PROPERNESS OF THE TORELLI MAP ON COMPACT TYPE CURVES

AARON LANDESMAN

1. INTRODUCTION

Let $\mathscr{M}_g$ denote the moduli stack of curves of genus g over $\operatorname{Spec}\mathbb{Z}$ and let $\mathscr{A}_g$ denote the moduli stack of principally polarized abelian varieties of dimension g over $\operatorname{Spec}\mathbb{Z}$. Let $\overline{\mathscr{M}}_g$ denote the Deligne-Mumford compactification of $\mathscr{M}_g$ and let $\mathscr{M}_g^c \subset \overline{\mathscr{M}}_g$ denote the open substack of stable compact type curves of genus g. Recall that $\mathscr{M}_g^c$, loosely speaking, parameterizes stable curves whose dual graph of irreducible components is a tree. The geometric points of $\overline{\mathscr{M}}_g$ lie in $\mathscr{M}_g^c$ precisely when the Jacobian of the corresponding curve is an abelian variety, as follows from [BLR90, §9.2, Example 8].

The goal of this expository note is to verify the well known fact that the Torelli map $\tau_g : \mathscr{M}_g \to \mathscr{A}_g$ sending a curve to its principally polarized Jacobian exists and extends to a proper map $\overline{\tau}_g : \mathscr{M}_g^c \to \mathscr{A}_g$ of stacks over $\operatorname{Spec}\mathbb{Z}$ in Theorem 4.1. We call this the "compactified Torelli map." Along the way, in Proposition 3.4, we also obtain an explicit description of the Theta divisor on compact type curves, which induces the compactified Torelli map.

The structure of this note is as follows: In § 2 we recall the construction of the usual Torelli map $\tau_g : \mathscr{M}_g \to \mathscr{A}_g$ on smooth curves. In §3 we show this extends uniquely to a Torelli map $\overline{\tau}_g : \mathscr{M}_g^c \to \mathscr{A}_g$ on compact type curves. In §4 we show $\overline{\tau}_g$ is proper.

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2. THE TORELLI MAP ON SMOOTH CURVES

We next wish to construct the Torelli map for smooth curves. The construction is also given in [Mum65, Proposition 6.9], using [Cor86, VII, Lemma 6.9] to deal with the case that the base is an algebraically closed field. We give a different, but similar construction in this section. Recall that the Torelli map is a map $\mathscr{M}_g \to \mathscr{A}_g$. So, given a family of smooth curves $C \to S$, by which we will always mean a smooth proper morphism of relative dimension 1 with geometrically connected fibers, we will have to construct a family of principally polarized abelian varieties. The family of abelian varieties will simply be $\operatorname{Pic}^0_{C/S} \to S$, but we will have to work a bit to construct the principal polarization, induced by the so-called Theta-divisor. In §2.1 we construct the Theta divisor and the induced map $\phi_\Theta : \widehat{\operatorname{Pic}^0_{C/S}} \to \operatorname{Pic}^0_{C/S}$. In §2.2 we verify the Abel-Jacobi map $C \to \operatorname{Pic}^1_{C/S}$ is an immersion.

In §2.3, we show ϕ_Θ induces a principal polarization over an algebraically closed field. Finally, in §2.4, we show the existence of the Torelli map by showing ϕ_Θ induces a principal polarization in general.

2.1. **Constructing the Theta divisor.** To the end of constructing the Theta divisor, begin with a relative curve $C \to S$. I.e., $C \to S$ is a smooth proper morphism of relative dimension 1 with geometrically connected fibers. Let $C^r := \underbrace{C \times_S C \times_S \cdots \times_S C}_{r}$ denote the r-fold fiber product of C with itself over S. There is a natural map $C^{g-1} \to \operatorname{Pic}^{g-1}_{C/S}$ sending $(p_1, \ldots, p_{g-1}) \mapsto \mathscr{O}(\sum_{i=1}^{g-1} p_i)$. Now, suppose $p : S \to C$ is a given choice of section to the projection $C \to S$. We can define a map functorially by

$$\begin{aligned} \alpha_p : C^{g-1} &\to \operatorname{Pic}^0_{C/S} \\ (p_1, \ldots, p_{g-1}) &\mapsto \mathscr{O}_C\left(\left(\sum_{i=1}^{g-1} p_i\right) - (g-1) \cdot p\right). \end{aligned} \tag{2.1}$$

Let Θ_p denote the schematic image of this map.

We are not sure whether the formation of the scheme Θ_p commutes with arbitrary base change on S, but we will only use that its formation commutes with flat base change and that its underlying topological space commutes with arbitrary base change on S.

Lemma 2.1. *For any $p : S \to C$ as above, the subscheme $\Theta_p \subset \operatorname{Pic}^0_{C/S}$ is an effective relative Cartier divisor.*

Proof. Recall that by the equivalence of various notions of relative Cartier divisor [BLR90, §8.2, Lemma 6], we just need to check that Θ_p is a Cartier divisor and each fiber is a Cartier divisor. Further, because $C \to S$ is pulled back via the universal curve over $\mathscr{M}_g$, regularity of $\mathscr{M}_g$ allows us to assume that the base S is regular. We can further assume S is finite type over $\operatorname{Spec} \mathbb{Z}$ and integral. On a regular finite type scheme, the notions of Cartier divisor and Weil divisor coincide, and so it suffices to verify that Θ_p is a divisor for any integral regular base S.

Note that Θ_p is then integral of dimension at most $\dim S + g - 1$, being the scheme theoretic image of an integral scheme of dimension $\dim S + g - 1$. Because Θ_p is integral, in order to show it is a divisor, it suffices to show it has dimension $\dim S + g - 1$. Note here, we are using that finite type schemes over $\operatorname{Spec} \mathbb{Z}$ are catenary, so that $\dim \Theta_p = \dim \operatorname{Pic}^0_{C/S} - \operatorname{codim}_{\operatorname{Pic}^0_{C/S}} \Theta_p$. In particular, having dimension $\dim S + g - 1$ will imply Θ_p has codimension 1 in $\operatorname{Pic}^0_{C/S}$.

We now verify that Θ_p has dimension $\dim S + g - 1$ by showing $\alpha_p : C^{g-1} \to \operatorname{Pic}^0_{C/S}$ has $g-1$ dimensional image in each fiber. So, we assume for the remainder of the proof that $S = \operatorname{Spec} k$ for k a field. This follows from the fact that the natural map $C^g \to \operatorname{Pic}^g_{C/S}$ is surjective, as follows from Riemann-Roch and Serre duality (i.e., every degree g divisor on a genus g curve is effective). Therefore, in the case S is the spectrum of a field, the image of C^{g-1} under α_p has dimension at least $g-1$ (since the image of $C^g \to \operatorname{Pic}^g_{C/S}$ has dimension g and $C^{g-1} \subset C^g$ has codimension 1). Since C^{g-1} has dimension $g-1$, the image of α_p also has dimension at most $g-1$. Hence, the image has dimension exactly $g-1$, as we wished to show □

Recall that given an abelian scheme $A \to S$ and an invertible sheaf $\mathscr{L}$ on A, we have an associated map $\phi_{\mathscr{L}}$ defined functorially by

$$\begin{aligned} \phi_{\mathscr{L}} \colon A &\to \widehat{A} \\ x &\mapsto t_x^* \mathscr{L} \otimes \mathscr{L}^\vee. \end{aligned}$$

Since Θ_p is a relative Cartier divisor by Lemma 2.1, $\mathscr{O}(\Theta_p)$ defines a line bundle on A and we can apply the ϕ construction to obtain a map $\phi_{\mathscr{O}(\Theta_p)} : \widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$. Note that usually the ϕ construction goes from an abelian variety to its dual, but there is a canonical isomorphism $\mathrm{Pic}^0_{C/S} \simeq \widehat{\widehat{\mathrm{Pic}^0_{C/S}}}$. Thus, here and in the future, we identify the two abelian schemes and write $\mathrm{Pic}^0_{C/S}$ as the target of $\phi_{\mathscr{O}(\Theta_p)}$. We next wish to verify that this does not depend on the choice of point p, which will allow us to construct the above map even when no section p exists.

