}$, it will be useful to have a more conceptual way of thinking about formal group laws. Let R be a commutative ring and let $f(x,y) \in R[[x,y]]$ be a formal group law over R. We let Alg_R denote the category of commutative R-algebras. We can associate to f a functor $\mathcal{G}_f : \mathrm{Alg}_R \to \mathrm{Ab}$ from R to the category of abelian groups: namely, we let $\mathcal{G}_f(A) = \{a \in A : (\exists n) a^n = 0\} \subseteq A$, with the group structure given by $(a,b) \mapsto f(a,b)$. Note that this expression makes sense: though f has infinitely many terms, if a and b are nilpotent then only finitely many terms are nonzero. We will call $\mathcal{G}_f$ the *formal group* associated to f.

Remark 3. The condition that $f \in R[[x,y]]$ define a formal group law is *equivalent* to the requirement that the above formula defines a group structure on $\mathcal{G}_f(A)$ for every R-algebra A.

Suppose that we are given two formal group laws $f, f' \in R[[x,y]]$ and an isomorphism $\alpha : \mathcal{G}_f \simeq \mathcal{G}_{f'}$ of the corresponding formal groups. In particular, for every R-algebra A, α determines a bijection α_A from the set $\{a \in A : a \text{ is nilpotent }\}$ with itself. To understand this bijection, let us treat the universal case where A contains an element a such that $a^{n+1} = 0$. This is the truncated polynomial ring $A = R[t]/t^{n+1}$. In this case, α carries t to another nilpotent element, necessarily of the form $b_0 t + b_1 t^2 + \ldots + b_{n-1} t^n$. Since α is functorial, it follows that for *any* commutative R-algebra A containing an element a with $a^n = 0$, we have $\alpha_A(a) = b_0 a + b_1 a^2 + \cdots + b_{n-1} a^n$. In particular, if $A = R[t]/t^n$, we deduce that $\alpha_A(t) = b_0 t + b_1 t^2 + \ldots + b_{n-2} t^{n-1}$. In other words, the coefficients b_i which appear are independent of n. We conclude that there exists a power series $g(t) = b_0 t + b_1 t^2 + \cdots$ such that $\alpha_A(a) = g(a)$ for every commutative ring a. Since α_A is a bijection for any A, we conclude that g is an invertible power series. Since α_A is a group homomorphism, we deduce that g satisfies the formula $f'(g(x), g(y)) = gf(x,y)$: that is, the formal group laws f and f' differ by the change-of-variable g.

Definition 4. Let R be a commutative ring. An *coordinatizable formal group* over R is a functor $\mathcal{G} : \mathrm{Alg}_R \to \mathrm{Ab}$ which has the form $\mathcal{G}_f$, for some formal group law $f \in R[[x,y]]$.

We regard the coordinatizable formal group laws (and isomorphisms between them) as a subcategory of the category $\mathrm{Fun}(\mathrm{Alg}_R, \mathrm{Ab})$ of functors from Alg_R to abelian groups. We have just seen that this subcategory admits a less invariant description: it is equivalent to a category whose objects are formal group laws $f \in R[[x,y]]$, and whose morphisms are maps g such that $f'(g(x), g(y)) = gf(x,y)$.

The coordinatizable formal group laws over R do *not* satisfy descent in R. Consequently, it is convenient to make the following more general definition:

Definition 5. Let R be a commutative ring. A *formal group law* over R is a functor $\mathcal{G} : \mathrm{Alg}_R \to \mathrm{Ab}$ satisfying the following conditions:

(1) The functor $\mathcal{G}$ is a sheaf with respect to the Zariski topology. In other words, if A is a commutative R-algebra with a pair of elements x and y such that $x + y = 1$, then $\mathcal{G}(A)$ can be described as the subgroup of $\mathcal{G}(A[\frac{1}{x}]) \times \mathcal{G}(A[\frac{1}{y}])$ consisting of pairs which have the same image in $\mathcal{G}(A[\frac{1}{xy}])$.

(2) The functor $\mathcal{G}$ is a coordinatizable formal group law locally with respect to the Zariski topology. That is, we can choose elements $r_1, r_2, \ldots, r_n \in R$ such that $r_1 + \cdots + r_n = 1$, such that each of the composite functors
$$\mathrm{Alg}_{R[\frac{1}{r_i}]} \to \mathrm{Alg}_R \to \mathrm{Ab}$$
has the form $\mathcal{G}_f$ for some formal group law $f \in R[\frac{1}{r_i}][[x,y]]$.

By definition, the moduli stack of the formal groups $\mathcal{M}_{\mathrm{FG}}$ is the functor which assigns to each commutative ring R the category of formal group laws over R (the morphisms in this category are given by isomorphisms).

