UNDERSTANDING POONEN RAINS CONJECTURES IN TERMS OF STACKY HEIGHTS

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1. SUMMARY OF RESULTS

In this note, we'd like to understand whether two conjectures on Selmer groups of elliptic curves are compatible. Specifically, [PR12] predicts that the average size of the n-Selmer groups of elliptic curves is a finite value. Of course, they do much more, and predict it is $\sum_{d|n} d$, and even predict much more about the distribution. On the other hand, [DY22] gives general predictions for the number of points of bounded height on sufficiently nice stacks, counted with respect to what they call raised line bundles. Their general prediction is of the form $cX^a(\log X)^b$, and they give values for a and b in terms of the situation, but do not attempt to predict the constant c. Our main result is that these two predictions are compatible.

Theorem 1.1. *There is a stack $\mathscr{S}^n$ with a raised line bundle (ω_n, c_n), see Definition 2.1 and Definition 5.4, so that the following hold over global fields of characteristic more than* 3.

1. *The Poonen-Rains heuristics* [PR12, Conjecture 1.4(b)] *predict that there is a constant D_n so that the number of points of height at most X is $D_nX^{10} + o_n(X^{10})$*
2. *A slight generalization of the Darda-Yasuda heuristics* [DY22, Conjecture 9.15] *(allowing for non-separated stacks and raised line bundles where the raising function can take value ∞) predict that there is a constant E_n so that the number of points of height at most X is $E_nX^{10} + o_n(X^{10})$.*

Proof. The Darda-Yasuda prediction is verified in Proposition 5.5.

It remains to verify the claim about the Poonen-Rains heuristics. However, when one unwinds the definitions given below, one finds the height of a Selmer element with respect to (ω_n, c_n) agrees with the usual Faltings height of the corresponding elliptic curve (which in Darda-Yasuda's terminology, would be the height with respect to the raised Hodge bundle (ω_1, c_1) on the moduli stack of elliptic curves, $\overline{\mathscr{M}}_{1,1}$.) Now, we know the number of elliptic curves in the function field setting of bounded discriminant Δ is a constant times $\Delta^{5/6}$. Since the discriminant is the height with respect to the twelfth power of the raised Hodge bundle, there are $X^{12 \cdot 5/6} = X^{10}$ elliptic curves of height at most X. In the number field setting, this asymptotic bound of elliptic curves counted by Faltings height is also proven, see [Hor16]. □

One can also deduce variants of the above compatibility in the setting of quadratic twist families of elliptic curves and the Cohen-Lenstra heuristics for average sizes of torsion in class groups of quadratic fields, see Remark 5.7.

Here is the outline of the remainder of this note. In § 2, we define the stack parameterizing Selmer elements, $\mathscr{S}^n$. We next analyze properties of the Selmer stack in §3, in order to explain why its points are good approximation for Selmer elements. Following this, in §4 we investigate what the Darda-Yasuda and Ellenberg-Satriano-Zureick-Brown heuristics predict for heights of points on $\overline{\mathscr{M}}_{1,1}$, and explain why they are correct. Finally, in §5 we investigate what their (modified form for non-separated stacks) would predict for $\mathscr{S}^n$ and show it is consistent with the Poonen-Rains heuristics for the number of Selmer elements.

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2. DEFINING THE SELMER STACK AND HEIGHTS ON IT

Let's try to understand the Poonen-Rains heuristics [PR12] in terms of stacky heights. First, we recall the setup. Throughout we will use terminology from [DY22] without recalling it.

We would like to count the number of Selmer elements of height up to d. The Poonen Rains heuristics predict that there should also be, up to a constant power of q, q^{10d} such Selmer elements when working over a curve over $\mathbb{F}_q$ and, up to a constant, X^{10} such elements when working over a characteristic 0 field. Of course the Poonen-Rains heuristics give much finer predictions, but let's just focus on this aspect of them.

