1. Introduction to Topological Groups

In this section, we introduce topological groups, maps of topological groups, and discuss properties of open and closed subgroups.

1.1. Definition of a topological group. We build up to the definition of a topological group in [Definition 1.5]. In order to define this, we will need to introduce the notion of a topology.

**Definition 1.1.** A topological space $X$ is a set (also called $X$) together with a collection of subsets $\{U_i\}_{i \in I}$ called open sets with $U_i \subset X$ satisfying the following conditions

1. The empty set $\emptyset \subset X$ is open.
2. The set $X$ is open.
3. An arbitrary union of open sets is open.
4. The intersection of two open sets is open.

The collection of open subsets is called a topology for $X$.

**Definition 1.2.** A continuous map of topological spaces $f : X \to Y$ is a map of underlying sets such that for any open $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open.

**Definition 1.3.** For $X$ and $Y$ two topological spaces, define the product topology by taking the topology whose open sets are unions of sets of the form $U \times V$ for $U \subset X$ open and $V \subset Y$ open.

**Exercise 1.4.** Verify that the product topology is a topology. 

**Definition 1.5.** A topological group is a topological space $G$ together with a multiplication map $m : G \times G \to G$ and an inversion map $i : G \to G$ satisfying the group axioms (i.e., multiplication is associative, every element has a unique two sided inverse, there is an identity element) such that $m$ and $i$ are both continuous as maps of topological spaces.

**Remark 1.6.** We will often write $g \cdot h$ in place of $m(g, h)$ and $g^{-1}$ in place of $i(g)$.
1.2. **Maps of topological groups.** We next introduce maps of topological groups.

**Definition 1.7.** Let $G$ and $H$ be topological groups. A **map of topological groups** $f : G \rightarrow H$ is a continuous map of topological spaces which is also a group homomorphism.

**Definition 1.8.** A continuous map of topological spaces $f : X \rightarrow Y$ is a **homeomorphism** if it has a continuous inverse map $f^{-1} : Y \rightarrow X$ (i.e., $f \circ f^{-1} = \text{id}_H, f^{-1} \circ f = \text{id}_G$).

**Definition 1.9.** A map of topological groups $f : G \rightarrow H$ is an **isomorphism** if it is a homeomorphism of topological spaces and an isomorphism of groups. I.e., $f$ has an inverse map of topological groups $f^{-1} : H \rightarrow G$ meaning $f \circ f^{-1} = \text{id}_H, f^{-1} \circ f = \text{id}_G$.

**Exercise 1.10.** Let $G$ be a topological group and let $g \in G$ be an element. Show that the left and right multiplication maps
\[
l_g : G \rightarrow G \quad \quad h \mapsto m(g, h)
\]
and
\[
r_g : G \rightarrow G \quad \quad h \mapsto m(h, g)
\]
define homeomorphisms of topological spaces. \(^2\)

1.3. **Open and closed subgroups.** We next discuss open and closed subgroups of topological groups and the pleasant relationship between them.

**Definition 1.11.** Let $X$ be a topological space. A subset $Z \subset X$ is **closed** if $X - Z$ is open.

**Exercise 1.12.** Let $G$ be a topological group and let $U \subset G$ be an open subset. Show that for $g \in G$,
\[
g \cdot U := \{m(g, u) : u \in U\}
\]
is again an open set.

Similarly, show that if $Z \subset G$ is a closed set, then $g \cdot Z$ is again a closed set.

**Exercise 1.13.** Show that an open subgroup of a topological group is necessarily also closed. \(^3\)
Definition 1.14. Let $G$ be a group. A subgroup $H \subset G$ has finite index if the quotient $G/H$ is finite (i.e., $H$ has only finitely many cosets in $G$).

Exercise 1.15. Let $H \subset G$ be a subgroup which is also closed. Suppose $H$ has finite index. Show that $H$ is also open. 4
2. EXAMPLES

Next, we’ll consider some examples of topological groups. To give relevant examples, we’ll topologize them via the subspace topology, which we now introduce.

Definition 2.1. If \( i : X \subset Y \) with \( X \) a set and \( Y \) a topological space, then the **subspace topology** on \( X \) (with respect to \( i \)) is the topology on \( X \) whose open sets are those of the form \( X \cap U \) for \( U \subset Y \) open.

Exercise 2.2. Verify that the subspace topology is indeed a topology.

Exercise 2.3. Define a topology on \( \mathbb{R}^n \) so that the open sets are unions of open balls. Show that this indeed defines a topology, which we call the **Euclidean topology**.

