

THE HODGE THEORY OF THE PUTMAN-WIELAND CONJECTURE

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To start, we will introduce three conjectures/problems in algebraic geometry: Prill's problem, Ivanov's conjecture, and the Putman-Wieland conjecture. The rough structure of relations between these is: (no Prill exceptional covers) implies (Putman-Wieland) which is equivalent to (Ivanov)

1. PRILL'S PROBLEM

Let's start with a simple to state, open question in geometry of algebraic curves:

Definition 1.1. Call a finite branched cover $f : X \rightarrow Y$ of smooth proper connected curves *Prill exceptional* if $h^0(X, \mathcal{O}(f^{-1}(p))) \geq 2$ for a general point $p \in Y$.

Question 1.2 (Prill's problem). Are there any Prill exceptional covers where Y has genus $g \geq 2$?

Despite its simplicity, not much is known about Prill's problem, and the expected answer in the literature seems to be that there are no Prill exceptional covers.

Theorem 1.3 (Landesman-Litt). *If Y is any smooth proper connected curve of genus 2, there is a finite étale cover $f : X \rightarrow Y$ which is Prill exceptional.*

However, it turns out to be closely connected to some important open conjectures in geometric topology.

Theorem 1.4 (Landesman-Litt). *Any counterexample to the Putman-Wieland conjecture is also Prill exceptional.*

Theorem 1.3 follows from Theorem 1.4 since there turns out to be a counterexample to Putman-Wieland in genus 2. So next, let me tell you about this Putman-Wieland conjecture.

2. IVANOV'S CONJECTURE AND THE PUTMAN-WIELAND CONJECTURE

Much of the original motivation for the Putman-Wieland conjecture was as an approach to proving Ivanov's conjecture, which is a major open question in the study of mapping class groups.

Definition 2.1. Recall that the mapping class group $\text{Mod}_{g,n}$ is the fundamental group of $\mathcal{M}_{g,n}$.

More usually, the mapping class group is defined as the group of orientation preserving diffeomorphisms of an n -punctured genus g Riemann surface, up to isotopy.

Conjecture 2.2 (Ivanov). $I_{g,n}$: There is no finite index subgroup of $\text{Mod}_{g,n}$ which surjects onto \mathbb{Z} for $g \geq 3, n \geq 0$.

Remark 2.3. Equivalent formulation: $I_{g,n}$ holds if and only if for any finite étale cover $\mathcal{M} \rightarrow \mathcal{M}_{g,n}$ has $H_1(\mathcal{M}, \mathbb{Q}) = 0$.

Exercise 2.4. Come up with a counterexample to $I_{2,n}$.

Another highly related conjecture to Ivanov's conjecture and Prill's problem is the Putman-Wieland conjecture. To state it consider the setup

$$(2.1) \quad \begin{array}{ccccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} & \longrightarrow & \mathcal{C}_{\text{univ}} \\ & \searrow \pi' & \downarrow \pi & \nearrow s_1, \dots, s_n & \downarrow \\ & & \mathcal{M} & \longrightarrow & \mathcal{M}_{g,n} \end{array}$$

where

- (1) \mathcal{X} has genus g' , \mathcal{Y} has genus g are smooth projective curves with geometrically connected fibers
- (2) $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a Galois H -cover
- (3) f is unramified away from the sections s_i
- (4) The induced map $\mathcal{M} \rightarrow \mathcal{M}_{g,n}$ is dominant

Conjecture 2.5. $\text{PW}_{g,n}^H$: For any group H as above, $\text{Jac}_{\mathcal{X}/\mathcal{M}}$ has no isotrivial isogeny factor, "i.e., no fixed part."

$\text{PW}_{g,n}$: For fixed g and all H , $\text{PW}_{g,n}^H$ holds.

Remark 2.6. An equivalent formulation, which is how Putman-Wieland is usually stated, is that for X a fiber of $\mathcal{X} \rightarrow \mathcal{M}$, $H^1(X, \mathbb{C})$ has no nonzero vectors with finite orbit under the action of $\pi_1(\mathcal{M})$.

The relation between Ivanov's and the Putman-Wieland conjecture is the following:

Theorem 2.7 ([PW13, Theorem 1.3]). For $g \geq 3, n \geq 0$, $\text{PW}_{g-1, n+1} \implies I_{g,n}$ and $I_{g, n+1} \implies \text{PW}_{g,n}$.

