TORSION LINE BUNDLES ON FINITE COVERS

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1. Introduction

Goal 1.1. For varying hyperelliptic curves $H \to \mathbb{P}^1$, understand $\text{Pic}(H)[n] := \{\mathscr{L} \in \text{Pic}(H) : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_H\}$. Number theory variant: Replace $H \to \mathbb{P}^1$ by $\text{Spec}\, \mathscr{O}_K \to \text{Spec}\, \mathbb{Z}$, for $K/\mathbb{Q}$ a quadratic extension.

More generally, let

$$\begin{aligned} L^{d,n}(B) &:= \left\{(X, \mathscr{L}) : g : X \to B \text{ finite locally free of degree } d\ \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X\right\} \\ &= \cup_{g:X\to B, \text{degree d}} \text{Pic}(X)[n]. \\ \widetilde{L}^{d,n}(B) &:= \left\{(X, \mathscr{L}, \alpha) : g : X \to B \text{ finite locally free of degree } d\ \alpha : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X\right\} \\ &= \cup_{g:X\to B, \text{degree d}} H^1(X, \mu_n). \end{aligned}$$

Can we understand $L^{d,n}(B)$ or $\widetilde{L}^{d,n}(B)$?

Warning 1.2. I'm thinking of this as a fibered category, but in no way an algebraic stack.

Remark 1.3. Motivation from the Cohen-Lenstra heuristics in number theory, which predict the average size for $\text{Pic}(\mathscr{O}_K)[n]$, for varying quadratic fields K. E.g., the chance $\text{Pic}(\mathscr{O}_K)[n] \simeq G$, for K imaginary quadratic, is inversely proportional to the automorphisms of G. For example, this implies the average size of $\text{Pic}(\mathscr{O}_K)[n]$ is the number of divisors of n.

To relate this to classical algebraic geometry, we'd like a map from $L^{d,n}(B)$ to some classical object, so that this map is often an isomorphism in cases of interest.

Let $\mathbb{P}^1 \xrightarrow{\nu_n} \mathbb{P}^n$ denote a rational normal curve and let Sec_n denote the complement of $\nu_n(\mathbb{P}^1)$ in the 2-secant variety to the rational normal curve. This carries an action of $\text{PGL}_2 = \text{Aut}_{\mathbb{P}^1}$.

Theorem 1.4. *There is a natural map* $\widetilde{L}^{2,n}(B) \to [\text{Sec}_n\ /\ \text{PGL}_2]\,(B)$*. This factors through an isomorphism* $L^{2,n}(B) \to [\text{Sec}_n\ /\ \text{PGL}_2]\,(B)$ *in many cases of interest, including when*

(1) *$B = \mathbb{P}^1_k$ for k an algebraically closed field or a finite field and n prime to twice the characteristic of k. [This uses the isomorphism $H^1(B, g_* \mathbb{G}_m / \mathbb{G}_m[n]) \simeq H^1(B, g_* \mathbb{G}_m / \mathbb{G}_m \xrightarrow{\times n} g_* \mathbb{G}_m / \mathbb{G}_m)$ and more cohomological diagram chasing. [The reason for the prime to the characteristic assumption is in the above identification, while the prime to 2 assumption has to do with analyzing the part of $H^2(\mu_n)$ coming from the kernel of $H^1(\mathbb{P}^1, \mathbb{G}_m)/n \to H^1(X, \mathbb{G}_m)/n$ since degree n sheaves on $\mathbb{P}^1$ pull back to degree $2n$ sheaves on X.]*

(2) *$B = \operatorname{Spec} \mathbb{Z}$ and n is odd. [Possible other cases when $H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.]*

Remark 1.5. A fun, but somewhat involved example of when it is not a bijection is when $B = \operatorname{Spec} \mathbb{R}$ and $n = 2$.

Outline:

(1) Describe the map above.
(2) Investigate the $n = 3$ case (which is related to solving the cubic equation)
(3) Describe the generalization to degree $d > 2$, time permitting.
(4) Describe $[\mathrm{Sec}_n \,/\, \mathrm{PGL}_2]$ in terms of a moduli space of genus 1 curves/divisors on hirzebruch surfaces.