Exercise 2.4. Show that \( S^1 := \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \) endowed with the subspace topology from the Euclidean topology on \( \mathbb{R}^2 \) is a topological group. Here, inversion is given by sending \( (x, y) \mapsto (x, -y) \) and is multiplication given by adding angles on the circle, or more formally

\[
(x, y) \cdot (z, w) = (xz - yw, yz + xw).
\]

Exercise 2.5. Identify \( n \times n \) real matrices with \( \mathbb{R}^{n^2} \). Give \( n \times n \) real matrices the Euclidean topology via this identification, with group operation given by matrix addition and inversion given by negating the matrix. Show that this is a topological group.

Exercise 2.6. In this exercise, we show certain matrix groups are topological groups.

1. Let \( \text{Gl}_2(\mathbb{R}) \) denote the set of \( 2 \times 2 \) invertible matrices with real entries, viewed as a subset of all \( 2 \times 2 \) invertible matrices. Topologize \( \text{Gl}_2(\mathbb{R}) \) with the subspace topology, giving all \( 2 \times 2 \) matrices the topology from Exercise 2.5 (which can itself be identified with the Euclidean topology on \( \mathbb{R}^4 \)). Show that \( \text{Gl}_2(\mathbb{R}) \) is a topological group, where multiplication is given by matrix multiplication and inversion is given by matrix inversion.

2. Let \( \text{Sl}_2(\mathbb{R}) \) denote the set of \( 2 \times 2 \) matrices with determinant 1. Topologize \( \text{Sl}_2(\mathbb{R}) \) with the subspace topology, giving all \( 2 \times 2 \) matrices the topology from Exercise 2.5 (which can itself be identified with the Euclidean topology on \( \mathbb{R}^4 \)). Show that \( \text{Sl}_2(\mathbb{R}) \) is a topological group.
(3) Generalize the preceding parts to show $\text{Gl}_n(\mathbb{R})$ ($n \times n$ invertible matrices) and $\text{Sl}_n(\mathbb{R})$ ($n \times n$ determinant 1 matrices) are topological groups.

**Exercise 2.7** (Optional, unimportant tricky exercise). In this exercise we explore what happens when you topologize the complex numbers with the cofinite topology, and try to make it a group under addition.

1. Give the complex numbers, $\mathbb{C}$, the **cofinite topology**, where the only closed sets other than $\mathbb{C}$ itself are the finite sets. Verify that this is indeed a topology.
2. Give $\mathbb{C}$ the structure of a group where the operation is addition of complex numbers and inversion is negation. Show that addition of complex numbers is not continuous with respect to the cofinite topology, so $\mathbb{C}$ is not a group under addition in the cofinite topology.
3. For $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ polynomials, in $n$ variables define

$$V(f_1, \ldots, f_m) := \{(a_1, \ldots, a_n) : f_1(a_1, \ldots, a_n) = \cdots = f_m(a_1, \ldots, a_n) = 0\}.$$

Define the Zariski topology on $\mathbb{C}^n$ as that topology where closed sets are precisely those of the form $V(f_1, \ldots, f_n)$. Show that this defines a topology, called the **Zariski topology**.
4. Show that the Zariski topology on $\mathbb{C}$ agrees with the cofinite topology.
5. Show that if we give $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ and $\mathbb{C}$ the Zariski topologies, then addition $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is indeed continuous. In some sense, this is a fix to the failure of addition on $\mathbb{C}$ to be continuous with respect to the cofinite topology.
3. QUOTIENTS OF TOPOLOGICAL GROUPS

In this section, we discuss quotients of topological spaces, the $T_1$ property for topological groups, and then show that the quotient of a topological group by a closed normal subgroup is again a topological group.

3.1. Quotients of topological spaces.

Definition 3.1. Let $Y$ be a topological space and let $Z$ be a set such that there is a surjective map of sets $f : Y \to Z$. Define the quotient topology on $Z$ by declaring the open sets in $Z$ to be those sets $U \subset Z$ such that $f^{-1}(U)$ is open.

Exercise 3.2. Verify that the quotient topology is indeed a topology. That is, show finite intersections of open sets in $Z$ are open and arbitrary unions of open sets in $Z$ are open.

3.2. $T_1$ and quotients.

Definition 3.3. A topological space $X$ is $T_1$ if every point $x \in X$ is closed.

Exercise 3.4. Let $Y$ be a topological space, $f : Y \to Z$ a surjective map, and give $Z$ the quotient topology. Show that $Z$ is $T_1$ if and only if for every $z \in Z$, the set $f^{-1}(z) \subset Y$ is closed as a subset of $Y$.

Exercise 3.5. Let $G$ be a topological group and $H \subset G$ be a closed subgroup. Topologize $G/H$ with the quotient topology.

1. Show that the map $G \to G/H$ is continuous.
2. Show that $G/H$ is $T_1$.  

3.3. Quotients of topological groups. In the next series of exercises, we show that for $G$ a topological group and $H \subset G$ a closed normal subgroup, $G/H$ can also be given the structure of a topological group.

