

Math 221: Problem Set 7

- (1) Let R be an arbitrary Noetherian ring. We say that R is *normal* if it is reduced and integrally closed in $K(R)$, where $K(R)$ is the total ring of fractions of R . Show that R is normal if and only if R is a finite product of integrally closed integral domains.
- (2) Let R be an integrally closed integral domain with fraction field K , and let L be a finite extension of K . For each element $x \in L$, let $p(t)$ denote the characteristic polynomial of the multiplication map $x : L \rightarrow L$ (as a map of K -vector spaces). Show that x is integral over R if and only if the coefficients of p belong to R .
- (3) Let R be a Noetherian integrally closed domain with fraction field K , and let L be a finite separable extension of K . Show that the integral closure of R in L is a finitely generated module over R . (Hint: if L/K is separable, then the trace pairing $(x, y) \mapsto \text{Tr}(xy)$ is a nondegenerate bilinear form on L with values in K .) Conclude that if R is a Dedekind domain, then the integral closure of R in L is a Dedekind domain.
- (4) Let R be a commutative ring. The K -group of R is defined to be the quotient of the free group on generators $[M]$, where M is a finitely generated projective R -module, by the relations $[M \oplus N] = [M] + [N]$. Show that if R is a Dedekind domain, then there is a canonical isomorphism $K(R) \simeq \mathbf{Z} \oplus \text{Pic}(R)$.