The first question is: what stack should we take? We now introduce notation to define the relevant stack. Let $\overline{\mathscr{M}}_{1,1}$ denote the moduli stack of semistable elliptic curves. Let $\mathscr{E}$ denote the universal smooth but not proper elliptic curve over $\overline{\mathscr{M}}_{1,1}$. Notably, the geometric of $\mathscr{E}$ fiber over the point corresponding to a singular elliptic curve is $\mathbb{G}_m$. Since $\mathscr{E}$ is a group scheme, we can make sense of $\mathscr{E}[n]$. This is a quasi-finite group scheme over $\overline{\mathscr{M}}_{1,1}$ which has degree n^2 over any point corresponding to a smooth elliptic curve and degree n over the point corresponding to a nodal elliptic curve.

Definition 2.1. Define $\mathscr{S}^n := [\overline{\mathscr{M}}_{1,1}/\mathscr{E}[n]]$, where $\mathscr{E}[n]$ is acting trivially on $\overline{\mathscr{M}}_{1,1}$.

It turns out that certain rational points of the stack $\mathscr{S}^n$ essentially parameterize Selmer elements on elliptic curves. We warn the reader that there are many more rational points of $\mathscr{S}^n$ than Selmer elements, but one can use a generalization of the formalism of Darda-Yasuda to just count those corresponding to Selmer elements. We would like to apply the heuristics of [DY22] to the above stack to count rational points on the above stack.

A slight hitch is that the stack is not separated, whereas the above heuristics only apply to separated stacks. However, it fails to be separated in a mild manner.

Lemma 2.2. *The quotient stack $\mathscr{S}^n$ is not separated, but is universally closed.*

Proof. By definition, a stack is separated if and only if its diagonal is proper. By definition, the inertia stack is the fiber product

$$\begin{tikzcd} \mathscr{S}^n \times_{\mathscr{S}^n \times \mathscr{S}^n} \mathscr{S}^n \arrow[r] \arrow[d] & \mathscr{S}^n \arrow[d] \\ \mathscr{S}^n \arrow[r] & \mathscr{S}^n \times \mathscr{S}^n \end{tikzcd} \tag{2.1}$$

Pulling back along the diagonal, we find that in order to show $\mathscr{S}^n$ is not separated, it is enough to show the inertia stack $\mathscr{S}^n \times_{\mathscr{S}^n \times \mathscr{S}^n} \mathscr{S}^n \to \mathscr{S}^n$ is not proper over $\mathscr{S}^n$. Indeed, this properness fails over a neighborhood of the point of $\mathscr{S}^n$ corresponding to the singular elliptic curve, as the preimage of such a neighborhood is two copies of $\mathscr{E}[n]$, neither of which is proper, since $\mathscr{E}[n]$ is quasi-finite but not finite because the degree drops from n^2 to n over the point corresponding to the singular elliptic curve.

To check universal closedness, we wish to show that given any dvr R with fraction field K and map $\operatorname{Spec} K \to \mathscr{S}^n$ there exists an extension of dvrs R'/R with fraction field K' so that the map $K' \to \mathscr{S}^n$ extends to a map $R' \to \mathscr{S}^n$.

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathscr{S}^n \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd} \tag{2.2}$$

Given a map $\operatorname{Spec} K \to \mathscr{S}^n$, let $\operatorname{Spec} K_1$ be the reduction of a connected component of the fiber product $\operatorname{Spec} K \times_{\mathscr{S}^n} \overline{\mathscr{M}}_{1,1}$. Then by the valuative criterion for properness for $\overline{\mathscr{M}}_{1,1}$ there is a further extension K' of K_1 so that the map $\operatorname{Spec} K' \to \operatorname{Spec} K_1 \to \overline{\mathscr{M}}_{1,1} \to \mathscr{S}^n$ extends to a map $\operatorname{Spec} R' \to \overline{\mathscr{M}}_{1,1} \to \mathscr{S}^n$, restricting to the given map $\operatorname{Spec} K' \to \mathscr{S}^n$, which by construction factors through the original map $\operatorname{Spec} K' \to \operatorname{Spec} K \to \mathscr{S}^n$. $\square$