Lemma 2.2. *Suppose $p, q : S \to C$ are two sections to the structure map $C \to S$. Then, the induced maps $\phi_{\mathscr{O}(\Theta_p)}, \phi_{\mathscr{O}(\Theta_q)} : \widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$ agree.*

Proof. Given a fixed section $p : S \to C$, we obtain an element of $\mathrm{Hom}(\widehat{\mathrm{Pic}^0_{C/S}}, \mathrm{Pic}^0_{C/S})(S)$. Therefore, "stringing these together" we obtain a map defined functorially by

$$\begin{aligned} C &\to \mathrm{Hom}_S(\widehat{\mathrm{Pic}^0_{C/S}}, \mathrm{Pic}^0_{C/S}) \\ t &\mapsto \phi_{\mathscr{O}(\Theta_t)}. \end{aligned}$$

Recall that $\mathrm{Hom}_S(\widehat{\mathrm{Pic}^0_{C/S}}, \mathrm{Pic}^0_{C/S})$ is unramified. This is because it is formally unramified, as follows from the rigidity lemma [Mum65, Proposition 6.1].

To verify independence of $\phi_{\mathscr{O}(\Theta_p)}$ on the choice of point p, we wish to show that this morphism agrees with the morphism which is the constant map sending C to $\phi_{\mathscr{O}(\Theta_p)}$. This equality will follow from the rigidity lemma [Mum65, Proposition 6.1] so long as we can verify equality upon restriction to one point $s \in S$.

So, we may assume $S = \mathrm{Spec}\, k$ for k an algebraically closed field. In this case, we find $\mathrm{Hom}_S(\widehat{\mathrm{Pic}^0_{C/S}}, \mathrm{Pic}^0_{C/S})$ is unramified, so is therefore a disjoint union of copies of $\mathrm{Spec}\, k$. Any map from the connected curve C to $\mathrm{Hom}_S(\widehat{\mathrm{Pic}^0_{C/S}}, \mathrm{Pic}^0_{C/S})$ is therefore constant, as we wished to show. □

Corollary 2.3. *For any smooth relative curve $C \to S$ (not necessarily possessing a section) there is a unique map $\phi_\Theta : \widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$ such that for any flat $T \to S$ such that $C_T \to T$ possesses a section p, the base change of ϕ_Θ to T agrees with $\phi_{\mathscr{O}(\Theta_p)}$.*

Proof. This is an application of faithfully flat descent. We will implicitly use below that the formation of Θ_p commutes with flat base change. Let $T \to S$ be an fppf surjection over which there exists a section of the base change $C_T \to T$. For example, we can take $T = C$ and the section to be the diagonal map. Then, to construct the map ϕ_Θ, via fppf descent, it suffices to construct a map $\widehat{\mathrm{Pic}^0_{C/S_T}} \to \mathrm{Pic}^0_{C/S}$ such that the two pullbacks along the projections $\pi_1, \pi_2 : T \times_S T \to T$ agree. In other words, we wish to find a map ϕ making the upper square of

$$\begin{tikzcd}
\widehat{\mathrm{Pic}^0_{C/S}}_{T\times_S T} \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & (\mathrm{Pic}^0_{C/S})_{T\times_S T} \arrow[d, shift left] \arrow[d, shift right] \\
\widehat{\mathrm{Pic}^0_{C/S}}_T \arrow[r, "\phi_{\mathscr{O}(\Theta_p)}"] \arrow[d] & (\mathrm{Pic}^0_{C/S})_T \arrow[d] \\
\widehat{\mathrm{Pic}^0_{C/S}} & \mathrm{Pic}^0_{C/S}
\end{tikzcd} \tag{2.2}$$

commute, for either the left or right choice of vertical maps induced by π_1 and π_2. This will yield the desired induced map $\widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$ via fppf descent.

Indeed, the composition given by first taking the vertical map induced by π_i and then composing with $\phi_{\mathscr{O}(\Theta_p)}$ corresponds to the section $\pi_i^*(p) : T \times_S T \to C_{T\times_S T}$. By Lemma 2.2, these two sections for $i = 1, 2$ yield the same map $\phi := \phi_{\mathscr{O}(\Theta_{\pi_1^*(p)})} = \phi_{\mathscr{O}(\Theta_{\pi_1^*(p)})} : \left(\widehat{\mathrm{Pic}^0_{C/S}}\right)_{T\times_S T} \to (\mathrm{Pic}^0_{C/S})_{T\times_S T}$. This implies that the above diagram indeed commutes, so we obtain our desired induced map $\phi_\Theta : \widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$.

The final statement follows from Lemma 2.2. □

2.2. **Showing the Abel-Jacobi map is an immersion.** Let $f : C \to S$ be a smooth proper relative curve of genus at least 1 with geometrically connected fibers. Recall that the Abel-Jacobi map defined functorially by

$$\begin{aligned} i : C &\to \mathrm{Pic}^1_{C/S} \\ x &\mapsto \mathscr{O}(x) \end{aligned}$$

In this subsection, we check this map is an immersion. This will be used in later checking the Theta divisor induces a principal polarization.

