STACKY HEIGHTS ON ELLIPTIC CURVES IN CHARACTERISTIC 3

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Today, we'll discuss the notion of heights on stacks. We'll describe the presentations as in Ellenberg, Satriano, Zureick-Brown [ESZB21] and Darda Yasuda [DY22]. These are able to recover many notions of height functions, but have some serious shortcomings, especially when wants to work in low characteristic. We explain the issue with these approaches for counting elliptic curves in characteristic 3, and present a germ of an idea for a potential fix.

1. MOTIVATION FOR HEIGHT ON STACKS

First, let's recall the setup for height on schemes and on projective space.

Definition 1.1. Given a point $[x_0, \ldots, x_n] \in \mathbb{P}^n(\mathbb{F}_q(t))$ we'd like to define its height. Here, $x_i \in \mathbb{F}_q(t)$ are rational functions with no common zero. Let $y_i \in \mathbb{F}_q[t]$ so that $[y_0, \ldots, y_n] = [x_0, \ldots, x_n]$, i.e. we have cleared denominators. The height of $[y_0, \ldots, y_n] = [x_0, \ldots, x_n]$ is $\max_i \deg y_i$.

Remark 1.2. The height of such a point is the maximum degree of any polynomial y_i. Upon homogenizing, one can also think of such a point as a map $\mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^n_{\mathbb{F}_q}$ and the height of the point is the degree of this map. This leads to an alternative definition.

A point $x \in \mathbb{P}^n(\mathbb{F}_q(t))$ is a map $\mathbb{F}_q(t) \to \mathbb{P}^n$, and extends uniquely to a map $\overline{x} : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^n$. The height of x is then $\deg \overline{x} = \deg \overline{x}^* \mathscr{O}(1)$.

One can extend this notion in general to heights on varieties. To simplify notation, from now on we'll let $K := \mathbb{F}_q(t)$.

Definition 1.3. Let X be a proper variety over $\mathbb{F}_q$ and $\mathscr{L}$ a line bundle on X. Let $x : \operatorname{Spec} K \to X$ be a point with extension $\overline{x} : \mathbb{P}^1_{\mathbb{F}_q} \to X$. Then the **height** of x with respect to $\mathscr{L}$ is by definition $\operatorname{ht}_{\mathscr{L}}(x) := \deg \overline{x}^* \mathscr{L}$

Remark 1.4. The Batyrev-Manin (Bat-Man) heuristics predict asymptotics for the number of such points. Namely, they predict the number of such points in terms of the geometry of X, using features such as the picard group and effective cone of X.

In another direction, Malle's conjecture predicts the number of G extensions. There is a stack called BG, and by definition a map $\operatorname{Spec} K \to BG$

corresponds to a G-extension of K. In this sense, Malle's conjecture and Bat-Man should both be special cases of a stacky Bat-Man conjecture that specializes to these two conjectures in the case that $X = BG$ and X is a scheme. For example, we also want heights on the moduli space of elliptic curves to correspond to the usual faltings height.

2. EXTENDED EXAMPLE: HEIGHT FOR G-EXTENSIONS

Let K be a number field or function field. There should be a notion of height which recovers the discriminant. A map $x : \operatorname{Spec} K \to BG$ corresponds to a G-cover $\operatorname{Spec} L \to \operatorname{Spec} K$. We would like to associate a height function which outputs the relative discriminant of L over K.

As a first attempt, one might naively try taking a line bundle on BG pulling it back to K, and taking the degree, as before. However, there is a serious issue with this.

Fact 2.1. There is a bijection beween rank r vector bundles on BG and r-dimensional G-representations. The key word for proving this is "descent data."

Example 2.2. Now, let's consider the case $G = A_5$. Then, if we want the height with respect to a line bundle, we need to specify a 1-dimensional representation of A_5. However, since A_5 has trivial abelianization, it has no line bundles, other than the trivial line bundle. The height with respect to the trivial line bundle should always be 0, so this is not a good candidate to recover the discriminant, and it seems we are stuck.