Remark 2.3 (Heuristically, no separated selmer stack should exist)**.** Morally, we feel $\mathscr{S}^n$ should be the best approximation to a stack parameterizing selmer elements. Indeed, any stack parameterizing Selmer elements should have automorphism group along smooth elliptic curves which is related to $E[n]$ (though perhaps an extension of that by $\operatorname{Aut}(E)$) but automorphism group along singular elliptic curves which is related to μ_n, the n-torsion in $\mathbb{G}_m$. If one believes this, the automorphism order drops from size n^2 to n, implying the intertia stack is not proper so any stack parameterizing Selmer elements cannot be separated. Of course this is not a rigorous argument, but at least shows that if you do have a proper stack parameterizing Selmer elements, its inertia groups must be different from what one expects.

Since $\mathscr{S}^n$ is not separated, we need a new definition of the height of a rational point with respect to a vector bundle or raised line bundle. We will also want to allow a generalization of raising functions, where we allow them to take the value ∞. This has the effect of prohibiting certain behavior.

Definition 2.4. A raised line bundle on a stack $\mathscr{X}$ will mean a line bundle $\mathscr{L}$ together with a function $c : \mathcal{I}_0\mathscr{X} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Remark 2.5. Ultimately, whether we allow ∞ or not will not matter much, because if we take the value on certain sectors extremely large, for asymptotic purposes of counting points of bounded height, it should be equivalent to taking the value on that sector to be ∞.

Definition 2.6. If K is a global function field, $x : \operatorname{Spec} K \to \mathscr{S}^n$ is a rational point, and $(\mathscr{L}, c)$ is a raised line bundle on $\mathscr{S}^n$, for each extension of x to a tame map $\overline{x} : \mathscr{C} \to \mathscr{S}^n$ (where here tame means the stabilizer orders are prime to the residue characteristic) where $\mathscr{C}$ is a stacky curve with generic point $\operatorname{Spec} K$, define

$$\operatorname{ht}_{(\mathscr{L},c)}(\overline{x}) := \deg \overline{x}^*\mathscr{L} + \sum_{\text{stacky places } v} c(\overline{x}_v),$$

where here $c(\overline{x}_v)$ indicates that we apply c to the sector to which the place v is sent under $\overline{x}$. Note that our tameness assumption on the map $\overline{x}$ ensures the residual gerbe over v is $B\mu_n$ for some n, and hence this map corresponds to a sector.

Define $\operatorname{ht}_{(\mathscr{L},c)}(x)$ to be the minimum height over all such extensions $\overline{x} : \mathscr{C} \to \mathscr{S}^n$ of x, for $\mathscr{C}$ a stacky curve with generic point $\operatorname{Spec} K$, restricting to x over $\operatorname{Spec} K$.

Remark 2.7. In the above definition, I am using additive notation, which conforms with notation for height for function fields. If one wishes to speak of heights with respect to number fields, one would typically use multiplicative notation, and adjust heights to Arakelov heights in order work with the place at infinity.

Remark 2.8. The above definition seems quite reasonable to us because for "most" elliptic curves, there is in fact a unique extension to a stacky curve. Indeed, this holds for any elliptic curve E over K whose discriminant is squarefree. Indeed, this will follow from Lemma 3.2 below. On the other hand, for the few at which there are multiple extensions, there are only finitely many such extensions. This holds because when one examines the ways to fill in the torsor $\mathscr{E}[n]$ over the singular fibers, there will be only finitely many ways to fill it in because it is a torsor for a quasi-finite group scheme.

3. Properties of the Selmer stack

There is a projection $\pi : \mathscr{S}^n \to \overline{\mathscr{M}}_{1,1}$ which we can think of as sending an "Selmer element" corresponding to its corresponding elliptic curve. More precisely, if we have a base scheme B and a map $f : B \to \overline{\mathscr{M}}_{1,1}$, we can ask about the set of lifts $g : B \to \mathscr{S}^n$ so that $f = \pi \circ g$. By definition of $\mathscr{S}^n$ the set of lifts are precisely the set of torsors $H^1(B, B \times_{\overline{\mathscr{M}}_{1,1}} \mathscr{E}[n])$ where $\mathscr{E}$ is the universal elliptic curve. Moreover, at least in the case B is the ring of integers of a global field, $B \times_{\overline{\mathscr{M}}_{1,1}} \mathscr{E}$ is the identity component of the Néron model of its generic fiber.