Exercise 3.6. Let $X$ and $Y$ be topological spaces and let $Z$ and $W$ be sets with surjective maps $f : Y \to Z$ and $g : X \to W$. Give $Z$ and $W$ and the quotient topology. We obtain an induced map $g \times f : X \times Y \to W \times Z$, where we give both $X \times Y$ and $W \times Z$ the product topologies. Show that the map $g \times f$ is continuous.

Exercise 3.7. Show that the map

$$\alpha : G \times G/H \to G/H$$

$$\alpha(g_1, g_2 \cdot H) \mapsto g_1 g_2 \cdot H$$

is continuous, where we give $G/H$ the quotient topology.
Exercise 3.8. Let $H \subset G$ be a closed normal subgroup. We have a surjective map $G \to G/H$ sending $g \mapsto g \cdot H$ and so we may endow $G/H$ with the quotient topology. In this exercise, we show $G/H$ is a topological group.

(1) Show that the induced multiplication map on $G/H$
\[
\bar{m} : G/H \times G/H \to G/H
\]
\[
g_1 \cdot H, g_2 \cdot H \mapsto g_1 \cdot g_2 \cdot H
\]
is continuous. 10

(2) Show that the induced inversion map
\[
\bar{i} : G/H \to G/H
\]
\[
g \cdot H \mapsto g^{-1} \cdot H
\]
is continuous.

(3) Show that $\bar{m}$ and $\bar{i}$ give $G/H$ the structure of a topological group.
4. THE CLOSURE OF THE IDENTITY

The main goal of this section is to show that for $G$ any topological group, the closure of the identity is again a subgroup.

**Definition 4.1.** Let $X$ be a topological space and $U \subset X$ a subset (not necessarily open or closed). The **closure of $U$** is the intersection of all closed sets containing $U$.

**Exercise 4.2** (Easy Exercise). Prove the following topological properties of the closure.

1. Verify that closure of any $U \subset X$ set is closed.
2. Show that the closure of $U$ is the smallest closed set containing $U$. That is, if $V \subset X$ is any closed set with $U \subset V$ then the closure of $U$ is contained in $V$.

**Definition 4.3.** A map of topological spaces $f : X \to Y$ is a **homeomorphism** if there is a continuous inverse map $f^{-1} : Y \to X$ so that $f \circ f^{-1} = \text{id}_Y$, $f^{-1} \circ f = \text{id}_X$.

**Exercise 4.4.** Let $G$ be a topological group. Show that the inverse map $i : G \to G$ is a homeomorphism. 11

**Exercise 4.5.** Let $G$ be a topological group and let $e \in G$ be the identity. Let $V$ denote the closure of $\{e\} \subset G$. In this exercise, we show that $V$ is a closed normal group.

1. Show that the map $i : G \to G$ restricts to a map $i_V : V \to V$. That is, show that for $v \in V$, $i(v) \in V$. 12
2. Show that the induced map $i_V : V \to V$ is continuous, where $V$ is given the subspace topology.
3. Show that if $g \in V$ then $g \cdot V$ is a closed set containing the identity. 13
4. Show that if $g \in V$ then $g \cdot V = V$. 14
5. Show that the multiplication map $m : G \times G \to G$ restricts to a multiplication map $m : V \times V \to V$. That is, show that if $v, w \in V$ then $m(v, w) \in V$. 15
6. Conclude that $V$ is a closed subgroup of $G$. In particular, $V$ is a topological group.
7. Show that $V$ is in fact a closed normal subgroup of $G$. 16

**Exercise 4.6.** Let $V \subset G$ be the closure of the identity. By **Exercise 4.5**, $V \subset G$ is a closed normal subgroup. Conclude that $G/V$ is a topological group.
5. HAUSDORFNESS

We now introduce the concept of a topological space being Hausdorff and explore the interaction between Hausdorffness and topological groups.

Definition 5.1. A topological space $X$ is **Hausdorff** if for any $x, y \in X$ with $x \neq y$, there are open sets $U, V \subset X$ with $x \in U, y \in V$ and $U \cap V = \emptyset$.

Exercise 5.2. Let $X$ be a topological space. Show that if $X$ is Hausdorff then $X$ is also $T_1$.

Exercise 5.3. Let $G$ be a $T_1$ topological group. We show that $G$ is also Hausdorff. Hence, a topological group is $T_1$ if and only if it is Hausdorff.

1. Let $g \in G$ with $g \neq e$ (for $e \in G$ the identity). Use that the multiplication map $m : G \times G \to G$ is continuous at $(e, e)$ to show that there are two open sets $U$ and $V$ with $e \in U, e \in V$ and $g \notin U \cdot V$.  

