

Math 221: Problem Set 9

- (1) Let R be a Noetherian commutative ring which is complete with respect to an ideal I (that is, the map $R \rightarrow R_I^\vee$ is an isomorphism). Let M be an R -module. Show that M is finitely generated if and only if the following conditions are satisfied:
 - (a) The quotient M/IM is finitely generated.
 - (b) The intersection $\bigcap I^n M$ is $\{0\}$.
- (2) Let R be a Noetherian ring containing an element a . Show that the power series $f(x) = x + a$ is not a zero divisor in $R[[x]]$. Give an example to show that this need not be true if R is not Noetherian.
- (3) Compute the Hilbert polynomial for the local ring $R_{\mathfrak{m}}$, where $R = \mathbf{C}[a, b, c, d]/(ad - bc)$ and $\mathfrak{m}$ is the maximal ideal of R generated by a, b, c , and d .
- (4) Let R be a commutative ring and let M be an R -module of finite length, so that there exists a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that each successive quotient $M_i/M_{i-1} \simeq R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i \subseteq R$. Show that for any other filtration

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n = M$$

having simple quotients $N_i/N_{i-1} \simeq R/\mathfrak{n}_i$, there exists a permutation σ of $\{1, \dots, n\}$ such that $\mathfrak{m}_i = \mathfrak{n}_{\sigma(i)}$ for $1 \leq i \leq n$.