

Math 221: Problem Set 3

Let A be a preordered set: that is, a set equipped with a relation $\leq$ such that $\alpha \leq \alpha$ for each $\alpha \in A$, and $\alpha \leq \beta \leq \gamma$ implies $\alpha \leq \gamma$. We say that A is *filtered* if it is nonempty and, for every pair of elements $\alpha, \beta \in A$, there exists $\gamma \in A$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. If R is a commutative ring and A is a filtered preordered set, then an A -indexed diagram of R -modules consists of an R -module M_α for each $\alpha \in A$ and an R -module homomorphism $\phi_{\alpha,\beta} : M_\alpha \rightarrow M_\beta$ for each $\alpha \leq \beta$ in A . These homomorphisms are required to satisfy the conditions $\phi_{\alpha,\alpha} = \text{id}_{M_\alpha}$ and $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma} \circ \phi_{\alpha,\beta}$ for $\alpha \leq \beta \leq \gamma$.

- (1) Let A be a filtered preordered set and let $\{M_\alpha, \phi_{\alpha,\beta}\}$ be an A -indexed diagram of R -modules for some commutative ring R . Define a relation $\sim$ on the disjoint union $\coprod_{\alpha \in A} M_\alpha$ as follows: for a pair of elements $x \in M_\alpha, y \in M_\beta$, we have $x \sim y$ if and only if there exists $\gamma \in A$ such that $\alpha \leq \gamma, \beta \leq \gamma$, and $\phi_{\alpha,\gamma}(x) = \phi_{\beta,\gamma}(y)$. Prove that $\sim$ is an equivalence relation. Set $\varinjlim M_\alpha = (\coprod_{\alpha \in A} M_\alpha) / \sim$, and for each $\alpha \in A$ let $\phi_\alpha : M_\alpha \rightarrow \varinjlim M_\alpha$ denote the composition

$$M_\alpha \hookrightarrow \coprod_{\alpha} M_\alpha \rightarrow (\coprod_{\alpha} M_\alpha) / \sim = \varinjlim M_\alpha.$$

Prove that there is a unique R -module structure on $\varinjlim M_\alpha$ for which each of the maps ϕ_α is an R -module homomorphism.

- (2) Let $\{M_\alpha, \phi_{\alpha,\beta}\}$ be as in Problem (1). Suppose we are given another R -module N and a collection of R -module homomorphisms $\psi_\alpha : M_\alpha \rightarrow N$ satisfying $\psi_\beta \circ \phi_{\alpha,\beta} = \psi_\alpha$ for $\alpha \leq \beta$. Show that there is a unique R -module homomorphism $\psi : \varinjlim M_\alpha \rightarrow N$ such that $\psi_\alpha = \psi \circ \phi_\alpha$ for each $\alpha \in A$.
- (3) Assume that R is an integral domain. Let S be a multiplicatively closed subset of R . For elements $s, t \in S$, write $s \leq t$ if $t = ss'$ for some $s' \in S$. Prove that $\leq$ is a preordering on S , and that S is filtered. Let M be any R -module, and define an S -indexed diagram of R -modules $\{M_s, \phi_{s,t}\}$ so that each $M_s = M$, and each $\phi_{s,t}$ is given by multiplication by s' , where $t = ss'$. Show that there is a canonical isomorphism of R -modules $\varinjlim M_s \simeq S^{-1}M$.
- (4) Let A be a filtered preordered set and let $\{M_\alpha, \phi_{\alpha,\beta}\}$ be an A -indexed diagram of R -modules. Let N be another R -module. Show that there is a canonical isomorphism of R -modules

$$(\varinjlim M_\alpha) \otimes_R N \simeq \varinjlim (M_\alpha \otimes_R N).$$

- (5) Let A be a filtered preordered set and let $\{M_\alpha, \phi_{\alpha,\beta}\}$ be an A -indexed diagram of R -modules. Show that if each M_α is flat, then $\varinjlim M_\alpha$ is flat.

An R -module M is said to be *finitely generated* if there exists a surjective R -module homomorphism $R^n \rightarrow M$, for some M .

- (6) Let M be an R -module, and let A be the collection of all finitely generated submodules of M . Show that A is a filtered partially ordered set (with respect to inclusion). For each $\alpha \in A$, let M_α denote the corresponding submodule of M . Show that there is a canonical isomorphism of R -modules $\varinjlim M_\alpha \simeq M$.
- (7) Let M be an abelian group, regarded as a module over the ring $\mathbb{Z}$ of integers. Show that M is flat if and only if it is torsion-free. (Hint: use the fact that every finitely generated abelian group is a direct sum of cyclic groups.)