

# NOTES FOR TALK AT PIMS ON STABILIZATION OF HURWITZ SPACES IN THE GROTHENDIECK RING

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## 1. INTRODUCTION AND PROBLEM STATEMENT

In this talk, let's explore various ways to make sense of the following question.

**Question 1.1.** How many degree  $d$  covers does  $\mathbb{P}^1$  have?

In order to broaden our perspectives of how we can answer this question, it will be useful to introduce a moduli space for parameterizing degree  $d$  covers of  $\mathbb{P}^1$ .

**Definition 1.2.** Define the **Hurwitz space**  $\mathcal{H}_{d,g}$  to be the stack parameterizing smooth proper geometrically connected degree  $d$  covers  $C \rightarrow \mathbb{P}^1$  of genus  $g$  with Galois group  $S_d$ . Define the **simply branched Hurwitz space**  $\mathcal{H}_{d,g}^\circ$  to be the open substack parameterizing so that over every fiber of  $C \rightarrow \mathbb{P}^1$  is simply branched, meaning each fiber is either smooth or has exactly one ramification point of order 2.

**Remark 1.3.** There are formulas for the number of covers after stabilization, which are essentially the same in the finite field, number rings, and Grothendieck ring setting.

Finite field point counting	Number Rings	Cohomology	Grothendieck Ring	Chow Ring
How many covers of $\mathbb{P}_{\mathbb{F}_q}^1$ are there?	How many covers of $\text{Spec } \mathbb{Z}$ are there?	What is the cohomology of $\mathcal{H}_{d,g}$ ?	What is the class of $\mathcal{H}_{d,g}$ ?	What is the rational Chow ring of $\mathcal{H}_{d,g}$ ?
For $d = 3$ , stabilizes in $g$ , due to Datskovsky-Wright (over arbitrary function fields), $d \leq 5$ due to Bhargava Shankar Wang, error term in $d = 3$ due to Zhao, Gunther, $d = 4$ work in progress generalization by Hast, Gunther, Matei	Stabilizes when counted by discriminant $d \leq 5$ , due to Bhargava, and in $d = 3$ Davenport and Heilbronn	A basepointed variant of $\mathcal{H}_{d,g}^\circ$ stabilizes (though it is not known what it stabilizes to) in $g$ when $d = 3$ , Ellenberg-Venkatesh-Westerland, maybe $d = 4$	Stabilizes in $g$ for $d \leq 5$	Patel-Vakil stabilization of rational chow ring $d = 3$ , Deopurkar-Patel stabilization of rational Picard groups $d \leq 5$ Canning-Larson stabilization of rational Chow ring $d \leq 5$ .

TABLE 1. Interpretations of counting covers of  $\mathbb{P}^1$ 

## 2. THE TOPOLOGICAL CONNECTION

Let's next describe the actual formulas for the number of simply branched covers of  $\mathbb{P}^1$ . First, we recall the definition of the Grothendieck ring of varieties.

**Definition 2.1.** The Grothendieck ring of varieties  $K_0(\text{Var}_k)$  is the abelian group generated by isomorphism classes of finite type schemes over  $k$  so that for every such scheme  $X$  and open  $U \subset X$

$$[X] = [U] + [X - U].$$

We'll use  $\mathbb{L} := [\mathbb{A}^1]$ .

Let  $\text{Conf}^n(X)$  denote the configuration space of  $n$  points on  $X$ .

**Question 2.2.** For  $d \geq 2$ , and  $\text{char}(\mathbb{F}_q) \nmid d!$ ,

$$\lim_{g \rightarrow \infty} \frac{[\mathcal{H}_{d,g}^\circ]}{\mathbb{L}^{\dim \mathcal{H}_{d,g}^\circ}} = \frac{\text{Conf}^{2g+2d-2}}{\mathbb{L}^{\dim \mathcal{H}_{d,g}^\circ}} = 1 - \mathbb{L}^{-2}.$$

In case we run out of time before the talk ends, let me also record the conjecture for the number of points of the full Hurwitz space:

**Question 2.3.** For  $d \geq 2$ , and  $\text{char}(\mathbb{F}_q) \nmid d!$ ,

$$\lim_{g \rightarrow \infty} \frac{[\mathcal{H}_{d,g}]}{\mathbb{L}^{\dim \mathcal{H}_{d,g}}} = \frac{1}{1 - \mathbb{L}^{-1}} \left( \prod_{x \in \mathbb{P}^1} (1 - \mathbb{L}^{-1}) \left( \sum_{\lambda \vdash d} \mathbb{L}^{d-|\lambda|} \right) \right).$$

where  $|\lambda|$  is the number of parts of the partition  $\lambda$  (and can also be thought of geometrically as the degree of ramification of a corresponding scheme).

Here, the Euler product means: For  $\mathcal{A} = (a_i)_{i \in I}$  a collection of classes in  $\mathbf{K}_0(\text{Var}_k)$  over  $X$ , define the *motivic Euler product*

$$(2.1) \quad \prod_{x \in X/S} \left( 1 + \sum_{i \in I} a_{i,x} t_i \right) := \sum_{\mu \in \mathcal{P}(I)} C_{X/S}^\mu(\mathcal{A}) \mathbf{t}^\mu \in \mathbf{K}_0(\text{Var}_k)[(t_i)_{i \in I}],$$

where  $C_{X/S}^\mu(\mathcal{A})$  is the  $\mu$  configuration space (associated to a tuple of points  $\mu$  on  $X$  a generalized partition (marking collections of points by colors in  $X$ ). When  $\mathcal{A}$  is effective, this is class of the usual  $\mu$  configuration space. When  $\mathcal{A}$  is a general (not necessarily effective) class, one defines  $\text{Sym}^\bullet(\mathcal{A})$ , which is the unique multiplicative extension of  $\text{Sym}^\bullet$  to the Grothendieck ring, i.e.,  $\text{Sym}(ab) = \text{Sym}(a) \text{Sym}(b)$  with  $\text{Sym}(a) = \sum_n \sum^n(a) t^n$ . From this, one can define  $\text{Sym}^\mu = \prod_{i \in I} \text{Sym}^{m_i}(a)$  and then  $C_{X/S}^\mu$  as the restriction to the open locus in  $U \subset \text{Sym}^\mu(X)$  where the points in  $a$  map to distinct points of  $X$ .

More generally, for  $\mathcal{X} = (X_i)_{i \in I}$  a collection of varieties  $X_i$  over  $X$ , and  $\mu = (m_i)_{i \in I}$  a generalized partition, define  $C_{X/S}(\mathcal{X})$  as the preimage of  $C_{X/S}^\mu \subset \prod_{i \in I} \text{Sym}^{m_i}(X)$  under the projection  $\prod_{i \in I} \text{Sym}^{m_i} X_i \rightarrow \prod_{i \in I} \text{Sym}^{m_i}(X)$ . As in [BH21, Defn. 6.1.7], one can extend this definition to make sense of  $C_{X/S}^\mu(\mathcal{A})$  as an element of  $\mathbf{K}_0(\text{Var}_k)$  where  $\mathcal{A} = (a_i)_{i \in I}$  with  $a_i$  in  $\mathbf{K}_0(\text{Var}_k)$  over  $X$ .

We will work in a modified version of the Grothendieck ring where

- (1) We invert universally bijective morphisms
- (2) We invert  $\mathbb{L}$ .
- (3) We complete along the dimension filtration, i.e., a sequence of classes approaches 0 if its dimension approaches  $-\infty$ .

**Theorem 2.4** (L-Vakil-Wood). *The the above questions indeed hold true for  $d \leq 5$ .*

**Remark 2.5.** The reason for the “degree 5” barrier is that there are explicit parameterizations of covers of degree up to 5. For example, degree 2 covers can be essentially specified by their branch points. Degree 3 covers always live inside a  $\mathbb{P}^1$  bundle. Degree 4 covers are an intersection of 2 quadrics inside a  $\mathbb{P}^2$  bundle and degree 5 covers can be described as a vanishing locus of Pfaffians inside a  $\mathbb{P}^3$  bundle.