Theorem 2.4. *For $f : C \to S$ a smooth proper relative curve of genus at least 1 with geometrically connected fibers, the Abel-Jacobi map is an immersion.*

Proof assuming Lemma 2.5. To see $i : C \to \mathrm{Pic}^1_{C/S}$ is an immersion, it is equivalent to verify it is a proper monomorphism. Properness is automatic, so we only need verify it is a monomorphism. To check this, we can verify it on each geometric fiber over S, and hence assume $S = \operatorname{Spec} k$ for k an algebraically closed field. To check a map of proper varieties over an algebraically field is a closed immersion, it suffices to check it is injective on closed points and surjective on cotangent spaces at each closed point. First, we check it is injective on points. We wish to check that for any two distinct k-points x and y, we have $\mathscr{O}(x) \not\simeq \mathscr{O}(y)$. Indeed, $\mathscr{O}(x) \simeq \mathscr{O}(y)$ implies $h^0(C, x) > 1$, which means C has a degree 1 map to $\mathbb{P}^1$, and hence necessarily has genus 0. So, to conclude the proof, we just need to check surjectivity of the Abel-Jacobi map on tangent vectors. In other words, we want to show $i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k} \to \Omega_{C/k}$ is surjective. Since we are over an algebraically closed field, $\mathrm{Pic}^1_{C/k} \simeq \mathrm{Pic}^0_{C/k}$ and $\Omega^1_{\mathrm{Pic}^0_{C/k}/k} \simeq H^0(\mathrm{Pic}^0_{C/k}, \Omega^1_{\mathrm{Pic}^0_{C/k}/k}) \otimes \mathscr{O}_{\mathrm{Pic}^0_{C/k}}$. This implies $i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k} \simeq H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \otimes \mathscr{O}_C$. Now, using the pushforward-pullback adjunction for f applied to the natural map $i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k} \to \Omega_{C/k}$, we obtain a commutative square

$$\begin{tikzcd}
H^0(C, i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \otimes \mathscr{O}_C \arrow[r] \arrow[d] & H^0(C, \Omega_{C/k}) \otimes \mathscr{O}_C \arrow[d] \\
i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k} \arrow[r] & \Omega_{C/k}.
\end{tikzcd} \tag{2.3}$$

Using the above description of $i^*\Omega^1_{\mathrm{Pic}^1_{C/k}/k}$, we can identify the square with

$$\begin{tikzcd}
H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \otimes \mathscr{O}_C \arrow[r] \arrow[d, "\mathrm{id}"] & H^0(C, \Omega_{C/k}) \otimes \mathscr{O}_C \arrow[d] \\
H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \otimes \mathscr{O}_C \arrow[r] & \Omega_{C/k}.
\end{tikzcd} \tag{2.4}$$

We wish to show the bottom map is surjective. Note that the right vertical map is surjective because $\Omega_{C/k}$ is basepoint free, as follows from Serre duality because the genus of C is at least 1. So, to conclude the proof, it suffices to show the natural map $H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \to H^0(C, \Omega_{C/k})$ induced by the adjunction is an isomorphism. We check this in Lemma 2.5. □

Lemma 2.5. *Let $f : C \to S$ be a smooth proper relative curve of genus at least 1 with geometrically connected fibers and let $i : C \to \mathrm{Pic}^1_{C/S}$ denote the Abel-Jacobi map. Then the natural map $H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k}) \to H^0(C, \Omega_{C/k})$ induced by i is an isomorphism.*

Proof. Note first that both vector spaces are g-dimensional. Indeed, $H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1_{C/k}/k})$ is g-dimensional because it is identified with the cotangent space to $\mathrm{Pic}^1_{C/k}$ at a point, while $H^0(C, \Omega_{C/k})$ is g-dimensional by Serre duality. Therefore, we only need verify the natural map is injective.

Choose a point $p \in C$. Via translation, note that we have a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "x \mapsto \mathscr{O}(x)"] \arrow[d, "x \mapsto (x, p, \ldots, p)"] & \mathrm{Pic}^1_{C/k} \arrow[d, "\mathscr{L} \mapsto \mathscr{L} \otimes \mathscr{O}((g-1)p)"] \\
C^g \arrow[r, "(x_1, \ldots, x_g) \mapsto \mathscr{O}(x_1 + \cdots + x_g)"'] & \mathrm{Pic}^g_{C/k}
\end{tikzcd} \tag{2.5}$$

Taking induced maps on global sections cotangent sheaves, we wish to show that the upper map is an isomorphism. The right map induces an isomorphism of schemes, and hence certainly an isomorphism on global sections of cotangent sheaves. Therefore, it suffices to show the composition of the left and lower maps induces an injection on global sections of cotangent sheaves. We now analyze this composition explicitly.

Note in fact that the composition factors as $C \to C^g \to \mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$, where the first map sends $x \mapsto (x, p, \ldots, p)$, the second sends $(x_1, \ldots, x_g) \mapsto [x_1, \ldots, x_g]$ and the last sends $[x_1, \ldots, x_g] \mapsto \mathscr{O}(x_1 + \cdots + x_g)$. We want to verify injectivity of the induced composition

$$H^0(C, \Omega_{C/k}) \leftarrow H^0(C^g, \Omega_{C^g/k}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k}) \leftarrow H^0(\mathrm{Pic}^g_{C/k}, \Omega_{\mathrm{Pic}^g_{C/k}/k}).$$

The map $\mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ is a birational map of smooth projective varieties, and hence induces an isomorphism on global differentials. Therefore, we just want to check the composition

$$H^0(C, \Omega_{C/k}) \leftarrow H^0(C^g, \Omega_{C^g/k}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k})$$

is injective.

As a first step, we will show $H^0(C^g, \Omega_{C^g/k}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k})$ is injective. Because $h : C^g \to \mathrm{Sym}^g_{C/k}$ is generically separable, being a quotient by the generically freely acting étale (and even constant) group S_g, we see $h^*\Omega_{\mathrm{Sym}^g_{C/k}/k} \hookrightarrow \Omega^1_{C^g/k}$ is an injection of sheaves. This implies $h_*h^*\Omega_{\mathrm{Sym}^g_{C/k}/k} \hookrightarrow h_*\Omega^1_{C^g/k}$ is an injection, and therefore the composition $\Omega_{\mathrm{Sym}^g_{C/k}/k} \to h_*h^*\Omega_{\mathrm{Sym}^g_{C/k}/k} \hookrightarrow h_*\Omega^1_{C^g/k}$. is also an injection. Note here that the first map is an injection because it is locally given by $\mathscr{O}_{\mathrm{Sym}^g_{C/k}} \to h_*\mathscr{O}_{C^g}$, which is an injection of sheaves because the finite map $C^g \to \mathrm{Sym}^g_{C/k}$ does not factor through a closed subscheme of the target. Further pushing this forward to $\mathrm{Spec}\, k$ and taking global sections, we obtain that $H^0(C^g, \Omega_{C^g/k}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k})$ is an injection.

Because the image of $H^0(C^g, \Omega_{C^g/k}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k})$ is S_g invariant via the natural action of S_g induced on $H^0(C^g, \Omega_{C^g/k})$ coming from the action permuting the g copies of C, we obtain an injection $H^0(C^g, \Omega_{C^g/k})^{S_g} \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}/k})$. Here, the superscript S_g indicates to take the S_g invariants. Under the identification $H^0(C^g, \Omega_{C^g/k}) \simeq \oplus_{i=1}^g H^0(C, \Omega_{C/k})$, the S_g invariants are simply the terms with the same differential in each of the g factors. Therefore, to conclude, it suffices to check that these invariant differentials are mapped injectively to $H^0(C, \Omega^1_{C/k})$ via the map induced by $C \to C^g$ sending $x \mapsto (x, p, \ldots, p)$. Indeed, we claim on differentials, this map corresponds to the map

$$\begin{aligned} H^0(C^g, \Omega_{C^g/k}) \simeq \oplus_{i=1}^g H^0(C, \Omega_{C/k}) &\to H^0(C, \Omega^1_{C/k}) \\ (\omega_1, \ldots, \omega_g) &\mapsto \omega_1, \end{aligned}$$

and so it is indeed injective on the diagonal differentials.

