

Math 221: Problem Set 2

- (1) Let R be a commutative ring, let $I \subseteq R$ is an ideal, and let $\pi : R \rightarrow R/I$ be the quotient map. Show that the construction $J \mapsto \pi^{-1}J$ determines a bijection from the set of ideals of R/I to the set of ideals in R which contain I . Show that an ideal $J \subseteq R/I$ is prime if and only if $\pi^{-1}J$ is prime. Show that the induced map $\mathbf{Spec} R/I \rightarrow \mathbf{Spec} R$ induces a homeomorphism of $\mathbf{Spec} R/I$ with a closed subset of $\mathbf{Spec} R$.
- (2) Let X be a topological space. A nonempty closed subset $K \subseteq X$ is *irreducible* if, for any pair of closed subsets $K_0, K_1 \subseteq K$ such that $K = K_0 \cup K_1$, we have $K = K_0$ or $K = K_1$. Show that for any point $x \in X$, the closure $\overline{\{x\}}$ is an irreducible closed subset of X . Show that if $X = \mathbf{Spec} R$ for some commutative ring R , then the converse holds: every irreducible closed subset has the form $\overline{\{x\}}$ for some point $x \in \mathbf{Spec} R$.
- (3) Let R be a commutative ring and let $f : M \rightarrow N$ be a map of R -modules. Show that f is surjective if and only if, for every maximal ideal $\mathfrak{m} \subseteq R$, the map of localizations $M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is surjective.
- (4) Let X be a compact Hausdorff space and let $C(X)$ denote the ring of continuous functions from X to the real numbers. Show that the construction $x \mapsto \mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$ determines a homeomorphism from X onto its image in $\mathbf{Spec} C(X)$.