**Question 2.6.** Should we expect any of these to continue in degree 6?

## 3. A BRIEF OUTLINE OF THE PROOF

**Fact 3.1.** • Every degree 3 cover  $C \rightarrow \mathbb{P}^1$  has a canonical embedding

$$(3.1) \quad \begin{array}{ccc} C & \xrightarrow{\iota} & \mathbb{P}^{\mathcal{E}} \\ & \searrow & \swarrow \pi \\ & \mathbb{P}^1 & \end{array}$$

for  $\mathcal{E}$  a degree  $g + 2$  rank 2 vector bundle on  $\mathbb{P}^1$  such that  $C$  lies in a particular linear system  $\mathcal{L}_{\mathcal{E}}$  on  $\mathbb{P}^{\mathcal{E}}$ , only depending on the genus of  $C$ .

- If  $\alpha, \beta \in H^0(\mathbb{P}^{\mathcal{E}}, \mathcal{L}_{\mathcal{E}})$  are two sections with vanishing locus isomorphic to  $C$ , then there is some automorphism of  $\mathcal{E}$  taking  $\alpha$  to  $\beta$ .

Using the above, we can count all trigonal curves by

- (1) Summing the above linear systems over different canonical bundles  $\mathcal{E}$
- (2) Dividing by the automorphism groups of the bundles
- (3) Multiplying by the proportion of smooth sections in each such linear system

The trickiest step is to make sense of the proportion of smooth sections. This typically has two steps

- (1) Compute the class of sections singular in a fiber (over the dual numbers) at a point
- (2) “Multiply these probabilities together” using a motivic Euler product (aka inclusion-exclusion).

## 4. COUNTING THE NUMBER OF SQUAREFREE POLYNOMIALS

Let’s now understand the machinery Ravi and Melanie developed for counting the class of squarefree polynomials, and more generally the class of smooth hyperplane sections, in the Grothendieck ring. To make sense of the formulas we’ll write in the rest of the talk, we’ll invert the class of  $[\mathbb{A}^1] =: \mathbb{L}$ . We’ll also invert  $\mathbb{L}^n - 1$  for all  $n$  and work in the “dimension filtration” meaning that we will say two sequences converge if their difference lives in high codimension. This was implicit above.

Jordan described one way to count squarefree polynomials using cohomology, which he said is the “hardest” way. We’ll now describe another way, which might be considered slightly easier.

First, let's recall the setup over finite fields. Recall that the Zeta function of a scheme  $X$  satisfies

$$\frac{1}{Z_X(q^{-s})} = \prod_{x \in X \text{ closed}} \frac{1}{1 - q^{-s}} = \prod_{x \in X \text{ closed}} \left( \sum_{n \geq 0} q^{ns} \right) = \sum_{n \geq 0} \# \text{Sym}_{\mathbb{P}^1}^n t^n.$$

**Lemma 4.1.** *The*

$$\lim_{d \rightarrow \infty} \frac{\# \text{smooth degree } d \text{ homogeneous polynomials in 2 variables defining a smooth divisor on } \mathbb{P}^1}{\text{all degree } d \text{ polys in 2 variables}}$$

$$= \frac{1}{Z_X(q^{-2})}$$

*Proof.* Idea 1 This is the idea described by Wei in her first talk: The polynomial  $f(s, t)$  of degree  $d$  is singular at  $s = 0$  in  $\mathbb{P}^1$  when the coefficients of  $t^d$  and  $t^{d-1}s$  both vanish. And so at 0, there is a  $1/q^2$  chance that the polynomial is singular. In general, the chance it is singular at  $x$  is  $1/q_x^2$  where  $q_x := \#\kappa(x)$ . So the chance of smoothness is  $1 - 1/q_x^2$ . Then, after much work, you can show the chances of smoothness at different points are independent, so the chance a hypersurface is smooth is  $\prod_{x \in \mathbb{P}^1} (1 - q_x^{-2}) = \frac{1}{Z_{\mathbb{P}^1}(q^{-2})}$ .