The issue here is that the discriminant is really purely local data, which takes into account ramification points of the cover, whereas these line bundles are much more global. There are two proposed fixes to this:

DY For each place of K where $\operatorname{Spec} L \to \operatorname{Spec} K$ is ramified, add a local height function and sum those local height functions.

ESZB Instead of working with line bundles, work with vector bundles, and attach local height functions to these vector bundles.

Both approaches have their advanteages and disadvantages. The setup of DY really only works in good characteristics, but in those cases it generalizes ESZB. On the other hand, ESZB works in arbitrary characteristic. Let's discuss the ESZB case. It turns out that if one takes the vector bundle correpsonding to the regular representation, this recovers the discriminant. Essentially, local inertia groups at ramified places under the regular representation tell one about the discriminant of the cover.

Definition 2.3. In the above setup, the map $\operatorname{Spec} K \to BG$ may not extend to a map $\operatorname{Spec} \mathscr{O}_K \to BG$. This is because, unlike in the case of varieties, the valuative criterion for stacks is somewhat more subtle. However, $x : \operatorname{Spec} K \to BG$ does extend to a map from the stack quotient $\overline{x} : [\operatorname{Spec} \mathscr{O}_L/G] \to BG$ since the G cover is $\operatorname{Spec} \mathscr{O}_L$. There is a natural "coarse space map" $\pi : [\operatorname{Spec} \mathscr{O}_L/G] \to \operatorname{Spec} \mathscr{O}_K$. One would like to define the height to be $\deg \pi_*(\overline{x}^* \mathscr{V})$, where this pushfoward under π is recovering the local ramification contributions. For technical reasons, it is better to define $\operatorname{ht}_{\mathscr{V}}(x) := -\deg \pi_*(x^* \mathscr{V}^\vee)$.

Remark 2.4. In general, ESZB make a similar definition whenever $\mathscr{X}$ (in place of BG is a proper Artin stack with finite diagonal.

3. EXTENDED EXAMPLE: ELLIPTIC CURVES

Start with an elliptic curve E over $\mathbb{F}_q(t)$, where $\operatorname{char} q \neq 2$ (just for simplicity). One can write this in minimal Weierstrass form at $zy^2 = x^3 + ax^2z + bxz^2 + cz^3$ where there is some d for which $\deg a = 2d, \deg b = 4d, \deg c = 6d$. This d is called the *Faltings height* of E.

One can also think of this elliptic curve as a point $x : \operatorname{Spec} \mathbb{F}_q(t) \to \overline{\mathscr{M}}_{1,1}$. We define $h_f(x)$ to be the faltings height of the corresponding elliptic curve E_x.

A natural question is:

Question 3.1. Is there some vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ so that $\operatorname{ht}_{\mathscr{V}}(x)$ is the Faltings height $h_f(x)$?

It turns out that that when the characteristic of K is more than 3, the answer is indeed yes. In fact, there Picard group of $\overline{\mathscr{M}}_{1,1}$ is $\mathbb{Z}$, and is generated by a certain line bundle called ω, the Hodge bundle. Similarly to the case of $\mathscr{O}(1)$ on $\mathbb{P}^1$ every vector bundle is a sum of powers of ω. This uses that the characteristic is not 2 or 3.

Lemma 3.2. *For $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ corresponding to an elliptic curve over K with everywhere semistable reduction, we have $\operatorname{ht}_\omega(x) = h_f(x)$.*

Sketch. Assuming the elliptic curve E corresponding to x has everywhere semistable reduction, the map $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ extends to a map $\mathbb{P}^1_{\mathbb{F}_q} \to \overline{\mathscr{M}}_{1,1}$. (In the case there are places of additive reduction, one needs to use a stacky curve, as in the number field case, and take into account local contributions.) Intuitively, the degree of the line bundle ω is roughly counting the number of places of $\mathbb{P}^1_{\mathbb{F}_q}$ where the elliptic curve has bad reduction.