2. To prove that $G$ is Hausdorff, show that it suffices to show that for any $x \in G$ there is some $U, V$ with $e \in U, x \in V$ and $U \cap V = \emptyset$.

3. Let $U, V$ be as in the first part and define $W := (U \cap V) \cap i(U \cap V)$. Show that $W \cap gW = \emptyset$ with $e \in W$ and $g \in gW$.  

4. Conclude that $G$ is Hausdorff.

Exercise 5.4. Let $G$ be a topological group, let $e \in G$ be the identity, and let $V$ denote the closure of $e$. Show that $G/V$, equipped with the quotient topology, is Hausdorff.  

Exercise 5.5. If $G$ is a Hausdorff topological group and $H \subset G$ is a closed subgroup, show that $G/H$ with the quotient topology is also Hausdorff.  

5.1. The $T_0$ property. Finally, we include a bonus subsection on the $T_0$ property.

Definition 5.6. A topological space $X$ is $T_0$ if for any two points $x, y \in X$ there is an open set $U \subset X$ so that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Exercise 5.7. Show that a $T_1$ topological space is $T_0$.

Exercise 5.8. Show that a $T_0$ topological group is also $T_1$. Conclude that a topological group is $T_0$ if and only if it is $T_1$ if and only if it is Hausdorff.
Notes

1 Hint: You need to show that an intersection of two sets of the form \((V_1 \times U_1) \cap (V_2 \times U_2)\), for \(U_i \subset X, V_i \subset Y\), can be written as a union of open sets of the form \(W_i \times Z_i\) for \(U_i \subset X, Z_i \subset Y\).

2 Hint: To show \(l_g\) is continuous, write \(l_g\) as the composition of the map \(\phi: G \rightarrow G \times G\) with the multiplication map \(m\). Show that \(\phi\) is continuous and that a composition of continuous maps is continuous.

3 Hint: If \(H \subset G\) is an open subgroup, show that \(U := \bigcup_{g \in H} g \cdot H\) is an open subset with \(G - U = H\).

4 Hint: Use the same idea as in Exercise 1.13

5 Hint: The key step to check is that an intersection of two open balls \(U\) and \(V\) is again a union of open balls. To show this, for each point \(x \in U \cap V\) find an open ball containing \(x\) and contained in \(U \cap V\).

6 Hint: Show multiplication and inversion are continuous by expressing them as a polynomial in the variables of the two matrices you are multiplying.

7 Hint: Use Exercise 3.4 and Exercise 1.12

8 Hint: Argue that it suffices to verify that for \(U \subset W, V \subset Z\) we have \((g \times f)^{-1}(U \times V) = g^{-1}(U) \times f^{-1}(V) \subset X \times Y\) is open. Then, show this from the definition of product topology.

9 Hint: Show that the diagram

\[
\begin{array}{ccc}
G \times G & \xrightarrow{m} & G \\
\downarrow{\text{id} \times \pi} & & \downarrow{\pi} \\
G \times G/H & \xrightarrow{\alpha} & G/H
\end{array}
\]

commutes. Show using Exercise 3.6 (taking \(X = W\) and \(g : X \rightarrow W\) the identity map there) that \(\alpha\) is continuous if \(m\) is.

10 Hint: Use the same idea as in Exercise 3.7

11 Hint: Show that \(i\) is its own inverse.

12 Hint: If \(i(v) \notin V\), then show \(V \cap i(V)\) is a closed set containing \(e\) strictly contained in \(V\).

13 Hint: Use Exercise 1.12 and that \(g^{-1} \in V\).

14 Hint: If \(g \cdot V \neq V\), show that \(g \cdot V \cap V\) is a strict subset of \(V\) containing the identity.

15 Hint: Use the previous part.

16 Hint: If \(g \in G\) show that \(gVg^{-1}\) is a closed subset containing the identity. If \(gVg^{-1} \neq V\), show that \(gVg^{-1} \cap V\) is a closed proper subset of \(V\), and reach a contradiction.

17 Hint: Since \(G\) is \(T_1\), \(G - g\) is open. Consider \(m^{-1}(G - g)\).

18 Hint: Show that \(e \notin m(gW \times W)\) but if \(a \in gW \cap W\) then \(e = a \cdot a^{-1} \in m(gW \times W)\).

19 Hint: Use Exercise 4.6, Exercise 3.4, and Exercise 5.3

20 Hint: Use the same idea as in Exercise 5.3.
21 Hint: Reduce to showing that if there is an open set containing $e$ but not $p$, then there is an open set containing $p$ but not $e$. If there is an open set $U$ containing $e$ but not $p$, then consider $U \cap i(U)$ and translate.