Idea 2 Try to do inclusion exclusion on the points. The smooth polynomials are all polynomials, minus those smooth as some points, but then you add back those singular at two points, and so on. □

The first idea isn't really feasible in the Grothendieck ring, but the second is. But the first should guide our intuition.

**Definition 4.2.** The motivic Zeta function of a variety  $X$  is  $Z_X(t) := \sum_n \text{Sym}_X^n t^n$ .

**Example 4.3.** We have  $Z_{\mathbb{P}^1}(t) = Z_{pt}(t) \cdot Z_{\mathbb{A}^1}(t) = \frac{1}{1-t} \frac{1}{1-\mathbb{L}t}$ .

**Theorem 4.4.**

$$\lim_{d \rightarrow \infty} \frac{[\text{smooth degree } d \text{ homogeneous polynomials in 2 variables defining a smooth divisor on } \mathbb{P}^1]}{[\text{all degree } d \text{ polys in 2 variables}]}$$

$$= \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})}.$$

*Sketch.* Let's follow our nose, and try to compute how many smooth sections there are. Let  $H^0(\mathbb{P}^1, \mathcal{O}(n))$  denote the vector space of degree  $n$  homogeneous polynomials in  $s, t$ .

**Definition 4.5** (By example). Let  $S_{2,2,1}$  denote the class of polynomials together with two marked points the first color, 2 marked points of the second

color, and 1 marked point of the third color 3 so that  $\mathbb{P}^1$  is singular at these  $2 + 2 + 1$  points and nowhere else.

Let  $S_{\geq,2,2,1}$  denote the class of polynomials together with two marked points the first color, 2 marked points of the second color, and 1 marked point of the third color 3 so that  $\mathbb{P}^1$  is singular at these  $2 + 2 + 1$  points and possibly elsewhere.

Now, let's try to do inclusion exclusion.

$$[\text{smooth polynomials}] = [\text{all polynomials}] - (S_1 + S_2 + S_3 + \cdots)$$

Then, doing this again

$$S_1 = S_{\geq,1} - S_{1,1} - S_{1,2} + \cdots$$

Similarly, we can write

$$S_2 = S_{\geq,2} - S_{2,1} + \cdots$$

When one combines this into an inclusion exclusion, one finds

smooth polynomials

$$= \text{all polynomials} - S_{\geq,1} - S_{\geq,2} + S_{\geq,1,1} - S_{\geq,3} + S_{\geq,2,1} - S_{\geq,1,1,1} + S_{\geq,1,2} + \cdots$$

Now the total space of polynomials of degree  $n$  has class  $\mathbb{L}^n$ , and the polynomials singular at a specific marked point has class  $\mathbb{L}^{n-2}$ . So, for example, for  $n$  large, the class  $S_{\geq,2,2,1} = \text{Conf}^{2,2,1}(\mathbb{P}^1) \times \mathbb{L}^{n-2(2+2+1)}$  where  $\text{Conf}^{2,2,1}$  means configurations of points where 2 are marked a first color, 2 are marked a second color, and 1 is marked a third color.

If we plug in the above formula, we find

$$\begin{aligned} & [\text{smooth polynomials}] \\ &= \sum_{\lambda \text{ partitions}} (-1)^{\text{number of colors}} \text{Conf}^\lambda(\mathbb{P}^1) \times \mathbb{L}^{n+1-2(\text{number of points})} \\ &= \sum_{\lambda \text{ partitions}} (-1)^{\text{number of colors}} \text{Sym}^\lambda(\mathbb{P}^1) \times \mathbb{L}^{n+1-2(\text{number of points})} \\ &= \frac{\mathbb{L}^{n+1}}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} \end{aligned}$$