3. OUR MAIN RESULT

The main new statement we can prove toward the Putman-Wieland conjecture is:

Theorem 3.1 (Landesman-Litt, Asymptotic Putman-Wieland). *For fixed H, n the Putman-Wieland conjecture for H -covers $PW_{g,n}^H$ holds whenever $g^2 > \#H$.*

4. COUNTEREXAMPLES IN LOW GENUS TO PUTMAN-WIELAND

Example 4.1 (Counterexample to $PW_{0,4}$, the Eierelegende Wollmilchsau). Consider the family of cyclic $\mathbb{Z}/4\mathbb{Z}$ covers of \mathbb{P}^1 given by curves $y^4 = x(x-1)(x-\lambda)$, for $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$. Recall that $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ parameterizing the position of the fourth point and the resulting family dominates $\mathcal{M}_{0,4}$. To give a counterexample, it suffices to show that every curve in this family possesses a copy of the same elliptic curve in its Jacobian. So it is enough to produce a map $y^4 = x(x-1)(x-\lambda)$ to a fixed elliptic curve. Indeed, we can rewrite the family via a change of variables as $y^4 = x^4 + \eta x^2 + 1$. I.e. we reparameterize the four marked points. Then there is a map from this curve to $y^4 = z^2 + \eta z + 1$, given by $z \mapsto x^2$, and the resulting elliptic curve has 4 automorphisms generated by $y \mapsto iy$. Since there is a unique elliptic curve with 4 automorphisms, this shows up as a fixed factor in the Jacobian of this family.

Example 4.2 (Counterexample to $PW_{1,1}$, the Eierelegende Wollmilchsau revisited). Again the family of cyclic $\mathbb{Z}/4\mathbb{Z}$ covers of \mathbb{P}^1 given by curves $C_\lambda : V(y^4 = x(x-1)(x-\lambda))$, for $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$. This time there is a map to the genus 1 curve $E_\lambda : V(y^2 = x(x-1)(x-\lambda))$, which is a double cover ramified over the 4 two-torsion points. By composing this map with multiplication by 2 on the elliptic curve, the resulting map $C_\lambda \rightarrow E_\lambda$ is only ramified over the identity of the elliptic curve. Again, this family has an isotrivial isogeny factor, since it is the same family of curves as in the previous example.

Remark 4.3. Include story of Daniel and my attempts in genus 2.

Example 4.4. For every genus 2 curve Y , there is a certain degree 72 cover $X \rightarrow Y$ so that X has a nonconstant map to the genus 1 curve $E_0 := V(y^2 = x^3 - 1)$, due to Bogomolov and Tschinkel. So every member of this family has a factor of E_0 in the Jacobian of X .

5. NO DIMENSION 1 IRREPS IN THE KERNEL

We'll now move onto some of the proofs.

Remark 5.1. Include story of two false attempts in genus at least 3.

Here is the starting point for the idea of a geometric proof we had.

Consider our family of covers $\mathcal{X} \rightarrow \mathcal{Y}$ over \mathcal{M} . What does it mean to have an isotrivial isogeny factor? There is a connection

$$\nabla : H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}) \otimes \Omega_{\mathcal{M}}.$$

Which describes how the first part of the Hodge structure $H^0(X, \omega_X) \subset H^1(X, \mathbb{C})$ is varying as we move around in \mathcal{M} . The Hodge filtration, with graded pieces $H^0(X, \omega_X)$ and $H^1(X, \mathcal{O}_X)$ induces a $\mathcal{O}_{\mathcal{M}}$ linear map

$$\theta : H^0(X, \omega_X) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(Y, \omega_Y^{\otimes 2}).$$

Saying there is an isotrivial isogeny factor means this has a kernel, since the vector corresponding to that tangent direction does not change as one varies the point in \mathcal{M} .

By fiddling with the above map and using Serre duality, it turns out to be adjoint to the natural multiplication and trace map

$$\begin{aligned} \text{Sym}^2 H^0(X, \omega_X) &\rightarrow H^0(X, \omega_X^{\otimes 2}) \rightarrow H^0(Y, \omega_Y^{\otimes 2}) \\ s \otimes t &\mapsto \sum_{h \in H} h^*(st). \end{aligned}$$

Now, suppose there were some vector in the kernel. This means there is some s so that its pairing with every t vanishes. Observe that $H^0(X, \omega_X)$ has an H action, so we can decompose it into irreps, and we want to show s lies in a large dimensional irrep, which will force H to be large. Let's just explain how to rule out 1-dimensional irreps.

