

Math 221: Problem Set 4

- (1) Let R be a commutative ring. An element $e \in R$ is said to be *idempotent* if $e^2 = e$. Show that if $I \subseteq R$ is a radical ideal, then the closed subset $V(I) = \{\mathfrak{p} \in \mathbf{Spec} R : I \subseteq \mathfrak{p}\}$ of $\mathbf{Spec} R$ is open if and only if I is the radical of (e) for some idempotent element $e \in R$. Conclude that $\mathbf{Spec} R$ is connected if and only if the only idempotent elements of R are 0 and 1.
- (2) Let R be a commutative ring containing elements $a_1, \dots, a_n \in R$ which generate the unit ideal (1). Let $f : M \rightarrow N$ be an R -module homomorphism. Show that if each of the induced maps $M[\frac{1}{a_i}] \rightarrow N[\frac{1}{a_i}]$ is injective, surjective, or an isomorphism, then f has the same property.
- (3) Let K be a field and let $v : K \rightarrow \mathbb{R} \cup \{\infty\}$ be a valuation on K , where $\mathbf{R}$ denotes the field of real numbers. For $x, y \in K$, define $d(x, y) = \exp -v(x - y)$ (here we adopt the convention that $\exp -\infty = 0$, so that $d(x, x) = 0$). Prove that (K, d) is a metric space. Let $K^\vee$ be the completion of K with respect to the metric d . Prove that the addition and multiplication on K have a unique continuous extension to addition and multiplication maps $K^\vee \times K^\vee \rightarrow K^\vee$, which endow $K^\vee$ with the structure of a field.
- (4) Let $v : K \rightarrow \mathbb{R} \cup \{\infty\}$ be as in (3), and let $R = \{x \in K : v(x) \geq 0\}$. Show that every nonzero prime ideal of R is maximal.
- (5) Let n be a square-free odd integer. Prove that the ring $\mathbf{Z}[\sqrt{n}]$ is integrally closed if n is congruent to 3 mod 4. Describe its integral closure when n is congruent to 1 mod 4.