Now, let us restrict to elliptic curves over global fields with multiplicative reduction. These correspond to maps $B \to \overline{\mathscr{M}}_{1,1}$ for B the spectrum ring of integers of a global field K.

Remark 3.1. If the reduction were additive, we would have to take B to be a stacky curve in order to extend the map. It is possible to analyze how such extensions relate to the Selmer group, but we have not written out the details of this, as morally, such points will be extremely sparse.

We would like to compare $H^1(B, B \times_m \mathscr{E}[n])$ to the n-Selmer group of E, defined to be the generic fiber $E := \operatorname{Spec} K \times_{\overline{\mathscr{M}}_{1,1}} \mathscr{E}$. We have the following results relating these two groups.

Lemma 3.2 ([Ces16, Proposition 5.4(c)])**.** *If E has reduced discriminant, or more generally the component group of its Néron model has order prime to n, in the function field case, or order prime to $2n$ in the number field case, then $\# \operatorname{Sel}_n(E) = \# H^1(B, B \times_m \mathscr{E}[n])$.*

One can also obtain a bound on the difference in the sizes of the above groups even when the reduction type of E is especially bad.

Lemma 3.3 ([Lan21, Proposition 3.26]). *In the case K is a global function field,* $\# \operatorname{Sel}_n(E) \leq \# H^0(B, \mathscr{E}[n]) \cdot H^1(B, \mathscr{E}^0[n])$ *where* $\mathscr{E}^0$ *denote the identity component of the Néron model of E. In particular,* $\# \operatorname{Sel}_n(E) \leq n^2 \cdot H^1(B, \mathscr{E}^0[n])$

For some more detailed analysis of the relation between the Selmer group and $H^1(B, \mathscr{E}^0[n])$, see [Lan21, §3].

4. The prediction for $\overline{\mathscr{M}}_{1,1}$

In order to analyze $\overline{\mathscr{M}}_{1,1}$ according to the description of [DY22] we need to carefully describe its stack of twisted 0-jets, $\mathcal{I}_0\overline{\mathscr{M}}_{1,1} := \coprod_\ell \operatorname{Hom}(B\mu_\ell, \overline{\mathscr{M}}_{1,1})$.

Definition 4.1. We observe $\mathcal{I}_0\overline{\mathscr{M}}_{1,1}$ has 7 nontrivial components which we label $Y_{1/6}, Y_{2/6}, Y_{4/6}, Y_{5/6}, Y_{1/4}, Y_{3/4}, Y_{1/2}$.

To be carefully specify the meanings of these components, let us describe these sectors explicitly in terms of automorphisms of elliptic curves. To start, we must choose a compatible system of roots of unity. Take $\zeta_6, \zeta_3, \zeta_4$ a compatible system of roots of unity, meaning that $\zeta_6^2 = \zeta_3$. The sector $Y_{1/2}$ corresponds to the map sending $\zeta_2 = -1$ to the automorphism of $y^2 = f$ sending $y \mapsto -y$. The sector $Y_{i/6}$ corresponds to the map sending ζ_6^i to the automorphism of $y^2 = x^3 + 1$ sending $y \mapsto (-1)^i y, x \mapsto (\zeta_3)^i x$. The sector $Y_{i/4}$ corresponds to the map sending ζ_4 to the automorphism of $y^2 = x^3 + x$ sending $y \mapsto (\zeta_4)^i y, x \mapsto (-1)^i x$.

Recall we work away from characteristics 2 and 3 throughout.

Lemma 4.2. *Upon identifying* $\overline{\mathscr{M}}_{1,1} \simeq \mathscr{P}(4,6)$*, the Hodge bundle corresponds to* $\mathscr{O}_{\mathscr{P}(4,6)}(1)$ *and the tangent bundle corresponds to* $\mathscr{O}_{\mathscr{P}(4,6)}(10)$.