The second equality is a tricky combinatorial lemma. The last equality follows from expanding a power series [probably skip the proof here]

$$\begin{aligned} & \frac{1}{Z_{\mathbb{P}^1}(t)} \\ &= \frac{1}{\sum_k \text{Sym}^k(\mathbb{P}^1) t^k} \\ &= \sum_m \left( - \sum_k [\text{Sym}^k(\mathbb{P}^1)] t^k \right)^m \\ &= \sum_{\lambda \text{ partitions}} (-1)^{\text{number of colors}} \text{Sym}^\lambda(\mathbb{P}^1) \times t^{(\text{number of points})} \end{aligned}$$

□

**Remark 4.6.** Important remark: We should think of

$$\frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} = \prod_{x \in \mathbb{P}^1} (1 - \mathbb{L}^{-2})$$

as finding the global probability of smoothness by extrapolating from independence of the local probabilities at each point. This is some sort of “infinitesimal gluing.”

## 5. PROOF SKETCH OF THE CLASS OF COVERS OF $\mathbb{P}^1$

For the rest of the talk, we’ll work over a fixed field  $k$  of characteristic at least 5.

**5.1. Warm up: counting hyperelliptic curves.** Since  $\dim \mathcal{H}_{2,g} = 2g + 2$ , we’d like to show class of hyperelliptic curves of genus  $g$  over a field  $k$  of characteristic not 2 is  $\mathbb{L}^{2g+2} - \mathbb{L}^{2g}$ . A hyperelliptic curve is essentially specified by its branch points.

So, morally, the class  $[\mathcal{H}_{2,g}]$  should equal  $[\text{Conf}^{2g+2}]$ , since  $2g + 2$  is the number of branch points by Riemann Hurwitz. Since two sections define the

same divisor exactly when they differ by scaling, and  $[\mathbf{G}_m] = [\mathbb{L} - 1]$ ,

$$\begin{aligned}
\lim_{g \rightarrow \infty} [\mathcal{H}_{2,g}] &= \lim_{g \rightarrow \infty} [\text{Conf}^{2g+2}] \\
&= \frac{[\text{smooth sections of } H^0(\mathbb{P}^1, \mathcal{O}(2g+2))]}{\mathbb{L} - 1} \\
&= \frac{1}{\mathbb{L} - 1} [\text{all sections}] [\text{proportion of smooth sections}] \\
&= \frac{1}{\mathbb{L} - 1} \mathbb{L}^{2g+3} \prod_{x \in \mathbb{P}^1} [\text{proportion of smooth sections smooth at } x] \\
&= \frac{1}{\mathbb{L} - 1} \mathbb{L}^{2g+3} \prod_{x \in \mathbb{P}^1} (1 - \mathbb{L}^{-2}) \\
&= \frac{1}{\mathbb{L} - 1} \mathbb{L}^{2g+3} \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} \\
&= \frac{1}{\mathbb{L} - 1} \mathbb{L}^{2g+3} (1 - \mathbb{L}^{-1}) (1 - \mathbb{L}^{-2}) \\
&= \mathbb{L}^{2g+2} (1 - \mathbb{L}^{-2}).
\end{aligned}$$

**5.2. The degree 3 case.** The degree 3 case follows a similar Euler product idea, but it is significantly more involved. We essentially follow Joe Gunther's outline (though he does the more general case that the base is  $\mathbb{P}^1$ ). Again, we work over finite fields, but everything can be made to work in the Grothendieck ring.

Following our outline in the degree 2 case, we first need a description of trigonal curves analogous to that of hyperelliptic curves in terms of branched covers. Here, trigonal curves are not uniquely determined by their branch data.

**Fact 5.1.**      • Every degree 3 cover  $C \rightarrow \mathbb{P}^1$  has a canonical embedding

$$(5.1) \quad \begin{array}{ccc} C & \xrightarrow{\iota} & \mathbb{P}^{\mathcal{E}} \\ & \searrow & \swarrow \pi \\ & & \mathbb{P}^1 \end{array}$$

for  $\mathcal{E}$  a degree  $g + 2$  rank 2 vector bundle on  $\mathbb{P}^1$  such that  $C$  lies in a particular linear system  $\mathcal{L}_{\mathcal{E}}$  on  $\mathbb{P}^{\mathcal{E}}$ , only depending on the genus of  $C$ .

