A PROOF OF THE SHIODA-TATE FORMULA OVER FINITE FIELDS

AARON LANDESMAN

We give a brief proof of the well known Shioda-Tate formula for non-constant elliptic surfaces over finite fields. Although I haven't seen this particular proof elsewhere, it's quite likely others have come up with it before. The main result is stated in Theorem 2, and is proven at the end in 7. In order to state Theorem 2, we introduce the following notation.

Notation 1. Fix a finite field k. Let C be a smooth proper geometrically connected curve over k. Let E be a smooth elliptic curve over the function field $K := K(C)$ which is non-constant, i.e., not a pullback from k. Let $\mathscr{E}$ denote the Néron model of E over C with identity component $\mathscr{E}^0$. Let $f : X \to C$ denote the minimal regular proper model of E over C. Let Q denote the quotient algebraic space $\operatorname{Pic}_{X/C} / \operatorname{Pic}^0_{X/C}$. Let $\operatorname{NS}(X) := \operatorname{Pic}(X) / \operatorname{Pic}^0(X)$ denote the Néron-Severi group of X. If m_x is the number of components in the fiber over a closed point $x \in C$, define $m := \sum_{x \in C}(m_x - 1)$. For a more precise definition of m, see 4.

The main result is the following formula, computing the rank of the Néron-Severi group.

Theorem 2 (Shioda-Tate)**.** *With notation as in Notation 1,* $\operatorname{rk}_{\mathbb{Z}} \operatorname{NS}(X) = m + \operatorname{rk}_{\mathbb{Z}}(E(K)) + 2$.

Our first step to proving Theorem 2 is to compute the rank of $\operatorname{Pic}_{X/C}(C)$ in Proposition 5. In order to accomplish this, we need the following lemma.

Lemma 3. *With notation as in Notation 1, the quotient* $T := H^0(C, Q) / \operatorname{im} H^0(C, \operatorname{Pic}_{X/C})$ *is a finite group.*

Proof. Observe that the long exact sequence associated to $\operatorname{Pic}^0_{X/C} \to \operatorname{Pic}_{X/C} \to Q$ identifies T with $\ker\left(H^1(C, \operatorname{Pic}^0_{X/C}) \to H^1(C, \operatorname{Pic}_{X/C})\right)$. There is a natural degree map $\deg : \operatorname{Pic}_{X/C} \to \mathbb{Z}$ sending an invertible sheaf to its degree when restricted to the generic fiber of $X \to C$. Let $P_{X/C} := \ker(\deg : \operatorname{Pic}_{X/C} \to \mathbb{Z})$. Note this degree map is surjective on global sections because $X \to C$ has a section. Then, since $H^1(C, \mathbb{Z}) = 0$, as $\mathbb{Z}$ is torsion free, we obtain $H^1(C, P_{X/C}) \simeq H^1(C, \operatorname{Pic}_{X/C})$ is an isomorphism. Further, by [BLR90, §9.5, Theorem 4(a)], there is a natural map $P_{X/C} \to \mathscr{E}$. So, using the maps $\operatorname{Pic}^0_{X/C} \to P_{X/C} \to \mathscr{E}$ and the identification $\operatorname{Pic}^0_{X/C} \simeq \mathscr{E}^0$ [BLR90, §9.5, Theorem 4(b)], we find

$$\begin{aligned} \ker\left(H^1(C, \operatorname{Pic}^0_{X/C}) \to H^1(C, \operatorname{Pic}_{X/C})\right) &= \ker\left(H^1(C, \operatorname{Pic}^0_{X/C}) \to H^1(C, P_{X/C})\right) \\ &\subset \ker\left(H^1(C, \mathscr{E}^0) \to H^1(C, \mathscr{E})\right). \end{aligned}$$

So, it suffices to show the latter is finite. By taking cohomology associated to

$$0 \longrightarrow \mathscr{E}^0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}/\mathscr{E}^0 \longrightarrow 0, \tag{1}$$

we obtain an exact sequence

$$0 \longrightarrow \frac{H^0(C, \mathscr{E}/\mathscr{E}^0)}{H^0(C, \mathscr{E})} \longrightarrow H^1(C, \mathscr{E}^0) \longrightarrow H^1(C, \mathscr{E}). \tag{2}$$