Proof. Note that the Hodge bundle is by definition given via the invariant differential, so it is a line bundle whose fiber over the elliptic curve $y^2 = x^3 + axz^2 + bz^3$ is dx/y. The action of the weighted projective space acts on a with weight 4 and b with weight 6. We wish to find how this acts on the invariant differential dx/y. Indeed, suppose $u \in \mathbb{G}_m$, then u sends the elliptic curve above to $y^2 = x^3 + au^4xz^2 + bu^6z^3$. Now, we can also realize this action by sending $x \mapsto u^{-2}x, y \mapsto u^{-3}y$, and then multiplying the whole equation by u^6. This implies that u acts on the invariant differential by sending $dx/y \mapsto d(u^{-2}x)/(u^{-3}y) = u \cdot dx/y$, and hence it acts with weight 1. This identifies the Hodge bundle with $\mathscr{O}(1)$.

To compute the action on the tangent bundle, we proceed via the Euler exact sequence. Since the action on the coordinates has weights 4 and 6, we have an exact sequences

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}(4) \oplus \mathscr{O}(6) \longrightarrow T_{\mathscr{P}(4,6)} \longrightarrow 0 \tag{4.1}$$

and taking determinants gives $T_{\mathscr{P}(4,6)} \simeq \mathscr{O}(4+6) = \mathscr{O}(10)$. □

Our next goal is to compute ages for the above bundles.

Lemma 4.3. *Letting $\{x\}$ denote the fractional part of x, the ages with respect to the Hodge bundle ω are: $a(Y_{i/6}, \omega) = \{-i/6\}, a(Y_{i/4}, \omega) = i/4$. The ages with respect to the tangent bundle are $a(Y_{i/6}) = \{2i/6\}, a(Y_{i/4}) = \{i/2\}$.*

Proof. The statement for the tangent bundle follows from that for the Hodge bundle because the tangent bundle is the 10th tensor power of the Hodge bundle, and the ages of the tangent bundle are the fractional part of 10 times the ages of the Hodge bundle.

To compute ages of the Hodge bundle, we study the action on the invariant differential at elliptic curves with extra automorphisms. For the elliptic curve $y^2 = x^3 + 1$, the automorphism $y \mapsto (-1)^i y, x \mapsto (\zeta_3)^i x$ sends dx/y to $(-\zeta_3)^i dx/y$, which is ζ_6^{-i}, for ζ_6 the primitive sixth root of unity squaring to ζ_3.

Similarly, for the elliptic curve $y^2 = x^3 + x$, the automorphism $y \mapsto \zeta_4^i y, x \mapsto (-1)^i x$ sends $dx/y \mapsto \zeta_4^i dx/y$, and so has age $i/4$. □

For the next proposition, we define the raised Hodge bundle to be the raised line bundle (ω, c) where c is the raising function $c(Y_\alpha) := a(Y_\alpha, \omega)$, as computed in Lemma 4.3.

Proposition 4.4. *The Darda-Yasuda heuristics* [DY22, Conjecture 9.15] *predict there should exist a constant C so that there are $CX^{10} + o(X^{10})$ points of height at most X on $\overline{\mathscr{M}}_{1,1}$ with respect to the raised Hodge bundle via* [DY22, Example 4.10].

Proof. We begin by showing that the class of the non-raised Hodge bundle lies in the effective cone, and any ray in the effective cone has non-negative coefficient for the non-raised Hodge bundle. By non-raised Hodge bundle, we mean the Hodge bundle with trivial raising function. In this case, note that the effective cone of $\overline{\mathscr{M}}_{1,1}$ contains the class of $\mathscr{O}(1)$ with trivial raising function, because the the 12th tensor power of the Hodge bundle is the divisor corresponding to the pullback of a point from the coarse space, which is indeed effective. Moreover, any ray in the effective cone must have non-negative coefficient for $\mathscr{O}(1)$ because we can take the covering family given trivializing the 5 torsion of the elliptic curve, which has trivial stacky structure.