It only remains to check the claim regarding the induced map on differentials. Under the above identification, we just have to check that the map is given by $(\omega_1, 0, \ldots, 0) \mapsto \omega_1$ and $(0, \ldots, 0, \omega_i, 0, \ldots, 0) \mapsto 0$ for $i > 1$. The first claim that $(\omega_1, 0, \ldots, 0) \mapsto \omega_1$ holds because the map $C \to C^g$ is a section to the projection map, and so the composition $C \to C^g \to C$

induces the identity on differentials. The second claim that $(0,\ldots,0,\omega_i,0,\ldots,0)\mapsto 0$ holds because the map on differentials is induced by the image of the composition $C\to C^g\to C$ where the first map is that given by $x\mapsto(x,p,\ldots,p)$ and the second is the natural projection onto the ith factor. This composition agrees with the constant map contracting C to the point p. Therefore, it induces the 0 map on global differentials because it factors through $\Omega_{p/k}$ which is trivial. ☐

2.3. **The case of an algebraically closed field.**

Proposition 2.6. *Let A be an abelian variety over a field k and suppose $\mathscr{L}=\mathscr{O}_A(D)$ for some effective Cartier divisor D. If $t_x^*\mathscr{L}\not\simeq\mathscr{L}$ for every $x\neq e$, $x\in A(\overline{k})$ and additionally $h^0(A,\mathscr{L})=1$, Then $\phi_{\mathscr{L}}$ is a principal polarization.*

Proof. Recall that $\phi_{\mathscr{L}}$ is symmetric by [CF17, Proposition 7.4.3].

By [Mum08, §16, p. 150, The Vanishing Theorem], if since we are assuming the kernel of $\phi_{\mathscr{L}}$ is finite and $H^0(A,\mathscr{L})\neq 0$, it follows that $H^i(A,\mathscr{L})=0$ for all $i>0$. Therefore, $\chi(\mathscr{L})=1$. By [Mum08, §16, p. 150, The Riemann-Roch Theorem], it follows that $1=\chi(\mathscr{L})^2=\deg\phi_{\mathscr{L}}$, and so $\phi_{\mathscr{L}}$ has degree 1, and is therefore an isomorphism.

To check that $\phi_{\mathscr{L}}$ is a principal polarization, it suffices to check $(1,\phi_{\mathscr{L}})^*\mathscr{P}$ is ample, for $\mathscr{P}$ the Poincaré bundle on $A\times\widehat{A}$. Since D is effective, it follows from [CF17, Proposition 3.4.8], and finiteness of $\ker\phi_{\mathscr{L}}$ that $\mathscr{L}$ is ample. Finally, $(1,\phi_{\mathscr{L}})^*\mathscr{P}$ is ample by [CF17, Remark 7.5.5]. ☐

In order to apply Proposition 2.6 to show the Theta divisor determines a principal polarization, the key geometric input is the following:

Lemma 2.7. *Let C be a curve over an algebraically closed field k, let p be a k point of C, and let $\Theta_p\subset\operatorname{Jac}(C)$ denote the associated Theta divisor. Then, $h^0(\operatorname{Jac}(C),\Theta_p)=1$.*

Proof. Let $X:=\operatorname{Pic}^{g-1}_{C/k}$. Because C has a k-point, $X\simeq\operatorname{Jac}(C)$. By abuse of notation, let Θ_p denote the divisor parameterizing points of $\operatorname{Pic}^{g-1}_{C/k}$ corresponding to effective divisors on C. Using Riemann Roch, one can see that a general divisor D of degree g has $h^0(C,D)=1$. For such D there is an immersion

$$\begin{aligned} i_D\colon C&\to X\\ x&\mapsto K_C-D+x, \end{aligned}$$

which is an immersion because it is a translate of the Abel-Jacobi map, shown to be an immersion in Theorem 2.4.

Lemma 2.8. *In the setup above, suppose $h^0(X,\Theta_p)=r+1$ and D is a degree d divisor divisor such that $h^0(C,D)=1$. Then, $i_D^{-1}(\Theta_p)=D$ as a divisor on C and the subspace of sections in $H^0(X,\Theta_p)$ vanishing on $i_D(C)$ has dimension r.*

Proof. Note first that such divisors D of degree g with $h^0(C,D)=1$ can be seen to exist using Serre duality and Riemann Roch. Since Θ_p parameterizes effective divisors, $i_D^{-1}(\Theta_p)$ parameterizes points $x\in C$ with $h^0(C,i_D(x))>0$. Since $h^0(C,D-x)=h^0(C,K_C-D+x)=h^0(C,i_D(x))>0$, using Serre duality, $i_D^{-1}(\Theta_p)$ parameterizes points $x\in C$ with $h^0(C,D-x)>0$. That is, it precisely parameterizes those x contained in the support of some divisor linearly equivalent to D. Since $h^0(C, D) = 1$, we find x lies in the support of D.

Because we have a map $i_D : C \to X$, we obtain an induced map $H^0(X, \Theta_p) \to H^0(C, \Theta_p|_C) = H^0(C, D)$. Since $H^0(C, D)$ is 1-dimensional, and the map is surjective, (as follows because $C \not\subset \Theta_p$, and there is a section of $H^0(C, \Theta_p)$ vanishing precisely on Θ_p) we conclude that the subspace of $H^0(X, \Theta_p)$ vanishing on $i_D(C)$ has codimension 1. $\square$

Say then $h^0(X, \Theta_p) = r + 1$ with $r > 0$. We want to reach a contradiction. For this, define Φ, π_1 and π_2 by
(2.6)

$$\begin{tikzcd} & \Phi := \left\{ (E, \Psi) : E \in \mathrm{Sym}^g_{C/k}, \Psi \in \mathbb{P}H^0(X, \Theta_p), i_E(C) \subset \Psi \right\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Pic}^g_{C/k} & & \mathbb{P}H^0(X, \Theta_p). \end{tikzcd}$$

Here, $\mathrm{Sym}^g_{C/k}$ denotes the gth symmetric power of C/k, i.e., the scheme quotient C^g/S^g, which exists as C is projective. The map π_1 is induced by the natural map $\mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ coming from sending a divisor $p_1 + \cdots + p_g \mapsto \mathscr{O}(p_1 + \ldots + p_g)$

Now, we will show that the generic fiber of π_2 has dimension $g - 1$, assuming $r > 0$. Using Lemma 2.8, the subspace of sections vanishing on a general E has dimension r, so its projectivization has dimension $r - 1$, meaning the fibers of π_1 generically have dimension $r - 1$. Therefore, assuming $r > 0$ so that the generic fibers is nonempty, $\dim \Phi = g + r - 1$. This implies that the fibers of π_2 are $g - 1$ dimensional.