Lemma 5.2. *There are no 1-dimensional irreps in the kernel of θ*

Proof. For 1-dimensional irreps, the map is $s \otimes t \mapsto \sum_{h \in H} \chi(h)(st)$, for $\chi(h) \in \mathbb{C}^\times$. By assumption, this vanishes for every t . So, we get a vanishing element of $\text{Sym}^2 H^0(X, \omega_X)$. In other words, this is a quadric vanishing on X under the canonical embedding. Using the above description, it has rank 2, corresponding to the s and t components. But any rank 2 quadric is a union of hyperplanes, and so cannot contain a canonical curve. \square

Example 5.3. For fun, let's see what the above analysis buys us in the Eierlegende Wollmilchsau example. There, we can explicitly compute that on the curve $y^4 = x(x-1)(x-\lambda)$ the differential forms

$$\frac{dx}{y^2}, \frac{x dx}{y^3}, \frac{dx}{y^3}$$

form a basis. We can decompose these into representations for the group $\mathbb{Z}/4\mathbb{Z}$ and we find $H^0(Y, \omega_Y) \simeq \chi_2 \oplus \chi_3 \oplus \chi_3$, where χ_n is the character

so that 1 acts by multiplication by the complex number i^n . Dually, one may compute that $H^1(Y, \mathcal{O}_Y)$ decomposes as $\chi_2 \oplus \chi_1 \oplus \chi_1$. Therefore, the connection

$$\bar{\nabla} : H^0(Y, \omega_Y) \rightarrow H^1(Y, \mathcal{O}_Y) \otimes \Omega_{\mathbb{A}^1 - \{0, \lambda\}}$$

must kill the χ_3 irreps, and so we get a 2-dimensional fixed part.

6. IDEA OF PROOF OF ASYMPTOTIC PUTMAN-WIELAND

Consider our family of covers $\mathcal{X} \rightarrow \mathcal{Y}$ over \mathcal{M} . What does it mean to have an isotrivial isogeny factor? There is a connection

$$\nabla : H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}) \otimes \Omega_{\mathcal{M}}.$$

Which describes how the first part of the Hodge structure $H^0(X, \omega_X) \subset H^1(X, \mathbb{C})$ is varying as we move around in \mathcal{M} . The Hodge filtration, with graded pieces $H^0(X, \omega_X)$ and $H^1(X, \mathcal{O}_X)$ induces a $\mathcal{O}_{\mathcal{M}}$ linear map

$$\bar{\nabla} : H^0(X, \omega_X) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(Y, \omega_Y^{\otimes 2}).$$

Saying there is an isotrivial isogeny factor means this has a kernel, since the vector corresponding to that tangent direction does not change as one varies the point in \mathcal{M} .

Definition 6.1. A vector bundle V on a curve Y is not generically globally generated if there is a proper subbundle $U \subset V$ so that $H^0(Y, U) \rightarrow H^0(Y, V)$ is an isomorphism.

Lemma 6.2. *Any $v \in \ker \bar{\nabla}$ implies there is a proper subbundle $U \subset f_*\omega_X$ inducing an isomorphism on global sections.*

Proof. As above, fix a point $m \in \mathcal{M}$, whose fiber yields a finite étale cover $f_m : X \rightarrow Y$. The map $\bar{\nabla}_m$ may be identified as a map

$$\bar{\nabla}_m : H^0(X, \omega_X) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(Y, \omega_Y^{\otimes 2}).$$

Using Serre duality and the identification $H^0(X, \omega_X) \simeq H^0(Y, f_*\omega_X)$, this can be identified with the map

$$(6.1) \quad \theta : H^0(Y, f_*\omega_X) \rightarrow \text{Hom}(H^0(Y, f_{m*}\omega_X), H^0(Y, \omega_Y^{\otimes 2})) \\ s \mapsto (t \mapsto \text{tr}(s \otimes t)).$$

By assumption θ has a non-zero kernel. Any non-zero element of the kernel yields a nonzero map $\alpha : f_*\omega_X \rightarrow \omega_Y^{\otimes 2}$, inducing the zero map on global sections. Taking $U = \ker \alpha$, the map $H^0(Y, U) \rightarrow H^0(Y, f_*\omega_X)$ is an isomorphism, so all global sections of $f_*\omega_X$ factor through a subbundle, and $f_*\omega_X$ is not generically globally generated. \square

Lemma 6.3 (Non-ggg lemma). *Suppose V is a slope $2g - 2$ semistable vector bundle on a smooth curve X of genus g . If V is not generically globally generated, then $\text{rk } V \geq g$.*

Proof. Suppose $U \subset V$ induces an isomorphism on H^0 . This follows from clifford's theorem for vector bundles, which says $h^0(C, U)$ is not too big.

More precisely, $h^0(C, U) \leq \deg U/2 + \text{rk } U$. We have

$$\deg V + (1 - g) \text{rk } V \leq h^0(V) = h^0(U) \leq \text{rk } U + \deg U/2 \leq (\text{rk } V - 1) + (g - 1)(\text{rk } V - 1).$$

Solving for $\text{rk } V$ gives $\text{rk } V \geq g$. \square

Combining the above lemmas, we find $\text{rk } f_*\omega > g$, which means $\#H > g$. Doing a slightly more detailed analysis by decomposing this into summands indexed by H -irreps in fact yields $\#H > g^2$.