- If  $\alpha, \beta \in H^0(\mathbb{P}^{\mathcal{E}}, \mathcal{L}_{\mathcal{E}})$  are two sections with vanishing locus isomorphic to  $C$ , then there is some automorphism of  $\mathcal{E}$  taking  $\alpha$  to  $\beta$ .
- Further,  $\dim H^0(\mathbb{P}^{\mathcal{E}}, \mathcal{L}_{\mathcal{E}}) = \dim \mathcal{H}_{3,g} + 4$ .

**Remark 5.2** (Skip). The bundle  $\mathcal{E} \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$  for  $a + b = g + 2$ . Further,  $C$  is concretely cut out by the equation

$$r_1(s, t)x^3 + r_2(s, t)x^2y + r_3(s, t)xy^2 + r_4(s, t)y^3$$

with  $\deg r_i = (2 - i)a + (-1 + i)(g + 2 - a)$ .

**Remark 5.3** (Skip). The canonical embedding can be described as follows. Explicitly, if  $\rho : T \rightarrow \mathbb{P}^1$  is a trigonal curve of genus  $g$ , then  $T$  has a canonical embedding into  $\mathbb{P}(\rho_*\mathcal{O}_T/\mathcal{O}_{\mathbb{P}^1})^\vee$ . One finds that the bundle  $\rho_*\mathcal{O}_T/\mathcal{O}_{\mathbb{P}^1}$  has degree  $g + 2$ , and so by the classification of projective bundles over  $\mathbb{P}^1$ , every such surface is isomorphic to  $X_a := \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(g + 2 - a))$  for some integer  $a$ .

In order to count all trigonal curves, we will also need to calculate the “local factors.” The local factors are described by counting the proportion of sections of  $a(s, t) + b(s, t)\varepsilon \in H^0(\mathbb{P}_D^1, \mathcal{O}_D(3))$  (for  $a(s, t), b(s, t)$  degree 3 polynomials) defining subschemes with 1-dimensional tangent spaces.

**Lemma 5.4.** *The proportion of sections  $a(s, t) + b(s, t)\varepsilon \in H^0(\mathbb{P}_D^1, \mathcal{O}_D(3))$  defining subschemes with 1-dimensional tangent spaces is*

$$(1 + \mathbb{L}^{-1} + \mathbb{L}^{-2})(1 - \mathbb{L}^{-1}) \frac{[\mathrm{PGL}_2]}{\mathbb{L}^3}.$$

*The proportion of sections which are restrictions of simply branched curves is  $(1 + \mathbb{L}^{-1})(1 - \mathbb{L}^{-1}) \frac{[\mathrm{PGL}_2]}{\mathbb{L}^3}$ .*

*Proof.* We can split this up by ramification type. After quotienting by a factor of  $(\mathbb{L} - 1)$ , we can count subschemes of  $\mathbb{P}_D^1$  in place of sections (this is the factor of  $(1 - \mathbb{L}^{-1})$ ). In fact, the number of subschemes corresponding to unramified covers is  $\mathbb{L}^3[\mathrm{PGL}_2]$ , the number of simply ramified subschemes is  $\frac{\mathbb{L}^3}{\mathbb{L}}[\mathrm{PGL}_2]$ , and the number of subschemes corresponding to covers with ramification order 3 is  $\frac{\mathbb{L}^3}{\mathbb{L}^2}[\mathrm{PGL}_2]$ .