To achieve a contradiction and deduce $r = 0$, we will now show that the generic fiber of π_2 has dimension at most $g - 2$. By upper semicontinuity of fiber dimension, it suffices to show the fiber over $[\Theta_p]$ is at most $g - 2$ dimensional. So, we want to show that the subscheme parameterizing E with $i_E(C) \subset \Theta_p$ is at most $g - 2$ dimensional. By Lemma 2.8, if $[E] \in \mathrm{Sym}^g_{C/k}$ satisfies $h^0(C, E) = 1$, we know $i_E^{-1}(\Theta_p) = E$ and so $i_E(C) \not\subset \Theta_p$. So, to conclude, we just need to show all $[E] \in \mathrm{Sym}^g_{C/k}$ with $h^0(C, E) > 1$ are contained in a subscheme of $\mathrm{Sym}^g_{C/k}$ of codimension 2.

To show the $[E] \in \mathrm{Sym}^g_{C/k}$ with $h^0(C, E) > 1$ have codimension at least 2, note that the locus of such E with $h^0(C, E) > 1$ are precisely the locus of points of $\mathrm{Sym}^g_{C/k}$ not isolated lying in positive dimensional fibers over $\mathrm{Pic}^g_{C/k}$. Said another way, the fiber of $h : \mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ over $\mathscr{O}(E)$ can be identified with $\mathbb{P}H^0(C, E)$. But now, recall that the map $h : \mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ is a birational map of smooth connected varieties of dimension g, [BLR90, §9.3, Lemma 5]. Therefore, there cannot be a codimension 1 subscheme of $\mathrm{Pic}^g_{C/k}$ on which h has positive dimensional fibers. Indeed, since the generic fiber of the map is 0 dimensional, having a codimension 1 point with a 1-dimensional fiber would imply $\mathrm{Sym}^g_{C/k}$ has at least 2 components of dimension g. This implies that the locus in $\mathrm{Pic}^g_{C/k}$ where the fibers of h are positive dimensional has codimension at least 2 in $\mathrm{Pic}^g_{C/k}$, completing the proof. This is a contradiction, so $r = 0$. $\square$

Corollary 2.9. *If A is an abelian variety over an algebraically closed field, $\phi_\Theta : A \to \widehat{A}$ is a principal polarization.*

Proof. This follows by using Lemma 2.7 to verify the hypotheses of Proposition 2.6. $\square$

2.4. **The Torelli map in general.** Using the algebraically closed case verified in §2.3, we now show that in general, the map induced by the Theta divisor is a principal polarization.

Theorem 2.10. *The map $\phi_\Theta : \widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$ of Corollary 2.3 is a principal polarization. In particular, this functorially defines a map $\mathscr{M}_g \to \mathscr{A}_g$ by sending a smooth proper relative dimension 1 morphism $C \to S$ with geometrically connected fibers to $\left(\mathrm{Pic}^0_{C/S}, \phi_\Theta\right)$.*

Proof. By the definition of principal polarization, we wish to show ϕ_Θ is a symmetric fiberwise ample isomorphism. Both $\widehat{\mathrm{Pic}^0_{C/S}}$ and $\mathrm{Pic}^0_{C/S}$ are flat over S, and so to show ϕ_Θ is an isomorphism, it suffices to verify it is an isomorphism on fibers. Further, by faithfully flat descent, it suffices to check this induces an isomorphism on geometric fibers. Similarly, fiberwise ampleness can be verified on geometric fibers, and the symmetry of the map ϕ_Θ can be expressed as an equality of ϕ_Θ and $\widehat{\phi_\Theta} \circ \iota$, where ι is the canonical isomorphism $\widehat{\mathrm{Pic}^0_{C/S}} \simeq \widehat{\widehat{\widehat{\mathrm{Pic}^0_{C/S}}}}$. Note that here and before, we have been implicitly identifying $\widehat{\widehat{\mathrm{Pic}^0_{C/S}}} \simeq \mathrm{Pic}^0_{C/S}$. Then, equality of these two maps can be verified on geometric fibers using that abelian schemes are separated over S combined with the rigidity lemma [Mum65, Proposition 6.1]. Hence, we can reduce to the case that S is the spectrum of an algebraically closed field, in which case the result follows from Proposition 2.6 and Lemma 2.7. $\square$

3. EXTENDING THE MAP TO COMPACT TYPE CURVES

Let $\mathscr{M}_g^c$ denote the moduli stack of compact type curves over $\operatorname{Spec} \mathbb{Z}$ and let $\mathscr{C} \to \mathscr{M}_g^c$ denote the universal curve over the moduli stack of compact type curves. Over $\mathscr{M}_g \subset \mathscr{M}_g^c$, we have $\mathscr{C}|_{\mathscr{M}_g}$ and as we have seen the Theta divisor induces a polarization $\mathrm{Pic}^0_{\mathscr{C}|_{\mathscr{M}_g}/\mathscr{M}_g} \to \widehat{\mathrm{Pic}^0_{\mathscr{C}|_{\mathscr{M}_g}/\mathscr{M}_g}}$. We next verify that the Torelli map extends from $\mathscr{M}_g$ to $\mathscr{M}_g^c$.

Proposition 3.1. *The principal polarization $\phi|_{\mathscr{M}_g} : \mathrm{Pic}^0_{\mathscr{C}|_{\mathscr{M}_g}/\mathscr{M}_g} \to \widehat{\mathrm{Pic}^0_{\mathscr{C}|_{\mathscr{M}_g}/\mathscr{M}_g}}$ uniquely extends to a principal polarization $\phi : \mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c} \to \widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}$.*

Note here that the map ϕ is the inverse of ϕ_Θ used in Theorem 2.10.

Remark 3.2. The same proof more generally shows that a polarization defined on an open over a normal noetherian base uniquely extends to a polarization on the whole base.

Proof assuming Lemma 3.3. First let us show that $\phi|_{\mathscr{M}_g}$ extends to a unique map ϕ. To construct the extension, by properness of $\widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}$ over $\mathscr{M}_g^c$, the map $\phi|_{\mathscr{M}_g}$ extends uniquely over all points of codimension 1. Therefore, since $\mathscr{M}_g^c$ is normal, by Weil's rigidity theorem

[BLR90, §4.4, Theorem 1], the map $\phi|_{\mathscr{M}_g}$ extends to a morphism $\phi : \mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c} \to \widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}$. This extension is unique by separatedness of $\widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}$.

It remains to check the constructed ϕ is a principal polarization. First, let us verify ϕ is symmetric. Symmetry is equivalent to showing commutativity of

$$\begin{tikzcd}
\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c} \arrow[rr] \arrow[dr] & & \widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}} \arrow[dl] \\
& \widehat{\widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}}. &
\end{tikzcd} \tag{3.1}$$

Because this restricts to $\phi|_{\mathscr{M}_g}$ over $\mathscr{M}_g$, which we showed is symmetric in Theorem 2.10, the diagram indeed commutes over this restriction. Since all stacks above are separated an representable by schemes over $\mathscr{M}_g^c$, it follows that two maps agreeing on a dense open must agree, so the diagram commutes and ϕ is symmetric.

Next, we check ϕ is an isomorphism. Indeed, just as we constructed ϕ as an extension of $\phi|_{\mathscr{M}_g}$, we can similarly construct an extension ψ of $\phi^{-1}|_{\mathscr{M}_g}$. We then have $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$ using separatedness, because these equalities hold upon restriction to $\mathscr{M}_g$. Therefore, ϕ is an isomorphism.