Then, $\frac{H^0(C,\mathscr{E}/\mathscr{E}^0)}{H^0(C,\mathscr{E})} = \ker\left(H^1(C,\mathscr{E}^0) \to H^1(C,\mathscr{E})\right)$ is finite because $\mathscr{E}/\mathscr{E}^0$ is a finite group scheme. □

4 (Structure of Q)**.** Retain notation as in Notation 1. For the purpose of proving Proposition 5 below, we review the explicit description of the algebraic space $Q = \operatorname{Pic}_{X/C} / \operatorname{Pic}^0_{X/C}$, essentially provided in [BLR90, §9.2, Corollary 13-14]. Note that in fact Q is isomorphic to the constant sheaf $\mathbb{Z}$ away from the singular fibers of $X \to C$, but is not a separated sheaf (so different points in the fibers "come together" over the generic point of C).

We next analyze the singular fibers. Suppose $v_1, \ldots, v_r \in C$ are the singular fibers, let m_i denote the number of irreducible components of the fiber X_{v_i}, and let $\overline{m}_i$ denote the number of irreducible components of $X_{\overline{v}_i}$, for $\overline{v}_i$ a geometric point over v_i. Then, by [BLR90, §9.2, Corollary 13-14], the geometric fiber $Q_{\overline{v}_i}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^{\oplus(\overline{m}_i - 1)}$. We single out the first coordinate as the component through which the identity section passes. If the geometric components of X_{v_i} are $X_{i,1}, \ldots, X_{i,\overline{m}_i}$ and $X_{i,j}$ has multiplicity $r_{i,j}$, then the isomorphism above is given explicitly with the jth coordinate sending a line bundle $\mathscr{L} \mapsto \frac{\deg_{X_{i,j}}(\mathscr{L})}{r_{i,j}}$. The action of $\operatorname{Gal}(\overline{v}_i/v_i)$ acts on $\mathbb{Z} \oplus \mathbb{Z}^{\oplus(\overline{m}_i - 1)}$ by permuting the factors of $\mathbb{Z}^{\oplus(\overline{m}_i - 1)}$, corresponding to its action permuting the geometric components $X_{i,2}, \ldots, X_{i,\overline{m}_i}$. Let $m := \sum_{i=1}^r (m_i - 1)$. Taking into account the Galois action described above and the explicit description of the restriction of Q to fibers given above, we find $H^0(C,Q) \simeq \mathbb{Z}^{m+1}$.

Using the description of Q above in conjunction with Lemma 3, we can compute the rank of $\operatorname{Pic}_{X/C}(C)$, which is the key ingredient in proving Shioda-Tate.

Proposition 5. *With notation as in Notation 1, we have* $\operatorname{rk}_{\mathbb{Z}} H^0(C, \operatorname{Pic}_{X/C}) = m + 1 + \operatorname{rk}_{\mathbb{Z}} E(K)$.

Proof. We have a left exact sequence

$$0 \longrightarrow H^0(C, \operatorname{Pic}^0_{X/C}) \longrightarrow H^0(C, \operatorname{Pic}_{X/C}) \longrightarrow H^0(C, Q), \tag{3}$$

where the last map has finite cokernel by Lemma 3. Recall we showed $\operatorname{rk}_{\mathbb{Z}} H^0(C,Q) = m + 1$ in 4.

To complete the proof, we only need verify $\operatorname{rk}_{\mathbb{Z}} H^0(C, \operatorname{Pic}^0_{X/C}) = \operatorname{rk}_{\mathbb{Z}} E(K)$. Since $\operatorname{Pic}^0_{X/C} \simeq \mathscr{E}^0$ [BLR90, §9.5, Theorem 4(b)], there is an injection $H^0(C, \operatorname{Pic}^0_{X/C}) \to H^0(C, \mathscr{E})$ whose cokernel is contained in the finite group $H^0(C, \mathscr{E}/\mathscr{E}^0)$. Therefore, $H^0(C, \operatorname{Pic}^0_{X/C})$ is finitely generated of rank $\operatorname{rk}_{\mathbb{Z}} E(K)$ if and only if the same holds for $H^0(C, \mathscr{E})$. Indeed, $H^0(C, \mathscr{E})$ is finitely generated of rank $\operatorname{rk}_{\mathbb{Z}} E(K)$ because $H^0(C, \mathscr{E}) = H^0(K(C), E)$ via the Néron mapping property. □