7. IDEA OF PRILL

We saw in Lemma 6.2 that any counterexample to Putman-Wieland $f : X \rightarrow Y$ has $f_*\omega_X$ which is not generically globally generated. Therefore, it is enough to show that such a non-ggg bundle yields a Prill exceptional cover. This is essentially geometric Riemann Roch

Lemma 7.1. *Let $f : X \rightarrow Y$ be a finite Galois H -cover of smooth proper connected curve. whose Galois closure has Galois group H . The following are equivalent:*

- (1) *The map f is Prill exceptional*
- (2) *$f_*\omega_X$ is not generically globally generated*

Proof. Let $F := f_*\omega_X$. We wish to show F is not generically globally generated if and only if $h^0(Y, f_*\mathcal{O}_X(p)) > 1$ for every point p . Note that F is not generically globally generated if and only if, for a general $p \in Y$, we have an exact sequence

$$(7.1) \quad 0 \rightarrow H^0(Y, F(-p)) \rightarrow H^0(Y, F) \xrightarrow{\beta} H^0(Y, F|_p)$$

which is not right exact.

The long exact sequence gives

$$(7.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y, f_*\omega(-p)) & \longrightarrow & H^0(Y, f_*\omega) & \xrightarrow{\beta} & H^0(Y, f_*\omega|_p) \\ & & & & & & \swarrow \\ & & H^1(Y, f_*\omega(-p)) & \longrightarrow & H^1(Y, f_*\omega) & \longrightarrow & 0 \end{array}$$

Since the $h^1(Y, f_*\omega) = 1$, the map β is not surjective precisely when precisely when $h^1(Y, f_*\omega(-p)) > 1$. By Serre duality, this is equivalent to $h^0(Y, f_*\mathcal{O}_X(p)) > 1$. \square

8. HIGHER PRYM REPRESENTATIONS

We learned from Ben Wieland, that he originally had a stronger conjecture in mind, and Putman-Wieland could be viewed as evidence toward it. Namely, as above, given an H -cover $\Sigma_{g',n'} \rightarrow \Sigma_{g,n}$, we have a monodromy action of $\text{Mod}_{g,n+1}$ on $H_1(\Sigma_{g'})$. Putman-Wieland predicts this has no nonzero finite orbit vectors, but in general, one might predict this action has image which is as big as possible. What does “as big as possible” mean? Let’s give some examples first.

Example 8.1. (1) Suppose $H = \text{id}$. Then the image is the full symplectic group $\text{Sp}(H_1(\Sigma_g))$.

(2) Suppose $H = \mathbb{Z}/2\mathbb{Z}$ and $g = 0$. Then the action of $\text{Mod}_{0,n+1}$ is essentially identified with that of the hyperelliptic mapping class group, and the action on $H^1(\Sigma_{g'})$ of the hyperelliptic mapping class group is again the full symplectic group.

In general, the image of the monodromy must be contained in $\text{Sp}(H_1(\Sigma_{g'}))$ it must commute with the H action, and it must be semisimple (by a general result of Deligne).

So, one might conjecture the upgraded form of Putman-Wieland conjecture.

Conjecture 8.2. For all $g \geq 3, n \geq 0$, the action of $\text{Mod}_{g,n+1}$ on $H_1(\Sigma_{g',n'})$ not only has no nonzero vectors with finite orbit, but moreover has Zariski closure with identity component equal to the commutator subgroup of the symplectic centralizer of H , $\text{Sp}(H_1(\Sigma_{g'}))^H$.

We can again prove an asymptotic version of this.

Theorem 8.3 (L-Litt-Sawin). *If $n = 0$ and $g \geq 2\sqrt{\#H} + 2$ or n is arbitrary and $g > \max(2\sqrt{\#H} + 1, \#H)$, then the above conjecture holds.*

The proof is again via analyzing the derivative of the associated period map, but substantially more involved than the previous one. A loose character is that the key issue is to prove irreducibility, as there are general classifications of the possible irreducible monodromy groups. If the image were reducible, the corresponding vector bundle would not be stable. On the other hand, we can show this bundle is stable by analyzing the derivative of the period map, similarly to our strategy for the Putman-Wieland conjecture.

In the paper, we have a more high-brow argument for this, showing that for ρ an irreducible H representation, we can actually deduce irreducibility of the ρ -part of the representation on H_1 from irreducibility of the irreducible representation ρ .

REFERENCES

- [PW13] Andrew Putman and Ben Wieland. Abelian quotients of subgroups of the mappings class group and higher Prym representations. *J. Lond. Math. Soc. (2)*, 88(1):79–96, 2013.