To complete the proof, we only need verify that $\mathscr{L} := (1 \times \phi)^* \mathscr{P}$ is relatively ample, for $\mathscr{P}$ the Poincaré bundle on $\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c} \times \widehat{\mathrm{Pic}^0_{\mathscr{C}/\mathscr{M}_g^c}}$. We know $\mathscr{L}$ is relatively ample when restricted to $\mathscr{M}_g$.

First, we claim that for each geometric point $x \in \mathscr{M}_g$, $h^0(\mathscr{C}_x, \mathscr{L}|_x) > 0$. Indeed, we know $\mathscr{L}|_x$ is ample, hence $\mathscr{L}^{\otimes n}|_x$ is very ample for some $n > 0$, and so $h^0(\mathscr{C}_x, \mathscr{L}^{\otimes n}|_x) > 0$. Then, [Mum08, Corollary, p. 159] implies $h^0(\mathscr{C}_x, \mathscr{L}|_x) > 0$.

By upper semicontinuity of cohomology, we obtain that for each geometric point $x \in \mathscr{M}_g^c$, $h^0(\mathscr{C}_x, \mathscr{L}|_x) > 0$. In particular, $\mathscr{L}|_x$ is effective. For the remainder of the proof, let A denote the abelian variety which is the Jacobian of the curve associated to x. Then, ϕ restricts to an isomorphism which we still call $\phi_x : A \to \widehat{A}$. We again let $\mathscr{P}_x$ denote the Poincaré bundle on $A \times \widehat{A}$. We just need to check that $\mathscr{L}|_x := (1 \times \phi_x)^* \mathscr{P}_x$ is ample.

Associated to $\mathscr{L}|_x$ we obtain a morphism $\phi_{\mathscr{L}|_x} : A \to \widehat{A}$ and another invertible sheaf $\mathscr{M} := (1 \times \phi_{\mathscr{L}|_x})^* \mathscr{P}_x$. Since $\mathscr{L}|_x$ is ample if and only if $\mathscr{L}^{\otimes 2}|_x$ is we will show the latter is ample. Using [CF17, Proposition 3.4.8] and effectivity of $\mathscr{L}^{\otimes 2}|_x$, it suffices to show $\phi_{\mathscr{L}^{\otimes 2}|_x}$ is an isogeny.

By [CF17, Lemma 7.5.2], there is some $\mathscr{N} \in \mathrm{Pic}^0_{A/\kappa(x)}$ so that $\mathscr{M} = (1 \times \phi_{\mathscr{L}|_x})^* \mathscr{P}_x = [2]^* \mathscr{L}|_x \otimes \mathscr{L}_x^{\otimes -2} \simeq \mathscr{L}_x^{\otimes 2} \otimes \mathscr{N}$. Now, since $\phi_{\mathscr{N}} = 0$ as $\mathscr{N} \in \mathrm{Pic}^0_{A/\kappa(x)}(\kappa(x))$ [CF17, Proposition 5.2.2] and $\phi_{\mathscr{L}|_x^{\otimes 2}} + \phi_{\mathscr{N}} = \phi_{\mathscr{M}}$ from the definitions, we see $\phi_{\mathscr{L}|_x^{\otimes 2}} = \phi_{\mathscr{M}}$. Therefore, to conclude the proof it suffices to check $\phi_{\mathscr{M}}$ is an isogeny. Since ϕ is an isogeny, $\phi_{\mathscr{M}} = [2] \circ \phi$ by Lemma 3.3, so $\phi_{\mathscr{M}}$ is also an isogeny. □

Recall that a morphism $\phi : A \to \widehat{A}$ is symmetric if $\phi = \widehat{\phi} \circ \iota$, for $\iota : A \to \widehat{\widehat{A}}$ the canonical map.

Lemma 3.3. *Let A be an abelian variety over a field k and let $\phi : A \to \widehat{A}$ be a symmetric morphism of abelian varieties. For $\mathscr{P}$ the Poincaré bundle on $A \times \widehat{A}$, we have $\phi_{(1,\phi)^*\mathscr{P}} = [2] \circ \phi$, where $[2] : A \to A$ denotes multiplication by 2.*

Proof. Let $\mathscr{M} := (1, \phi)^*\mathscr{P}$ and define $\mathscr{L} := (1 \times \phi_{\mathscr{M}})^*\mathscr{P}$. Let $m : A \times A \to A$ be multiplication and $p, q : A \times A \to A$ denote the projections onto the first and second factors respectively. The key fact we will use is that $(1 \times \phi_{\mathscr{M}})^*\mathscr{P} = m^*\mathscr{M} \otimes p^*\mathscr{M}^\vee \otimes q^*\mathscr{M}^\vee$ [CF17, Proposition 3.3.1].

Observe that the composition $A \xrightarrow{(e,\mathrm{id})} A \times A \xrightarrow{(1\times\phi_{\mathscr{M}})} A \times \widehat{A} \to \widehat{A}$ (for e the constant map to the identity section) is precisely $\phi_{\mathscr{M}}$. Therefore, it suffices to show this composition is $[2] \circ \phi$. For $a \in A(T)$ a T-point, we see that $a \mapsto \phi_{\mathscr{M}}(a)$ where $\phi_{\mathscr{M}}(a)$ is described as the line bundle $\mathscr{P}|_{A\times T} = \mathscr{L}|_{A\times T}$ using commutativity of

$$\begin{tikzcd}
A \times T \arrow[rr, "(\mathrm{id},a)"] \arrow[dr, "(\mathrm{id},\phi_{\mathscr{M}}(a))"'] & & A \times A \arrow[dl, "(\mathrm{id},\phi_{\mathscr{M}})"] \\
& A \times \widehat{A}. &
\end{tikzcd} \tag{3.2}$$

Therefore, it suffices to verify $\mathscr{L}|_{A\times T} = \phi(a) \otimes \phi(a)$. Using $\mathscr{L} \simeq m^*\mathscr{M} \otimes p^*\mathscr{M}^\vee \otimes q^*\mathscr{M}^\vee$, we find $\mathscr{L}|_{A\times T} = m^*\mathscr{M}|_{A\times T} \otimes p^*\mathscr{M}^\vee|_{A\times T} \otimes q^*\mathscr{M}^\vee|_{A\times T}$. Then, $m^*\mathscr{M}|_{A\times T}$ is the pullback of $\mathscr{P}$ along the map

$$\begin{aligned} A \times T &\to A \times \widehat{A} \\ (x, b) &\mapsto (x + a \circ b, \phi(x) + \phi(a \circ b)) \end{aligned}$$

while $p^*\mathscr{M}^\vee|_{A\times T}$ is similarly the pullback of $\mathscr{P}$ along

$$\begin{aligned} A \times T &\to A \times \widehat{A} \\ (x, b) &\mapsto (x, \phi(x)) \end{aligned}$$

and $q^*\mathscr{M}^\vee|_{A\times T}$ is the pullback of $\mathscr{P}$ along

$$\begin{aligned} A \times T &\to A \times \widehat{A} \\ (x, b) &\mapsto (a \circ b, \phi(a \circ b)). \end{aligned}$$

Therefore, since the addition law in $\widehat{A}$ corresponds to tensor product of line bundles, we find that $\mathscr{L}|_{A\times T}$ is the pullback of $\mathscr{P}$ along

$$\begin{aligned} A \times T &\to A \times \widehat{A} \\ (x, b) &\mapsto (x, \phi(a \circ b)) + (a \circ b, \phi(x)) \end{aligned}$$

using that

$$(x + a \circ b, \phi(x) + \phi(a \circ b)) - (x, \phi(x)) - (a \circ b, \phi(a \circ b)) = (x, \phi(a \circ b)) + (a \circ b, \phi(x)).$$

Equivalently, it is the tensor product of the pullbacks of $\mathscr{P}$ along the maps $(x, b) \mapsto (x, \phi(a \circ b))$ and $(x, b) \to (a \circ b, \phi(x))$.

