

Math 221: Problem Set 1

- (1) Let R be the commutative ring $\mathbf{Z}[\sqrt{-5}]$, and let $I \subseteq R$ be the ideal $(2, 1 + \sqrt{-5})$ generated by 2 and $1 + \sqrt{-5}$. Show that I is a maximal ideal of R (in particular, show that $I \neq R$). Show that I is not a principal ideal. Describe the residue field R/I .
- (2) Let R be a commutative ring and let $\mathfrak{p} \subseteq R$ be a prime ideal. Suppose that I and J are ideals of R such that $IJ \subseteq \mathfrak{p}$. Show that $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$.
- (3) Let R be a commutative ring and let $f : M \rightarrow N$ be a homomorphism of R -modules. The *coimage* of f is defined to be the quotient $M/\ker(f)$, where $\ker(f) = \{x \in M : f(x) = 0\}$. Show that there is a canonical isomorphism between the coimage of f and the image of f .
- (4) Let X be a compact Hausdorff space, and let $C(X)$ be the ring of continuous functions $X \rightarrow \mathbb{R}$. Show that every maximal ideal of $C(X)$ has the form $\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$ for some point $x \in X$. Describe the corresponding residue field.
- (5) Let X be a locally compact Hausdorff space which is not compact, and let $C(X)$ be as in (4). Show that there exists a maximal ideal of $C(X)$ which is not of the form $\{f \in C(X) : f(x) = 0\}$.