The other ramification profiles are similar. □

With the above tools in hand, we can now compute the number of trigonal curves.

$$\begin{aligned}
& \lim_{g \rightarrow \infty} [\mathcal{H}_{3,g}] \\
&= \lim_{g \rightarrow \infty} \sum_{\text{bundles } \mathcal{E} \text{ of degree } g+2} \frac{1}{[\text{Aut } \mathcal{E}]} [\text{smooth section of } \mathcal{L}_{\mathcal{E}} \text{ on } \mathbb{P}^{\mathcal{E}}] \\
&= \sum_{\text{bundles } \mathcal{E} \text{ of degree } g+2} \frac{1}{[\text{Aut } \mathcal{E}]} [H^0(\mathbb{P}^{\mathcal{E}}, \mathcal{L}_{\mathcal{E}})] [\text{probability that a section is smooth}] \\
&= \sum_{\text{bundles } \mathcal{E} \text{ of degree } g+2} \frac{1}{[\text{Aut } \mathcal{E}]} [H^0(\mathbb{P}^{\mathcal{E}}, \mathcal{L}_{\mathcal{E}})] \\
&\quad \left( \text{chance a section is smooth to first order at a geometric point } x \in \mathbb{P}^1 \right)^{\mathbb{P}^1} \\
&= \sum_{\text{bundles } \mathcal{E} \text{ of degree } g+2} \frac{1}{[\text{Aut } \mathcal{E}]} \mathbb{L}^{\dim \mathcal{H}_{3,g}+4} (1 + \mathbb{L}^{-1} + \mathbb{L}^{-2})(1 - \mathbb{L}^{-1}) \frac{[\text{SL}_2]}{\mathbb{L}^3} \\
&= \mathbb{L}^{\dim \mathcal{H}_{3,g}+4} \frac{1}{\mathbb{L} - 1} \left( \sum_{\text{projective bundles } \mathcal{E} \text{ of degree } g+2} \frac{1}{[\text{Aut } \mathbb{P}^{\mathcal{E}}]} \right) \left( \frac{[\text{SL}_2]}{\mathbb{L}^3} \right)^{\mathbb{P}^1} \\
&\quad \left( (1 + \mathbb{L}^{-1} + \mathbb{L}^{-2})(1 - \mathbb{L}^{-1}) \right)^{\mathbb{P}^1} \\
&= \mathbb{L}^{\dim \mathcal{H}_{3,g}+4} \mathbb{L}^{-3} \frac{1}{\mathbb{L} - 1} \left( (1 + \mathbb{L}^{-1} + \mathbb{L}^{-2})(1 - \mathbb{L}^{-1}) \right)^{\mathbb{P}^1} \\
&= \mathbb{L}^{\dim \mathcal{H}_{3,g}} \frac{1}{1 - \mathbb{L}^{-1}} \left( (1 + \mathbb{L}^{-1} + \mathbb{L}^{-2})(1 - \mathbb{L}^{-1}) \right)^{\mathbb{P}^1}
\end{aligned}$$

This was the prediction. Above the cancellation follows from the computation of the Tamagawa number of  $\text{SL}_2$ .

**Remark 5.5.** In fact, the cancellation can be viewed topologically. Then, the cancellation above amounts to

$$\frac{\sum_{\text{SL}_2 \text{ bundles}} \frac{1}{[\text{Aut } \mathcal{E}]} }{\mathbb{L}^{-\dim \text{SL}_2}} = \left( \frac{\mathbb{L}^{\dim \text{SL}_2}}{[\text{SL}_2]} \right)^{\mathbb{P}^1}.$$

Since  $[\text{Bun } \text{SL}_2(\mathbb{P}^1)] = \sum_{\text{SL}_2 \text{ bundles}} \frac{1}{[\text{Aut } \mathcal{E}]}$ , the above can be restated as

$$\frac{[\text{Bun}_{\text{SL}_2}(\mathbb{P}^1)]}{\mathbb{L}^{\dim \text{Bun}_{\text{SL}_2}(\mathbb{P}^1)}} = \left( \frac{[\text{Bun}_{\text{SL}_2}(x)]}{\mathbb{L}^{\dim \text{Bun}_{\text{SL}_2}(x)}} \right)^{\mathbb{P}^1}.$$

In fact, Gaitsgory and Lurie are able to prove this formula by relating this point counting to cohomology via the Lefschetz Trace formula, and then developing the theory of factorization homology, to show that in a certain sense, the homology of the left hand side “factors” as a product of the homologies of the right hand side.