By the definition of the Poincaré bundle, the pullback along the former is simply $\phi(a)$. So, it suffices to show the pullback of $\mathscr{P}$ along the latter map is also $\phi(a)$. This will follow from symmetry of ϕ. Indeed, restating the problem, we are trying to show the pullback of

$\mathscr{P}$ along $T \times A \xrightarrow{(\mathrm{id},\phi)} T \times \widehat{A} \xrightarrow{(a,\mathrm{id})} A \times \widehat{A}$ is $\phi(a)$, where $\phi(a)$ is viewed as a line bundle on $A \times T$. The pullback of the Poincaré bundle along (a, id) is, essentially by definition, the line bundle corresponding to the point $[a]$ viewed as a line bundle on $T \times \widehat{A}$. Then, we wish to compute $\phi^*([a])$. This is the same as $\widehat{\phi} \circ \iota$ where $\iota : A \to \widehat{\widehat{A}}$ is the canonical map. Recall that $\widehat{\phi} : \widehat{\widehat{A}} \to \widehat{A}$ precisely corresponds to pulling back line bundles along ϕ, under the identification given by ι. Using symmetry of ϕ, this tells us that $\phi^*([a]) = \widehat{\phi} \circ \iota([a]) = \phi(a)$, as we wished to show. □

As a bonus, we obtain the following explicit characterization of the Theta divisor on a compact type curve.

Proposition 3.4. *For C_0 a compact type curve over a field k with a smooth rational point x, the principal polarization on $\mathrm{Pic}^0_{C_0/k}$ is induced by $W_{g-1} - (g-1)x$ where $W_{g-1} \subset \mathrm{Pic}^{g-1}_{C_0/k}$ denotes the closed subscheme of effective line bundles on C_0.*

Moreover, if C_0 as above is a union of r smooth components $C_1, \ldots, C_r$ with genera $g_1, \ldots, g_r$, then the Theta divisor for C_0 also has r components. The ith such component parameterizes effective divisors which have degree $g_i - 1$ on C_i and, for every $j \neq i$, degree g_j on C_j.

Proof. After passing to a finite extension of k (which is harmless for the purposes of determining whether a divisor induces a polarization) we can find a family $C \to S$ for S a discrete valuation ring with special fiber C_0 and with smooth generic fiber. (This can be obtained by taking a discrete valuation ring mapping to $\mathscr{M}_g^c$ with closed fiber mapping to $[C_0]$ and generic fiber mapping to any point in $\mathscr{M}_g$. We can further assume $C \to S$ has a section. For example, this is possible since there is no harm in passing to the henselization of S, in which case the section on the special fiber necessarily lifts.

We know by the explicit description of the Theta divisor on smooth curves given in (2.1) (see also Theorem 2.10), that the Theta divisor on a smooth curve is given by W_{g-1} as above. We claim that over S the Theta divisor is given informally by the divisor corresponding to those points $[\mathscr{L}] \in \mathrm{Pic}^{g-1}_{C/S}$ with $H^0(\mathscr{L}) \neq 0$. More formally, we can construct the Theta divisor as follows: Take $C \times_S \mathrm{Pic}^{g-1}_{C/S} \xrightarrow{\pi} \mathrm{Pic}^{g-1}_{C/S}$ and let $\mathscr{P}$ denote the universal line bundle on $C \times_S \mathrm{Pic}^{g-1}_{C/S}$. Then, consider the reduced locus of points $x \in \mathrm{Pic}^{g-1}_{C/S}$ on which $H^0(C \times_S x, \mathscr{P}|_{C \times_S x}) \neq 0$. This is closed by the semicontinuity theorem. We claim this is a relative Cartier divisor. Because $\mathrm{Pic}^{g-1}_{C/S}$ is a regular scheme, once we verify this locus is irreducible, it will automatically be a relative Cartier divisor, since this will imply it is a Weil divisor, and additionally a Weil divisor in fibers. (In other words, the only worry we must assuage is that there may be components of W_{g-1} in the special fiber which are not in the closure of W_{g-1} from the generic fiber.) Once we check this irreducibility, $W_{g-1} - (g-1)x$ will define a map $\widehat{\mathrm{Pic}^0_{C/S}} \to \mathrm{Pic}^0_{C/S}$ (via the usual procedure of twisting down by a multiple of a section, taking the associated invertible sheaf, and applying the ϕ construction). However, in Proposition 3.1, we constructed another such extension (or more precisely, the extension there induces an extension in our setup via basechange to S), and so by separatedness the two extensions must agree and so this divisor will induce the principal polarization of Proposition 3.1.

In order to show the above divisor induces a principal polarization, it remains to check the divisor constructed is a relative Cartier divisor. On the generic fiber, this simply recovers the usual Theta divisor. On the special fiber, we can describe the locus in question as follows. Here, W_{g-1} on the special fiber is identified as the image of $\widetilde{C}^{g-1} \to \mathrm{Pic}^{g-1}_{C/k}$, where $\widetilde{C}$ denotes the normalization of C. Therefore, it is in particular a divisor on the special fiber. However, it may fail to have irreducible special fiber. To show this is actually a Weil divisor, we just need to verify that every component of the special fiber lies in the closure of the generic fiber. To this end, we may as well assume k is algebraically closed and S is strictly Henselian (via further base change). Then, a general k point of $\widetilde{C}_0$ lifts to an S-point of $\widetilde{C}$. The components of $W_{g-1} \subset \mathrm{Pic}^{g-1}_{C_0/k}$ are parameterized by multidegrees of divisors on each of the components, and we can then write a general such divisor of multidegree $(d_1, \ldots, d_m)$ as $\sum_{i=1}^m \left(\sum_{j=1}^{d_i} \mathscr{O}(p_{j,i})\right)$ where $p_{j,i}$ is a k point on the ith component of C_0. A general such divisor lifts to a divisor on C by lifting each of the k points of C_0 to S points of C. Therefore, each component lies in the closure of the Theta divisor on the generic fiber, as we wished to verify.

For the final statement regarding the number of components, observe that the Theta divisor is necessarily the unique member of its linear system. This holds because the Theta divisor is an effective divisor inducing a principal polarization, and so by Riemann Roch for abelian varieties and the vanishing theorem [Mum08, p. 150], we see the Theta divisor must be unique in its linear series.