We are nearly ready to prove Shioda-Tate, and to do so we recall the following standard fact about elliptic surfaces. One can find a more general statement implying it in [Ulm14, Proposition 4.2.4] (using [Con06, Example 2.2]), but we prefer to sketch a more elementary direct proof.

Lemma 6. *For $f : X \to C$ a non-constant elliptic surface, the map* $\operatorname{Pic}^0(C) \to \operatorname{Pic}^0(X)$ *is an isomorphism.*

Proof. Since $f : X \to C$ has a section, $\mathrm{Pic}^0_{C/k} \hookrightarrow \mathrm{Pic}^0_{X/k}$ is injective. Since $\mathrm{Pic}_{C/k}$ is smooth, we only need to prove the map $\mathrm{Pic}^0_{C/k} \to \mathrm{Pic}^0_{X/k}$ is an isomorphism on tangent spaces. For this, recall the tangent spaces as the identity are identified with $H^1(C, \mathscr{O}_C)$ and $H^1(X, \mathscr{O}_X)$. From the Leray spectral sequence associated to the map $f : X \to C$, since $f_*\mathscr{O}_X = \mathscr{O}_C$, the cokernel of $H^1(C, \mathscr{O}_C) \to H^1(X, \mathscr{O}_X)$ is a subgroup of $H^0(C, R^1 f_*\mathscr{O}_X)$. So, to demonstrate the isomorphism of tangent spaces, it suffices to check $R^1 f_*\mathscr{O}_X$ is an invertible sheaf of negative degree on C (so then $H^0(C, R^1 f_*\mathscr{O}_X) = 0$ implying $H^1(C, \mathscr{O}_C) \to H^1(X, \mathscr{O}_X)$ is an isomorphism). By Grothendieck duality, it is equivalent to check $f_*\omega_{X/C}$ has strictly positive degree. This is a standard fact from the theory of elliptic surfaces via Weierstrass equations. A proof in the case $C = \mathbb{P}^1$ is given in [dJ02, 4.5], and the proof for general curves C is analogous (see also [Del75, §1]).

Note that above we are using that X is not constant to rule out the possibility that $\deg f_*\omega_{X/C} = 0$, as we now explain. If $\deg f_*\omega_{X/C} = 0$, one can obtain a Weierstrass equation for X given by sections of tensor powers of $f_*\omega_{X/C}$, see [Del75, §1]. Since tensor powers of $f_*\omega_{X/C}$ have degree 0, these would be sections in $H^0(C, \mathscr{O}_C) \simeq k$. Hence, the resulting Weierstrass equation has coefficients lying in k, so X would be constant. □

We are now ready to prove Shioda-Tate via an application of Proposition 5.

7 (Proof of Shioda-Tate Theorem 2)**.** Because C is a curve, $\mathrm{rk}_{\mathbb{Z}}(\mathrm{Pic}(C)/\mathrm{Pic}^0(C)) = \mathrm{rk}_{\mathbb{Z}}\,\mathbb{Z} = 1$. Also, by Lemma 6, $\mathrm{Pic}^0(C) \simeq \mathrm{Pic}^0(X)$. Therefore,

$$\mathrm{rk}_{\mathbb{Z}}\,\mathrm{NS}(X) = \mathrm{rk}_{\mathbb{Z}} \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)} = \mathrm{rk}_{\mathbb{Z}} \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(C)} = 1 + \mathrm{rk}_{\mathbb{Z}} \frac{\mathrm{Pic}(X)}{\mathrm{Pic}(C)} = 1 + \mathrm{rk}_{\mathbb{Z}}\, H^0(C, \mathrm{Pic}_{X/C}).$$

The result then follows from Proposition 5.

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