2. Resultant 1

Theorem 2.1 (L)**.** *Let K be a quadratic field with $\operatorname{disc}(\mathscr{O}_K) = d$. Under the correspondence between quadratic forms $q = ax^2 + bxy + cy^2$, for $a, b, c \in \mathbb{Z}$ of discriminant d and line bundles on Spec $\mathscr{O}_K$, q corresponds to an element of $\operatorname{Pic}(\mathscr{O}_K)[n]$ if and only if there exists $f := \sum_{i=0}^n t_i x^i y^{n-i}$ with $\operatorname{Res}(q, f) = \pm 1$.*

Here,

$$\operatorname{Res}(q,f) := \begin{pmatrix} a & 0 & \cdots & 0 & t_0 & 0 \\ b & a & \cdots & 0 & t_1 & t_0 \\ c & b & \ddots & 0 & t_2 & t_1 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & t_{n-2} & t_{n-3} \\ 0 & 0 & \ddots & a & t_{n-1} & t_{n-2} \\ 0 & 0 & \ddots & b & t_n & t_{n-1} \\ 0 & 0 & \cdots & c & 0 & t_n \end{pmatrix}.$$

2.1. **The bijection between quadratic forms and line bundles.** Given q, we wish to produce a line bundle on a degree 2 extension. Note that $V(q)$ defines a degree 2 subscheme of $\mathbb{P}^1_{\mathbb{Z}}$. The degree 2 extension is $V(q)$ and the line bundle is the pullback of $\mathscr{O}_{\mathbb{P}^1}(1)$. Conversely, given $X \to \operatorname{Spec} \mathbb{Z}$ together with a line bundle $\mathscr{L}$, we get a map $X \to \mathbb{P}(g_* \mathscr{L}) \simeq \mathbb{P}^1_{\mathbb{Z}}$, and upon choosing a basis for $\mathbb{P}(g_* \mathscr{L})$, we get an embedding $X \to \mathbb{P}^1_{\mathbb{Z}}$, i.e., a quadratic form.

2.2. **Proof idea Theorem 2.1.** Let's just do one direction. Given q and f with $\operatorname{Res}(q, f) = \pm 1$, we want to show q is n-torsion. As above, q gives $\iota : X \to \mathbb{P}^1_{\mathbb{Z}}$ and a line bundle $\mathscr{L} := \iota^* \mathscr{O}(1)$. We want to show that f forces $\mathscr{L}$ to be trivial. Consider the n-Veronese $X \to \mathbb{P}^1 \to \mathbb{P}^n$, which is induced by $\mathscr{L}^{\otimes n}$. To show this is the trivial line bundle, it is enough to show there is a hyperplane not meeting X under this embedding. Indeed, $V(f)$ is such a hyperplane, since the resultant condition means $V(f)$ does not meet $V(q)$.

2.3. **The secant variety.** The following lemma explains why locally this gives a degree n polynomial f not meeting X in $\mathbb{P}^n$.

Lemma 2.2. *Suppose $X = V(q) \subset \mathbb{P}^1_B$. Given a degree n polynomial f, we obtain a point p on the secant variety to the rational normal curve in $\mathbb{P}^n$, which misses X if $\operatorname{Res}(q, f)$ is a unit on B.*

Conversely, given such a point p, we can locally lift it to an f.

Proof. We work in fibers over B. We have $X \to \mathbb{P}^1 \to \mathbb{P}^n$ by the n-Veronese. The polynomial f determines a hyperplane in $\mathbb{P}^n$. Draw the line M in $\mathbb{P}^n$ spanned by X. Then f meets this line at a unique point p. Since X is two points on the rational normal curve, p lies on the secant variety to the rational normal curve, but is disjoint from q if the resultant is a unit. □

We now give a construction taking a n-torsion line bundle and outputting the point on the secant variety to the rational normal curve.

3. The main construction in degree 2

We'd now like to describe a construction which takes as input a base B, a degree 2 cover $g : X \to B$ and $\mathscr{L}$ with $\mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$ and outputs a point on the secant variety to the rational normal curve, not on the rational normal curve.

