

Math 221: Problem Set 8

- (1) Prove the following version of Serre's criterion: a reduced Noetherian ring R (not necessarily an integral domain) is normal if and only if it satisfies the following condition: for every non-zero-divisor $x \in R$ and every prime ideal $\mathfrak{p}$ associated to R/xR , the localization $R_{\mathfrak{p}}$ is a discrete valuation ring.
- (2) Let $\mathfrak{p} \subseteq \mathbf{C}[x_1, \dots, x_n]$ be a prime ideal and let $R = \mathbf{C}[x_1, \dots, x_n]/\mathfrak{p}$. Use the Noether normalization to show that the integral closure of R (in its field of fractions) is a finitely generated R -module.
- (3) Let R be a Noetherian ring and let M be a finitely generated module over R . Suppose that $\text{Supp}(M) = \{\mathfrak{p} \in \mathbf{Spec} R : M_{\mathfrak{p}} \neq 0\}$ is the union $K_0 \cup K_1$, where $K_0, K_1 \subseteq \mathbf{Spec} R$ are closed subsets satisfying $K_0 \cap K_1 = \emptyset$. Prove that $M \simeq M_0 \oplus M_1$, where $\text{Supp}(M_0) = K_0$ and $\text{Supp}(M_1) = K_1$, and that this decomposition is unique up to isomorphism.
- (4) Let $\mathfrak{m}$ be a maximal ideal in a ring R . Prove that the completion $R_{\mathfrak{m}}^{\vee}$ is a local ring, and that there is a unique map of R -algebras $R_{\mathfrak{m}} \rightarrow R_{\mathfrak{m}}^{\vee}$.