By definition, there is a universal proper genus 1 curve $\mathscr{E} \to \overline{\mathscr{M}}_{1,1}$. Most fibers are smooth elliptic curve, but there is also one point where the fiber is a nodal elliptic curve. Call $\mathscr{E}_x$ the elliptic surface corresponding to x. We have a pullback square

$$\begin{tikzcd}
\mathscr{E}_x \arrow[r, "x'"] \arrow[d, "f'"] & \mathscr{E} \arrow[d, "f"] \\
\mathbb{P}^1_{\mathbb{F}_q} \arrow[r, "x"] & \overline{\mathscr{M}}_{1,1}
\end{tikzcd} \tag{3.1}$$

The Hodge bundle ω is by definition $f_*\omega_{\mathscr{E}/\overline{\mathscr{M}}_{1,1}}$. In this case, the definition of stacky height gives

$$\mathrm{ht}_\omega(x) = \deg x^*\omega = \deg x^* f_*\omega_{\mathscr{E}/\overline{\mathscr{M}}_{1,1}} = \deg f'_*(x'^*\omega_{\mathscr{E}/\overline{\mathscr{M}}_{1,1}}) = \deg f'_*\omega_{\mathscr{E}_x/\mathbb{P}^1}.$$

It's a bit beyond the scope of this talk, but it's not too difficult too show that this degree recovers the Faltings height. Basically, the sections in the Weierstrass equation come from powers of ω. □

Remark 3.3. This actually all works in characteristics 2 and 3 as well when there is semistable reduction. The issue there relates to additive reduction, as we'll see next.

4. THE ISSUE IN CHARACTERISTIC 3

Now, let's talk about what goes wrong with the above in characteristic 3. In the case of characteristic more than 3, we saw that the height with respec to the Hodge bundle always recovers Faltings height. Now, assume $k = \mathbb{F}_q$ has characteristic 3 and $K = k(t)$.

Remark 4.1 (Automorphism groups of elliptic curves in characteristic 3)**.** The geometric automorphism groups of elliptic curves in characteristic 3 are either $\mathbb{Z}/2\mathbb{Z}$ or G, where G is the order 12 dicyclic group, which is $G = \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Elliptic curves of j invariant 0 have automorphism group BG (at least geometrically) which means that its automorphism group is G, corresponding to j-invariant 0 (note $0 = 1728 \bmod 3$!).

Example 4.2. Consider the family of elliptic curves $y^2 = x^3 - x + f(t)$ for f a degree d polynomial. It turns out that when one calculates the stacky height of this curve with respect to ω, one obtains 0, which is less than the Faltings height.

The loose idea to see this is that this has additive reduction at places dividing the roots of f. When one passes to the extension generated by u satisfying $u^3 - u + f = 0$, one finds that substituing $x + u$ for x, we get $y^2 = (x+u)^3 - (x+u) + f = x^3 - x + f + u^3 - u = x^3 - x$. So when we pass to this cubic extension, we get an elliptic curve of fixed height. Therefore, the faltings height is concentrated at the local places, and we want to show these are trivial.

So we have an inclusion $BG \to \overline{\mathscr{M}}_{1,1}$ (really we should use $G = \mathbb{Z}/3\mathbb{Z} \rtimes \mu_4$ to get this rationally) for G the dicyclic group of order 12, as in Remark 4.1. The abelianization of G is $\mathbb{Z}/4\mathbb{Z}$ which shows that any line bundle on $\overline{\mathscr{M}}_{1,1}$, such as ω, restricts to a $\mathbb{Z}/4\mathbb{Z}$ representation on BG. One the other hand, the extension is a cyclic cubic extension, which has order prime to 4, and this forces the local height contributions to be 0.

However, unlike in characteristic more than 3, it turns out there are other interesting vector bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. This led to the question.

Question 4.3. Although $\mathrm{ht}_\omega(x) \neq \mathrm{ht}_f(x)$ in characteristic 3, does there exist some other vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ inducing Faltings height in characteristic 3?

It turns out the answer is no!

Theorem 4.4. *There is no vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, where K is a finite extension of $k(t)$.*

Another related question, originally posed to us by Jordan Ellenberg, is whether there exist Northcott stacky heights on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. We say a height function on the set of $k(t)$ points of a stack $\mathscr{X}$ satisfies the *Northcott property* if there are finitely many such points of bounded height, cf. [ESZB21, p. 4].