We now wish to compute the a and b constants from Darda and Yasuda. Since the canonical bundle is the dual of the tangent bundle, we obtain that it has class $\mathscr{O}(-10)$ in the rational Picard group, and hence using Lemma 4.3,

$$K_{\overline{\mathscr{M}}_{1,1},orb} = [-10\omega] - 1/3Y_{1/6} - 2/3Y_{2/6} - Y_{1/2} - 1/3Y_{4/6} - 2/3Y_{5/6} - 3/4Y_{1/4} - 1/4Y_{3/4}.$$

Next, we compute the class of the raised line bundle corresponding to the Hodge bundle. Again, using Lemma 4.3, this has class

$$Z := [\omega] + 5/6Y_{1/6} + 2/3Y_{2/6} + 1/2Y_{1/2} + 1/3Y_{4/6} + 1/6Y_{5/6} + 1/4Y_{1/4} + 3/4Y_{3/4}.$$

We wish to compute the minimum a so that $aZ + K_{\overline{\mathscr{M}}_{1,1,orb}}$ is in the effective cone. In particular, we need the class $[-10\omega + a\omega]$ to be effective, we have $a \geq 10$. We also find that all coefficients of the stacky sectors are then positive, and so indeed $a = 10$.

Since the coefficients of the stacky sectors are all strictly positive when $a = 10$, the location of $K_{\overline{\mathscr{M}}_{1,1,orb}} + aZ$ in the effective cone lies on a maximal face, implying $b = 10$. Therefore, the asymptotic is predicted to be, up to a constant, $X^{10} \log X^0 = X^{10}$, as claimed. □

We next explain why the count in Proposition 4.4 gives the correct prediction for the number of points on $\overline{\mathscr{M}}_{1,1}$, as can be separately computed directly.

Remark 4.5. Consider elliptic curves in Weierstrass form $y^2 = x^3 + axz^2 + bz^3$. In the function field case, we count with respect to the Faltings height. Concretely, if a is a homogeneous polynomial of degree $4d$ and b has degree $6d$, the Faltings height is d. This is also the height with respect to the Hodge bundle in the sense of [ESZB21] or the corresponding raised line bundle in the formalism of [DY22]. Since there are q^{4d+1} choices for a and q^{6d+1} choices for b, it is not too difficult to see there are, up to a constant power of q independent of d, q^{10d} elliptic curves of height at most d.

Here we ignore powers of q in this asymptotic notion of counting by height, i.e., the actual number may be closer to q^{10d+1}.

5. The prediction for $\mathscr{S}^n$

Next, we would like to understand the sectors associated to the Selmer stack. Note that over the Selmer stack, there is again a universal elliptic curve $\pi : \mathscr{E}_n \to \mathscr{S}^n$ and so the Selmer stack $\mathscr{S}^n$ has its own Hodge bundle $\omega_n := \pi_* \omega_{\mathscr{E}_n/\mathscr{S}^n}$. Note that $\mathscr{E}[n]$ also acts trivially on the Hodge bundle on $\overline{\mathscr{M}}_{1,1}$ because these automorphisms are given by translation, but the differential dx/y spanning the Hodge bundle is translation invariant. When one pulls this back along the cover $\overline{\mathscr{M}}_{1,1} \to \mathscr{S}^n$, one obtains the Hodge bundle ω on $\overline{\mathscr{M}}_{1,1}$ via flat base change. Similarly, the tangent bundle on $\mathscr{S}^n$ is $T_{\mathscr{S}^n} \simeq \omega_n^{\otimes 12}$ because the isomorphism $T_{\overline{\mathscr{M}}_{1,1}} \simeq \omega^{\otimes 12}$ is equivariant for the trivial action of $\mathscr{E}[n]$ on $\overline{\mathscr{M}}_{1,1}$.

Definition 5.1. We may label the twisted sectors $\pi_0^*(\mathcal{I}_0\mathscr{S}^n)$ as X_α, Z_α^β where X_α corresponds to sectors to which Y_α map under $\overline{\mathscr{M}}_{1,1} \to \mathscr{S}^n$ and Z_α^β are the remaining sectors mapping Y_α.