(1) Begin with $X \to \mathbb{P}(g_* \mathscr{L})$.
(2) Compose with the n-veronese to get $X \to \mathbb{P}(\operatorname{Sym}^n(g_* \mathscr{L}))$
(3) We have a surjection $\operatorname{Sym}^n(g_* \mathscr{L}) \to g_* \mathscr{L}^{\otimes n} \simeq g_* \mathscr{O}_X \to Q := \operatorname{coker}(\mathscr{O}_B \to g_* \mathscr{O}_X)$ which picks out a line M in $\mathbb{P}(\operatorname{Sym}^n g_* \mathscr{L})$ and a point p on that line.

(4) The composition $\mathscr{O}_B \to g_*\mathscr{O}_X \to Q$ being 0 implies that the point does not lie on the image of X. Indeed, on geometric fibers, $X = B_1 \cup B_2$ and the surjection to Q is the diagonal inclusion, which is different from the idempotent inclusions of the two points.
(5) The points p lies on the 2-secant variety to the rational normal curve, but does not meet the rational normal curve in any fiber.
(6) Locally, we can now choose a hyperplane, meeting the line M at the point p. This is the desried degree n polynomial, which does not meet X because p does not meet X.

4. 3 TORSION

Lemma 4.1. *Working over $\mathbb{C}$, there is a bijection between*

$$\left\{\textit{simply branched degree } 3 \textit{ covers} T \to \mathbb{P}^1_{\mathbb{C}}\right\} \to L^{2,3}(\mathbb{P}^1_{\mathbb{C}}).$$

Sketch. Given H hyperelliptic, a 3-torsion line bundle corresponds to a degree 3 finite étale cover $Y \to H$. This will typically be an S_3 cover, and the quotient by some transposition defines the associated trigonal curve.

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ H & & T \\ \searrow & & \swarrow \\ & \mathbb{P}^1_{\mathbb{C}}. & \end{array} \tag{4.1}$$

The simply branched condition can be checked locally, and corresponds to the étaleness of $Y \to H$. The reverse direction is similar. □

Example 4.2. For us, given a degree 2 cover with a 3-torsion line bundle, we get a point in $\mathbb{P}^3$ on the secant variety to the twisted cubic. It is a classical fact that this secant variety is all of $\mathbb{P}^3$. So this gives us a degree 3 polynomial, corresponding to a general point p in $\mathbb{P}^3$.

Question 4.3. How do we produce the corresponding general degree 3 subscheme of $\mathbb{P}^1$?

The answer is "apolarity." Specifically, in each fiber, there will be three planes (counted with multiplicity) which pass through p and are flex points of the rational normal curve (since the curve of flex hyperplanes is also a rational normal curve, where the point a, b is sent to the plane restricting to $(ax - by)^3$).

These three planes define a degree 3 subscheme of a relative $\mathbb{P}^1$ bundle, which gives the trigonal curve over $\mathbb{P}^1_{\mathbb{C}}$.

5. The main construction in degree 3

Starting with a degree 3 cover $g : X \to B$ and an n-torsion line bundle $\mathscr{L}$, we now wish to generalize the degree 2 construction. Here, instead of getting a single degree n form, we get a pencil of degree n forms which is simultaneously diagonalized in each fiber. Said another way, we get a line on the 3-secant variety to the n-Veronese surface. (For higher degrees, we get a hyperplane on d-secant variety to the n-Veronese $d-1$-fold.)

(1) Start with $g : X \to \mathbb{P}(g_*\mathscr{L})$ which is an embedding into a relative $\mathbb{P}^2$ bundle.
(2) Compose with the n-veronese to get a map $X \to \mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}(\mathrm{Sym}^n(g_*\mathscr{L}))$.
(3) We have a surjection $\mathrm{Sym}^n(g_*\mathscr{L}) \to g_*\mathscr{L}^{\otimes n} \simeq g_*\mathscr{O}_X \to Q := \mathrm{coker}(\mathscr{O}_B \to g_*\mathscr{O}_X)$ corresponding to a plane P and a line M on that plane.
(4) In geometric fibers, the line M does not meet the three points in X, which lie on the n-Veronese. The plane is spanned by the 3-points, and so is a 3-secant plane to the n-Veronese surface, and the line M lies on that plane.