Question 4.5 (Ellenberg)**.** Does there exist a vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ over a finite field of characteristic 3 whose induced height function $\mathrm{ht}_{\mathscr{V}}$ is Northcott?

I suspect the answer is "yes" but don't have a proof.

5. SOMETHING ABOUT THE PROOF

The basic idea is that dealing with vector bundles on $\overline{\mathscr{M}}_{1,1}$ is difficult, but vector bundles on BG are much easier to understand, as they are equivalent to G representations. So, we will try to study what G representations could induce Faltings height on j-invariant 0 elliptic curves, and show that there are none by studying G-representations in characteristic 3.

We now elaborate on the above idea. The substack $BG \subset \overline{\mathscr{M}}_{1,1}$ corresponding to elliptic curves with j-invariant 0 has geometric automorphism group G, where G is the dicyclic group of order 12. When we restrict $\mathscr{V}$ to BG, we obtain a G-representation ρ. We show that some element $g \in G$ of order 4 acts with a codimension 1 fixed space and no eigenvalues equal to -1. This is enough to deduce that ρ is a sum of 1-dimensional representations, and hence factors through the abelianization of G. We then show that any such vector bundle cannot detect nontrivial stacky heights associated to elliptic curves which are isotrivial cyclic cubic twists, essentially via the argument in Example 4.2.

6. A POSSIBLE FIX IN BAD CHARACTERISTICS

In bad characteristics 2 and 3, we saw the usual Hodge bundle ω does not induce Faltings height. However, there turns out to be a vector bundle on a different stack, which does!

Example 6.1. Consider the stack $\mathscr{M}$ parameterizing integral genus 1 curves with a marked point. This contains $\overline{\mathscr{M}}_{1,1}$ as a dense open, but also contains a point correpsonding to a cuspidal elliptic curve.

A K point $x : \operatorname{Spec} K \to \mathscr{M}$ gives us an elliptic curve, and it turns out we can always extend this to a map $\overline{x} : \mathbb{P}^1_{\mathbb{F}_q} \to \mathscr{M}$. This is in contrast to the general case of proper stacks, where we may only be able to extend it to a map from a stacky curve. The extension comes from taking the curve given by the minimal Weierstrass equation.

This stack $\mathscr{M}$ has a univeral family of elliptic curves $f : \mathscr{E} \to \mathscr{M}$. If one then takes $\omega := f_* \omega_{\mathscr{E}/\mathscr{M}}$ one can analogously define $\operatorname{ht}_\omega(x) := \deg \overline{x}^* \omega$. It turns out that this does agree with Faltings height, via an analogous computation to the one we did for elliptic curves in characteristic not 2 or 3.

Remark 6.2. The reason the above example is not covered by DY or ESZB is that the stack $\mathscr{M}$ is not proper. Namely, there are multiple extensions of x to a map $\mathbb{P}^1 \to \mathscr{M}$, corresponding to non-minimal Weierstrass equations. However, one can take the minimum degree over all extensions, which gives the minimal Weierstrass equation. This poses a germ of an idea to possibly fix the notion of stacky heights in bad characteristics. Namely, one can allow non-separated stacks.

Remark 6.3. Given that one can recover Malle's conjecture in terms of stacky heights, one might also hope to recover other conjectures in number theory in terms of stacky heights. Namely, one might hope to recover Cohen-Lenstra heuristics for torsion in class groups of number fields, and also Poonen-Rains conjectures for Selmer groups of elliptic curves. It turns out one can recover a weak form of both of these, where one only asks for asymptotics and not precise constants. (The obstruction here is that no prediction for the constants in stacky Bat-Man has been formulated.) However, for Poonen-Rains, it seems one again needs to deal with certain non-separated stacks, which still satisfy the existence part of the valuative criterion.

REFERENCES

[DY22] Ratko Darda and Takehiko Yasuda. The batyrev-manin conjecture for dm stacks. *arXiv preprint arXiv:2207.03645v2*, 2022.

[ESZB21] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *arXiv preprint arXiv:2106.11340v1*, 2021.