Lemma 5.2. *For any pair (α, β) we have corresponding to a twisted sector of $\mathscr{S}^n$, $a(Z_\alpha^\beta, \omega_n) = a(X_\alpha, \omega_n) = a(Y_\alpha, \omega)$.*

Proof. This holds because the action of $\mathscr{E}[n]$ on $\overline{\mathscr{M}}_{1,1}$, the Hodge bundle, and the tangent bundle are all trivial. □

Remark 5.3. We will not need this remark in what follows. In fact, one can show that the sectors Z_1^β are in bijection with divisors of n, as are the sectors $Z_{1/2}^\beta$, using the fact that a generic elliptic curve has Galois representation on its n-torsion with image containing $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. For the other values of α, if 2 or 3 does not divide n, this will still be the case, but when 2 or 3 divides n we may have additional sectors, as the 2 or 3 torsion of elliptic curves with extra automorphisms may have additional homomorphisms from μ_n.

Definition 5.4. Define the *raised Hodge bundle* on $\mathscr{S}^n$ to be the raised line (ω_n, c_n) where c_n is the raising function which sends each Z_α^β to ∞ and sends each X_α to $a(Y_\alpha, \omega)$, i.e., c_n takes the same value as raising function associated to the Hodge bundle on $\overline{\mathscr{M}}_{1,1}$ takes on Y_α.

Proposition 5.5. *A slight generalization of the Darda-Yasuda heuristics* [DY22, Conjecture 9.15] *(allowing for non-separated stacks and raised line bundles where the raising function can take value ∞) predict there should exist a constant $E_n X^{10} + o_n(X^{10})$ points of height at most X on $\mathscr{S}^n$ with respect to the raised Hodge bundle.*

Proof. The proof is basically the same as Proposition 4.4, and we now explain the details.

To start, we need to get a grasp on the effective cone of $\mathscr{S}^n$, but a mild grasp will do. As before, there is a cover of $\mathscr{S}^n$ which is a scheme, taking first the cover $\overline{\mathscr{M}}_{1,1}$ and then passing to a further cover of $\overline{\mathscr{M}}_{1,1}$ adding level structure corresponding to the 5-torsion. The only class in the orbifold Néron Severi group of $\mathscr{S}^n$ pulling back to a nontrivial class on this cover is that of the Hodge bundle, which pulls back to a positive integer, and hence is effective. Moreover this shows that any class in the effective cone must have positive coefficient of the Hodge line bundle.

Using Lemma 5.2, we can write

$$K_{\mathscr{S}^n,orb} = [-10\omega_n] - 1(\sum_\beta Z_1^\beta) + (-1/3(X_{1/6}\sum_\beta Z_{1/6}^\beta) - 2/3(X_{2/6} + \sum_\beta Z_{2/6}^\beta)$$
$$- (X_{1/2} + \sum_\beta Z_{1/2}^\beta) - 1/3(X_{4/6} + \sum_\beta Z_{4/6}^\beta) - 2/3(X_{5/6} + \sum_\beta Z_{5/6}^\beta)$$
$$- 3/4(X_{1/4} + \sum_\beta Z_{1/4}^\beta) - 1/4(X_{3/4} + \sum_\beta Z_{3/4}^\beta).$$

Next, we compute the class of the raised line bundle corresponding to the Hodge bundle. Again, using Lemma 5.2, this has class

$$[\omega_n] + \frac{5}{6}X_{1/6} + \frac{2}{3}X_{2/6} + \frac{1}{2}X_{1/2} + \frac{1}{3}X_{4/6} + \frac{1}{6}X_{5/6} + \frac{1}{4}X_{1/4} + \frac{3}{4}X_{3/4} + \infty \cdot \sum_{\alpha,\beta} Z_\alpha^\beta.$$

We wish to compute the minimum a so that $aZ + K_{\overline{\mathscr{M}}_{1,1},orb}$ is in the effective cone. From our observation above, we need the class $[-10\omega_n + a\omega_n]$ to be effective. Therefore, we have $a \geq 10$. We also find that all coefficients of the stacky sectors are then positive, and so indeed $a = 10$.