Example 5.1. Let's interpret this in terms of the classical recillas correspondence, which associates a degree 4 cover to a degree 3 cover with a 2-torsion line bundle.

Given a 2-torsion line bundle on a degree 3 cover, we get a line on a 3-secant plane to the 2-Veronese surface in $\mathbb{P}^5$. One can check that this line is general in the space of lines in $\mathbb{P}^5$, corresponding to a codimension 2 space of quadrics. (The argument here is that there is an 8-dimensional grassmannian and an 8-dimensional space of pairs (L, P) for L a line in a 3-secant 2-plane. A line cannot lie on 2 such planes because that would give 6-points on the 2-veronese spanning a $\mathbb{P}^3$, but the intersection of $\mathbb{P}^3$ with the 2-veronese is an intersection of 2 quadrics, so it is either a conic (not spanning $\mathbb{P}^3$) or 4 points, less than 6.)

This space of quadrics is dual to a pencil of quadrics, whose base locus is the associated degree 4 cover. To geometrically construct this pencil, we can proceed as follows. Consider the space of double lines in the 2-veronese. This corresponds to a 2-dimensional family of planes tangent to $\mathbb{P}^2$ along these lines. There will be 4 points in this family which contain the line L in the 3-secant 2-plane (since containing L is an intersection of 2 hyperplanes in the 2-Veronese of double lines). This gives 4 double lines on the 2-Veronese and 3 points (as the intersection). Dualizing we get 4 points and 3 lines, which gives four points in $\mathbb{G}_m \times \mathbb{G}_m$, corresponding to a torsor for the 2-torsion. One can also see this cohomologically. These 4 points specify the degree 4 cover, and there is a unique pencil of quadric through these 4 points. One can also obtain this pencil from the original 4 lines by pairing up the 6 intersection points of the lines, which correspond to the 3 reducible elements in the pencil of quadrics, and recovers the degree 3 cover via recillas, we can see this is the degree 3 cover in $\mathbb{G}_m \times \mathbb{G}_m$ since the lines spanned by $(\pm 1, \pm 1)$ meet at the three coordinate points.

6. GENUS 1 CURVES

Let's now return to see some alternate descriptions of $[\mathrm{Sec}_n \,/\, \mathrm{PGL}_2]$. exactly what these moduli spaces are parameterizing. It turns out they parameterize n-coverings of relative abelian varieties.

Here are two other related moduli stacks which are a bit easier to see a direct relation to.

Definition 6.1. Let $\mathscr{M}_{sing}^{(n)}$ denote the algebraic stack with

$$\begin{aligned}\mathscr{M}_{sing}^{(n)}(B) = (X, q, P) : X \to B &\text{ is a relative genus 1 curve,}\\ q : B \to X &\text{ is a point in the singular locus of } X \to B,\\ X \to P &\text{ is an embedding into a projective bundle}\end{aligned}$$

Definition 6.2. Let $\mathbb{F}_{n-2} := \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(n) \oplus \mathscr{O}(2))$, and let E denote the directrix (the unique effective class of negative self intersection) and F denote the fiber. Let $U \subset \mathbb{P}H^0(\mathbb{F}_{n-2}, E + nF)$ denote the open parameterizing smooth relative curves.

Proposition 6.3. *$[\mathrm{Sec}_n \,/\, \mathrm{PGL}_2]$ is equivalent to $\mathscr{M}_{\mathrm{sing}}^{(n)}$ and $[U/\,\mathrm{Aut}_{\mathbb{F}_{n-2}}]$.*

Proof. Given a point on the secant variety the the rational normal curve, we can project the rational normal curve from that point, and we get a genus 1 curve which is generically singular, and the preimage of the singularity is our original degree 2 cover.

To go from there to the divisor on the Hirzebruch surface, we can note that further projection from the singular point is the rational normal curve in $\mathbb{P}^{n-2}$. So the singular genus 1 curve lies on the cone over the rational normal curve, which ends up being a Hirzebruch surface. The blow up at the singular point defines the divisor. □

Remark 6.4. I call this stack the smile stack, and I hope it leaves you with a smile for the rest of the day.

REFERENCES