

Math 221: Problem Set 6

- (1) Let X be a topological space. We say that X is *Noetherian* if every ascending chain of open sets

$$U_0 \subset U_1 \subset U_2 \subset \cdots$$

has finite length. Show that if X is Noetherian, then every closed subset of X can be written as a finite union of irreducible closed subsets.

- (2) Let R be a factorial integral domain. Show that $R[x]$ is also factorial.
- (3) Let $\phi : R \rightarrow R'$ be a map of commutative rings. Then ϕ induces a group homomorphism $\theta : \text{Pic}(R) \rightarrow \text{Pic}(R')$, given by $M \mapsto M \otimes_R R'$. Show that if $R' = R/I$ for some ideal I such that $I^n = 0$, then θ is an isomorphism.
- (4) Suppose we are given a pair of surjective ring homomorphisms $\phi_0 : A_0 \rightarrow B$ and $\phi_1 : A_1 \rightarrow B$, and let A denote the fiber product $A_0 \times_B A_1 = \{(a, a') \in A_0 \times A_1 : \phi_0(a) = \phi_1(a')\}$. Show that there is an exact sequence of abelian groups

$$0 \rightarrow A^\times \rightarrow A_0^\times \oplus A_1^\times \rightarrow B^\times \rightarrow \text{Pic}(A) \rightarrow \text{Pic}(A_0) \oplus \text{Pic}(A_1) \rightarrow \text{Pic}(B).$$

Here $R^\times$ denotes the group of invertible elements of R .