Since the coefficients of the stacky sectors are all strictly positive when $a = 10$, the location of $K_{\overline{\mathscr{M}}_{1,1},orb} + aZ$ in the effective cone lies on a maximal face, implying $b = 0$. Therefore, the asymptotic is again predicted to be, up to a constant, $X^{10}\log X^0 = X^{10}$, as claimed. □

Remark 5.6. So far, we have computed what the Darda-Yasuda heuristics predict for the raised Hodge bundle, as we defined it above. To actually connect it to the Poonen-Rains heuristics, we should justify why counting rational points with respect to the raised Hodge bundle is closely related to counting Selmer elements, i.e., pairs of an elliptic curve E and an element of $\mathrm{Sel}_n(E)$.

To justify this, suppose we start with an elliptic curve E over K which has Néron model $\mathscr{E}$ over $\mathscr{O}_K$ and identity component $\mathscr{E}^0$. Using the description of §3, in particular Lemma 3.2 and Lemma 3.3, and the, at least heuristic sparsity of Selmer elements whose Jacobians have non-squarefree discriminant, it is enough to show that rational points $x : \operatorname{Spec} K \to \mathscr{S}^n$ which have height $d < \infty$ and whose Jacobians have squarefree discriminant additive reduction every where are in bijection with elements of $H^1(\mathscr{O}_K, \mathscr{E}^0[n])$, and the corresponding elements have the same height as their Jacobians. By our definition of the function c for the raised Hodge bundle, any place which maps to the nontrivial sector of $\mathscr{E}^0[n]$ is given infinite height. Therefore, the only ones with finite height must map to the trivial sector of $\mathscr{E}^0[n]$, meaning that it corresponds to an actual $\mathscr{E}^0[n]$ torsor, and hence an element of

$H^1(\mathscr{O}_K, \mathscr{E}^0[n])$. Moreover, each element has the same height as its Jacobian because the raising function we assigned to the Hodge bundle on $\mathscr{S}^n$ agrees with the raising function of the corresponding sector on $\overline{\mathscr{M}}_{1,1}$, and the Hodge bundle ω_n on $\mathscr{S}^n$ pulls back to ω on $\overline{\mathscr{M}}_{1,1}$.

Remark 5.7 (Quadratic twists and Cohen-Lenstra)**.** One can also make sense of the Poonen-Rains heuristics for Selmer groups in quadratic twist families and the Cohen-Lenstra heuristics by restricting the raised Hodge bundle (ω_n, c_n) on $\mathscr{S}^n$ to fibers over $\overline{\mathscr{M}}_{1,1}$. The Cohen-Lenstra heuristics correspond to restricting it to the fiber over the residual gerbe at the singular genus 1 curve and the quadratic twist family for E corresponds to restricting it to the fiber over the residual gerbe of E, viewed as a point of $\overline{\mathscr{M}}_{1,1}$.

REFERENCES

[Ces16] Kestutis Cesnavicius. Selmer groups as flat cohomology groups. *J. Ramanujan Math. Soc.*, 31(1):31–61, 2016.

[DY22] Ratko Darda and Takehiko Yasuda. The batyrev-manin conjecture for dm stacks. *arXiv preprint arXiv:2207.03645v2*, 2022.

[ESZB21] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *arXiv preprint arXiv:2106.11340v1*, 2021.

[Hor16] Ruthi Hortsch. Counting elliptic curves of bounded Faltings height. *Acta Arith.*, 173(3):239–253, 2016.

[Lan21] Aaron Landesman. The geometric average size of Selmer groups over function fields. *Algebra Number Theory*, 15(3):673–709, 2021.

[PR12] Bjorn Poonen and Eric Rains. Random maximal isotropic subspaces and Selmer groups. *J. Amer. Math. Soc.*, 25(1):245–269, 2012.