**Remark 5.6.** If instead of counting all degree 3 covers, we only counted simply branched ones, the formula would then be

$$\begin{aligned}
 [\mathcal{H}_{3,g}] &= \mathbb{L}^{\dim \mathcal{H}_{3,g}} \frac{1}{1 - \mathbb{L}^{-1}} \left(1 - \mathbb{L}^{-2}\right)^{\mathbb{P}^1} \\
 &= \mathbb{L}^{\dim \mathcal{H}_{3,g}} \frac{1}{1 - \mathbb{L}^{-1}} \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} \\
 &= \mathbb{L}^{\dim \mathcal{H}_{3,g}} \frac{1}{1 - \mathbb{L}^{-1}} (1 - \mathbb{L}^{-1})(1 - \mathbb{L}^{-2}) \\
 &= \mathbb{L}^{\dim \mathcal{H}_{3,g}} (1 - \mathbb{L}^{-2}).
 \end{aligned}$$

## 6. COUNTING ALL COVERS, AND THE LOCAL FACTORS

In general, we don’t have much traction on proving the conjecture for degree  $> 5$ , since we don’t know the explicit parameterizations of curves. However, analogously to Bhargava’s Mass formula, we are able to “compute the local probabilities” and use this to make the conjecture above.

In fact, the local probabilities can be uniformly computed in the Grothendieck ring. To understand the local probabilities, it turns out that every curve of degree  $d$  can be canonically embedded in a  $\mathbb{P}^{d-2}$  bundle over  $\mathbb{P}^1$ .

**Theorem 6.1.** *Fix a type of ramification, corresponding to a partition of  $d$  with  $d - k$  parts. Then, the class of the Hilbert scheme parameterizing infinitesimal schemes which are possible restrictions of curves in the Hurwitz stack under the canonical embedding in  $\mathbb{P}_D^{d-2}$  is  $[\mathrm{SL}_{d-1}] \mathbb{L}^{\dim \mathrm{SL}_{d-1} - k}$ .*

The proof is somewhat tricky, though not overly so, and boils down to the computation of certain classes  $BG$ , for  $G$  certain affine disconnected algebraic groups. This may be expected, since the number field case amounted to Serre’s mass formula. From this, one can deduce the point count version. Then, linearizing the line bundle (which introduces a factor of  $\mathbb{L} - 1$ ), taking the product over all primes  $x \in \mathbb{P}^1$ , and then quotienting out by the linearization yields the conjectured formula

$$\lim_{g \rightarrow \infty} \frac{[\mathcal{H}_{d,g}]}{\mathbb{L}^{\dim \mathcal{H}_{d,g}^\circ}} = \frac{1}{1 - \mathbb{L}^{-1}} \left( (1 - \mathbb{L}^{-1}) \left( \sum_{\lambda \vdash d} \mathbb{L}^{d-|\lambda|} \right) \right)^{\mathbb{P}^1}.$$

Again, the  $\mathrm{SL}_{d-1}$  factors will “cancel out” using the Tamagawa number formalism.

In the case of simple branching. We can replace the sum with local factors that are just  $1 - \mathbb{L}^{-2}$ , and this yields

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{[\mathcal{H}_{d,g}^\circ]}{\mathbb{L}^{\dim \mathcal{H}_{d,g}^\circ}} &= \frac{1}{1 - \mathbb{L}^{-1}} (1 - \mathbb{L}^{-2})^{\mathbb{P}^1} \\ &= \frac{1}{1 - \mathbb{L}^{-1}} \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} \\ &= \frac{1}{1 - \mathbb{L}^{-1}} (1 - \mathbb{L}^{-1}) (1 - \mathbb{L}^{-2}) \\ &= 1 - \mathbb{L}^{-2}. \end{aligned}$$

#### REFERENCES

- [BH21] Margaret Bilu and Sean Howe. Motivic Euler products in motivic statistics. *Algebra Number Theory*, 15(9):2195–2259, 2021.