Math 155 Midterm with Solutions

October 7, 2011

(1) Let us say that a string of letters is *valid* if it satisfies the following conditions:

- It does not contain any consecutive vowels (here the letter "y" is considered to be a consonant).
- The letter "y" does not appear after a vowel or at the end of the string.

For example, the strings "abcde" and "fryer" are valid, but the strings "bead", "grayer", and "fry" are not. Let c_n denote the number of valid strings of length n.

(a) Find a recurrence relation satisfied by the integers c_n.

(b) Use (a) to determine the generating function $f(x) = \sum_{n \geq 0} c_n x^n$.

(c) Give a closed-form expression for the integers c_n.

Solution: Suppose first that $n \geq 2$. Let X be the set of valid strings of length n. Let X_v denote the subset of X consisting of those strings which start with a vowel, and X_c the subset consisting of those strings which start with a consonant. To give an element of X_c, one must give the initial consonant together with a valid string of length $n-1$. We therefore obtain

$$|X_c| = 21c_{n-1}.$$

To give an element of X_v, one must give a vowel (5 choices in all), followed by a second character which cannot be a vowel or a "y" (20 choices in all), followed by an arbitrary valid string of length $n-2$. We therefore have $|X_v| = 100c_{n-2}$. Thus

$$c_n = |X| = |X_c| + |X_v| = 21c_{n-1} + 100c_{n-2}.$$

To start the recursion off, we will also need some initial values. Note that there is exactly one valid string of length 0 (the empty string) and 25 valid strings of length 1 (any letter except for "y").

Multiplying by x^n, we obtain

$$c_n x^n = 21c_{n-1}x^n + 100c_{n-2}x^n.$$

Summing over all $n \geq 2$, we get

$$\sum_{n \geq 2} c_n x^n = 21x\left(\sum_{n \geq 2} c_{n-1}x^{n-1}\right) + 100x^2\left(\sum_{n \geq 2} c_{n-2}x^{n-2}\right).$$

We can write this as

$$f(x) - 25x - 1 = 21x(f(x) - 1) + 100x^2 f(x),$$

or

$$(1 - 21x - 100x^2)f(x) = 1 + 4x.$$

Solving for $f(x)$, we get

$$f(x) = \frac{1+4x}{1-21x-100x^2} = \frac{1+4x}{(1+4x)(1-25x)} = \frac{1}{1-25x}.$$

Expanding this in a power series, we get

$$f(x) = 1 + 25x + 25^2x^2 + \cdots$$

so that $c_n = 25^n$.

(2) Let G be the group of rotational symmetries of a regular tetrahedron, and regard G as acting on the set X of edges of the tetrahedron.

  (a) Compute the cycle index polynomial $Z_G(s_1, s_2, \ldots)$.

  (b) Up to rotational symmetry, in how many ways can you color the edges of the tetrahedron using three colors?

  (c) In the situation of (b), how many colorings use each color exactly two times?

Solution: As we have seen in class, the group G is of order 12 and contains three types of elements:

(i) The identity element, which does not permute the edges and therefore has cycle monomial s_1^6.

(ii) Rotations which fix a face and an opposite vertex. There are 8 of these. Each has two orbits of size 3 on the set of edges, hence the cycle monomial is given by s_3^2.

(iii) Rotations of 180° which fix an edge. There are three of these: each one fixes a pair of opposite edges, and permutes the remaining edges in pairs. The corresponding cycle monomial is therefore $s_1^2 s_2^2$.

We therefore obtain the cycle index polynomial

$$Z_G(s_1, s_2, \ldots) = \frac{s_1^6 + 8s_3^2 + 3s_1^2 s_2^2}{12}.$$

According to Polya's theorem, the answer to (b) is given by

$$Z_G(3, 3, \ldots) = \frac{3^6 + 8 \times 3^2 + 3 \times 3^4}{12} = \frac{1044}{12} = 87.$$

To obtain the answer to (c), we need the more sophisticated version of Polya's theorem that keeps track of the number of colorings. The answer is given by the coefficient of $X^2Y^2Z^2$ in the expression

$$Z_G(X+Y+Z, X^2+Y^2+Z^2, \cdots) = \frac{1}{12}(X+Y+Z)^6 + \frac{2}{3}(X^3+Y^3+Z^3)^2 + \frac{1}{4}(X+Y+Z)^2(X^2+Y^2+Z^2)^2.$$

Using the multinomial theorem, the coefficient in the first summand is given by

$$\frac{1}{12}\frac{6!}{2!2!2!} = \frac{720}{12 \times 8} = \frac{15}{2}.$$

The coefficient in the second summand vanishes (since each of the variables has exponent at least 3). The coefficient in the third summand is given by adding the coefficient of X^2 in $\frac{1}{4}(X+Y+Z)^2$ times the coefficient of Y^2Z^2 in $(X^2+Y^2+Z^2)^2$ to two other (identical) terms, and is therefore given by $\frac{3}{4} \times 1 \times 2 = \frac{3}{2}$. Summing these coefficients, we see that there are

$$\frac{15}{2} + \frac{3}{2} = 9$$

ways to color the edges of a tetrahedron using each of three colors exactly twice, up to rotational symmetries.

(3) Let G be a finite group acting on a finite set X. Prove that the number of orbits of G on X is given by evaluating the polynomial $\frac{\partial Z_G(s_1, s_2, \ldots)}{\partial s_1}$ at $s_1 = s_2 = \cdots = 1$.

Solution: Let $g \in G$ be an element and

$$Z_g(s_1, s_2, \ldots) = s_1^{k_1} s_2^{k_2} \cdots$$

the corresponding cycle monomial. Then

$$\frac{\partial Z_g}{\partial s_1} = k_1 s_1^{k_1 - 1} s_2^{k_2} s_3^{k_3} \cdots$$

Evaluating at $s_1 = s_2 = s_3 = \cdots = 1$, we obtain the integer k_1, which is the number of fixed points for the action of g on the set X. We therefore have

$$\begin{aligned} \frac{\partial Z_G(s_1, s_2, \ldots)}{\partial s_1}|_{s_i=1} &= \frac{1}{|G|} \sum_{g \in G} \frac{\partial Z_g}{\partial s_1}|_{s_i=1} \\ &= \sum_{g \in G} |X^g|, \end{aligned}$$

which is equal to the number of orbits of G on X by virtue of Burnside's formula.