So, we only need to check W_{g-1} has r geometric components when C has r geometric components. We may as well assume k is algebraically closed. In this case, as mentioned above, W_{g-1} is a Cartier divisor and is identified as the image of $\widetilde{C}_0^{g-1} \to \mathrm{Pic}^{g-1}_{C/k}$, where $\widetilde{C}$ denotes the normalization of C. Therefore, the components of W_{g-1} are identified with the components of $\widetilde{C}_0^{g-1}$ whose image has dimension $g-1$ (and not a smaller dimension). Said another way, they are identified with the components of $\widetilde{C}_0^{g-1}$ which map generically finitely onto $\mathrm{Pic}^{g-1}_{C/k}$. The different components of $\widetilde{C}_0^{g-1}$ can be identified with tuples $(d_1, \ldots, d_r)$, and the corresponding component parameterizes divisors with degree d_j on component C_j, for $C_1, \ldots, C_r$ the irreducible components of C_0, with C_j having genus g_j. If any $d_j > g_j$ the map $C_j^{d_j} \to \mathrm{Pic}^{d_j}_{C_j/k}$ will have gerically positive dimensional fibers. Therefore, the only possible components which map generically finitely onto $\mathrm{Pic}^{g-1}_{C/k}$ must satisfy $d_j \leq g_j$ for every j. Since we also have the constraint that $\sum_{j=1}^r d_j = g-1$, the above conditions can only be achieved when a single $d_i = g_i - 1$ and all remaining $d_j = g_j$ with $j \neq i$. This was the claimed characterization of the irreducible compoents of the theta divisor in the statement of the proposition. □

4. PROPERNESS OF THE EXTENDED TORELLI MAP

We finally arrive at the main goal of this note, which is to verify that the extension of the Torelli map $\overline{\tau}_g : \mathscr{M}_g^c \to \mathscr{A}_g$ is in fact a proper map of stacks.

Theorem 4.1. *The extended Torelli map $\overline{\tau}_g : \mathscr{M}_g^c \to \mathscr{A}_g$ sending a curve to its principally polarized Jacobian exists and is proper.*

Proof. Existence of the Torelli map follows from Proposition 3.1. It remains to verify properness. First, both stacks are finite type and separated, so we only need verify the map is universally closed. For this, we use the valuative criterion of universal closedness. Let us recall what this says. In what follows, by trait, we mean the spectrum of a discrete valuation ring. We let T and T' denote traits, and let η and η' denote their generic points. Then, the valuative criterion says that, given a commutative diagram,

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & \mathscr{M}_g^c \arrow[d] \\ T \arrow[r] & \mathscr{A}_g \end{tikzcd} \tag{4.1}$$

we want to show there exists some extension of traits $T' \to T$ and a map $T' \to \mathscr{M}_g^c$ making

$$\begin{tikzcd} \eta' \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & \mathscr{M}_g^c \arrow[d] \\ T' \arrow[r] \arrow[rru] & T \arrow[r] & \mathscr{A}_g \end{tikzcd} \tag{4.2}$$

commute.

By properness of $\overline{\mathscr{M}}_g$, we can find an extension

$$\begin{tikzcd} \eta' \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & \mathscr{M}_g^c \arrow[r] \arrow[d] & \overline{\mathscr{M}}_g \\ T' \arrow[r] \arrow[rrru] & T \arrow[r] & \mathscr{A}_g. & \end{tikzcd} \tag{4.3}$$

and we want to show this factors through $\mathscr{M}_g^c$.

We now reinterpret what we have in terms of Néron models. The map $T' \to \mathscr{A}_g$ corresponds to a family of abelian varieties $\mathscr{A} \to T'$ while the map $T' \to \overline{\mathscr{M}}_g$ corresponds to a family of stable curves $\mathscr{C} \to T'$. Let $\mathscr{J} := \operatorname{Pic}^0_{\mathscr{C}/T'} \to T'$ denote the Jacobian of $\mathscr{C} \to T'$.

Lemma 4.2. *In the notation above, $\mathscr{J}$ is isomorphic over T' to $\mathscr{A}$.*

Proof. As a first step, because $\mathscr{A}$ is proper, we note that it is the Néron model of its generic fiber by [BLR90, §1.4, Proposition 2].

Next, we recall the general fact that for $\mathscr{G}$ a smooth separated semi-abelian group scheme over a trait with generic fiber an abelian variety, the natural map from $\mathscr{G}$ to the identity component of its Néron model $N(A)^0$ is an isomorphism [BLR90, §7.4, Proposition 3]. We briefly recall why this above fact is true. First, one shows $\mathscr{G} \to N(A)^0$ is quasi-finite. Since both schemes are semi-abelian, if ℓ is a prime invertible on the base, the ℓ-power torsion is dense, and hence to verify the map is quasi-finite, one only needs check injectivity of the ℓ-power torsion on the special fiber. One can reduce to the case of a strictly Henselian base. In this case, by the structure theorem for quasi-finite schemes over a Henselian base, the injectivity of the map on torsion points of the special fiber can be deduced from injectivity on R-points. Further, injectivity on R-points follows from the isomorphism on K-points using the Néron mapping property of $N(A)$, the Néron model of A. With quasi-finiteness having been established, the claimed isomorphism then follows from the fact that a quasi-finite birational map of normal separated schemes is an open immersion.

Since $\mathscr{J}$ is semi-abelian as $\mathscr{C}$ is stable, the above fact implies the natural map $\mathscr{J} \to \mathscr{A}$ is an isomorphism, as desired. □

Since $\mathscr{J}$ is isomorphic to $\mathscr{A}$, it follows that the closed fiber of $\mathscr{C} \to T'$ is a curve whose special fiber has proper Jacobian. This implies that the special fiber of $T' \to \overline{\mathscr{M}}_g$ factors through $\mathscr{M}_g^c$, since $\mathscr{M}_g^c \subset \overline{\mathscr{M}}_g$ is precisely the open locus parameterizing points with proper Jacobian. By the valuative criterion for open immersions, $T' \to \overline{\mathscr{M}}_g$ factors through $\mathscr{M}_g^c$, as desired.

We have produced a diagram (4.2) and we will be done once we verify it commutes. That is, we need to check the composition $T' \to \mathscr{M}_g^c \to \mathscr{A}_g$ agrees with the given map $T' \to \mathscr{A}_g$. We know the two maps agree on the generic point η'. Therefore, they agree on all of T' by separatedness of $\mathscr{A}_g$. □

REFERENCES

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[CF17] Brian Conrad and Tony Feng. Abelian varieties. *Course at Stanford University*, 2017. Available at `https://www.ams.org/open-math-notes/omn-view-listing?listingId=110657` August 11, 2017 version.

[Cor86] Arithmetic geometry. pages xvi+353, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984.

[Lan] Aaron Landesman. Does the compactified torelli map extend to a proper map of stacks? MathOverflow. URL:https://mathoverflow.net/q/337964 (version: 2019-08-09).

[Mum65] David Mumford. *Geometric invariant theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin, 1965.

[Mum08] David Mumford. Abelian varieties. 5:xii+263, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.