There is a canonical map of stacks $\mathcal{M}^s_{\mathrm{FG}} = \operatorname{Spec} L/G \to \operatorname{Spec} L/G^+ = \mathcal{M}_{\mathrm{FG}}$. To understand this map (and the failure of general formal groups to be coordinatizable) it is useful to introduce a definition.

Definition 6. Let $\mathcal{G}$ be a formal group over R. The *Lie algebra* of $\mathcal{G}$ is the abelian group $\mathfrak{g} = \ker(\mathcal{G}(R[t]/(t^2)) \to \mathcal{G}(R))$.

Note that if $\mathcal{G} = \mathcal{G}_f$ for some formal group law f, we get a group isomorphism $\mathfrak{g} \simeq tR[t]/(t^2) \simeq R$ (since $f(x,y) = x + y$ to order 2). In fact, $\mathfrak{g}$ is not just an abelian group: for each $\lambda \in R$, the equation $t \mapsto \lambda t$ determines a map from $R[t]/(t^2)$ to itself, which induces a group homomorphism $\mathfrak{g} \to \mathfrak{g}$. When $\mathcal{G}$ is coordinatizable, this is the usual action of R on itself by multiplication. It follows by descent that the above formula always determines an action of R on $\mathfrak{g}$. Since $\mathfrak{g} \simeq R$ locally for the Zariski topology, we deduce that $\mathfrak{g}$ is an *invertible* R-module: that is, it determines a line bundle on the affine scheme $\operatorname{Spec} R$.

Proposition 7. (1) *A formal group $\mathcal{G}$ over R is coordinatizable if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to R.*

(2) *The quotient stack $\mathcal{M}^s_{\mathrm{FG}}$ parametrizes pairs $(\mathcal{G}, \alpha)$, where $\mathcal{G}$ is a formal group and $\alpha : \mathfrak{g} \simeq R$ is a trivialization of its Lie algebra.*

Proof. We have already established that $\mathfrak{g} \simeq R$ when $\mathcal{G}$ is coordinatizable. Conversely, fix an isomorphism $\mathfrak{g} \simeq R$. After localizing $\operatorname{Spec} R$, the group $\mathcal{G}$ becomes coordinatizable: that is, we can write $\mathcal{G} \simeq \mathcal{G}_f$ for some $f \in R[[x, y]]$. Modifying f by the action of $\mathbb{G}_m$, we may assume that this isomorphism is compatible with our trivialization of $\mathfrak{g}$. The trouble is that these isomorphisms might not glue. The obstruction to gluing them determines a cocycle representing a class in $H^1_{\mathrm{Zar}}(\operatorname{Spec} R, G)$. We claim that this group vanishes. This is because the group G is an iterated extension of copies of the additive group $(A \in \mathrm{Alg}_R) \mapsto (A, +)$, which has no cohomology on affine schemes.

Assertion (2) is just a translation of the following observation: if $f, f' \in R[[x, y]]$ are formal group laws, then an isomorphism of formal groups $\mathcal{G}_f \simeq \mathcal{G}_{f'}$ respects the trivializations of the Lie algebras of $\mathcal{G}_f$ and $\mathcal{G}_{f'}$ if and only if it is given by a power series of the form $g(t) = t + b_1 t^2 + \cdots$ (a power series of the form $g(t) = b_0 t + \cdots$ acts on the Lie algebras by multiplication by the scalar b_0). □

We can think of the assignment $(R, \mathcal{G}) \mapsto \mathfrak{g}^{-1}$ as defining a line bundle ω on the moduli stack $\mathcal{M}_{\mathrm{FG}}$. In fact, $\mathcal{M}^s_{\mathrm{FG}}$ is just the total space of ω with the zero section removed (equivalently, the moduli stack of trivializations of ω).

We can now be a little bit more precise about the E_2-term of the Adams-Novikov spectral sequence. Translating our gradings into algebraic geometry, we get the following result:

Claim 8. *For any spectrum X, the bordism groups $\mathrm{MU}_{\mathrm{even}}(X)$ form a module over the Lazard ring $L \simeq \pi_* \mathrm{MU}$ which carries a compatible action of the group scheme G^+, and therefore determines a sheaf $\mathcal{F}^{\mathrm{even}}$ on $\mathcal{M}_{\mathrm{FG}} = \operatorname{Spec} L/G^+$. The E_2-term of the Adams-Novikov spectral sequence satisfies*

$$E_2^{2a,b} = \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}; \mathcal{F}^{\mathrm{even}} \otimes \omega^a).$$

Similarly, the odd homotopy groups $\mathrm{MU}_{\mathrm{odd}}(X)$ determine a sheaf $\mathcal{F}^{\mathrm{odd}}$ on $\mathcal{M}_{\mathrm{FG}}$ satisfying

$$E_2^{2a+1,b} = \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}; \mathcal{F}^{\mathrm{odd}} \otimes \omega^a).$$

In order to exploit Claim 8, we will need to understand the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}}$. This will be our goal in the next lecture.