A thesis of minimal degree: two

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

In this thesis, we study geometry and moduli spaces associated to low degree covers. There are two parts to this thesis. The first part comprises the bulk of the thesis and focuses on covers of degree 2. It is motivated by the Cohen-Lenstra heuristics in number theory. The second part contains three topics. The first relates to certain low degree covers of moduli spaces of elliptic curves, the second topic relates to heights on elliptic curves in characteristic 3, and the third part relates to the Casnati-Ekedahl structure theorems for covers. We now describe these two parts in some more detail. We have chosen to include some less technical stories describing how we came across the topics appearing in this thesis in §1.1, §8.1, §9.1, §10.1.

Part I

In the first part, our objective is to gain a better geometric understanding of n-torsion line bundles on degree 2 covers of a base scheme. We are especially interested in base schemes such as $\operatorname{Spec} \mathbb{Z}$ or $\mathbb{P}^1_{\mathbb{F}_q}$, where our study is intimately tied with the Cohen-Lenstra heuristics, predicting the distribution of torsion in class groups of quadratic fields. We study these by constructing a moduli stack which approximately parameterizes these objects. Rather, the construction we give parameterizes n coverings of families of singular genus 1 curves, which we are then able to relate to n-torsion on degree 2 covers in cases of arithmetic interest. The main result is Theorem 1.2.6, describing the moduli space of n-coverings of singular genus 1 curves. From this we deduce as special cases our other main results Theorem 1.2.5 and Theorem 1.2.4.

Part II

The second part of the thesis features a number of related topics investigating low degree covers and elliptic curve in algebraic and arithmetic geometry.

We first investigate several special covers of moduli spaces of elliptic curves. These relate classical constructions in algebraic geometry to the geometry of elliptic curves. For example, one construction relates the Recillas construction used in solving the quartic equation to elements of 2-Selmer groups of elliptic curves. We then describe analogous classical geometric constructions relating to special automorphisms of genus 1 curves, and connect them in other ways to Selmer elements in elliptic curves.

Second, we examine stacky heights on the moduli stack of elliptic curves in characteristic 3. It is known that in characteristic greater than 3, the hodge bundle induces a stacky height which agrees with the usual notion of (Faltings) height on elliptic curves. However, we show that there is no height function coming from a vector bundle in characteristic 3 which induces the usual height function. We do, however, show that there exist vector bundles inducing a Northcott height function. That is, there are only finitely many elliptic curves of bounded height over $\mathbb{F}_3(t)$ with respect to this vector bundle.

Finally, we probe structure theorems on covers due to Casnati-Ekedahl. After reviewing the results of Casnati-Ekedahl with some minor upgrades, we investigate a natural pairing on the terms in the Casnati-Ekedahl resolution, with a particular eye toward understanding covers of degree 6. We also give a description of an open in the stack of degree d covers as a partial compactification of the classifying stack BS_d. This whole chapter is joint with Ravi Vakil, while § 10.2 and § 10.3 are also joint with Melanie Wood.

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Contents

Part I

Degree 2 Covers

Chapter 1

Introduction

In this first part of the thesis, we investigate moduli spaces which approximately parameterize n-torsion line bundles on degree 2 covers. Our motivation stems from the Cohen-Lenstra heuristics, which are a collection of important conjectures in number theory, predicting the distributions of class groups of quadratic fields. After telling a story about how we came to investigate these moduli spaces, we review the history on the question of counting n-torsion elements in class group of quadratic fields.

1.1 Story of the project

I would like to take some space to tell the story of some the twist and turns I faced when working through the results of this part.

My investigations into arithmetic statistics began several years ago in relation to Selmer groups of elliptic curves, see § 8.1. When I was investigating these Selmer groups, I noticed there should be a close connection between the classical triality in the automorphisms of the Dynkin diagram of D_4, determining the second moment of 2-Selmer groups, and counting 2-torsion in the class groups of degree 4 extensions given by binary quartic forms. I mentioned this relation to Alex Smith, who offered me $2 if I were able to compute the average size of the second moment of the 2-Selmer group of elliptic curves over $\mathbb{Q}$, and that motivated me to look further into this phenomenon.

Upon delving further, via conversations with Anand Patel, we found a beautiful description of 2-torsion in extensions of binary quartic forms in terms of certain relative nets of quadrics in $\mathbb{P}^3_{\mathbb{Z}}$ over $\operatorname{Spec} \mathbb{Z}$ (or any base) which in each geometric fiber were simultaneously diagonalizable. The three quadrics generating the net gave the relation to triality mentioned above. Upon listening to a talk by Melanie Wood, I realized this description generalized: Relative dimension $d - 1$ vector spaces of degree n polynomials on a relative projective bundle of dimension $d-1$, which were simultaneously diagonalizable in each fiber could be related to counting n-torsion in degree d covers. As a particular case, taking $d = 2$, I realized many of the ideas simplified, and so one could relate the question of counting n-torsion in class groups of relative degree 2 extensions to degree n polynomials in $\mathbb{P}^1$ bundles over the base. Then the base is Spec $\mathbb{Z}$, these degree n polynomials are simply degree n binary forms. Of course, this is closely related to the Cohen-Lenstra heuristics. Up to this point, the above picture gives a way of describing n torsion of class groups in degree d number fields, but the condition the degree n polynomials are simultaneously diagonalizable can be quite difficult to parameterize. However, in degree 2, one simply needs to specify a single polynomial, and so the simultaneous diagonalization condition is vacuous.

When I discovered the above, I was extremely excited, and thought that in order to prove versions of the Cohen-Lenstra heuristics, I simply would have to learn Bhargava's averaging techniques. Last Spring, I ran an online seminar. As part of this seminar, I carefully went through the methods Bhargava introduced in the context of counting cubic number fields.

As I tried to apply Bhargava's methods to my idea above, I discovered there was an additional open condition, which in the context of quadratic fields translates to the condition that the resultant of the quadratic form q and degree n form ξ is a unit. Unfortunately, bounding field valued points in an open subscheme of affine space is typically not too difficult, but counting $\mathbb{Z}$ points when that open subscheme has complement of codimension 1 can be quite tricky. In our context, it translates to counting points on the "resultant = 1 hypersurface."

Nevertheless, even though I was unable to count points on this resultant 1 hypersurface, I thought I had found a workaround sufficient to bound the total number of n-torsion elements in class groups of quadratic fields of discriminant at most Y by $O(Y^{5/4})$, which is weaker than $O(Y)$ predicted by the Cohen-Lenstra heuristics, but stronger than previously known bounds for $n > 5$. I had written up a draft, and sent it to some experts. Fortunately, Arul Shankar and Jacob Tsimerman quickly responded to my draft, pointing out a serious issue in my argument, which I have been unable to fix. Nevertheless, the geometric description for n-torsion in class groups in quadratic fields I present here remains a new perspective which we hope will be useful for arithmetic statisticians.

It would also be quite interesting to work out the details of how "simultaneously diagonalized linear systems of n-ics" mentioned above relate to n-torsion in class groups of number fields, but unfortunately we have not had the time to do so.

1.2 Introduction to moduli spaces of degree 2 covers

1.2.1 Background on the Cohen-Lenstra heuristics

The study of class groups of quadratic fields dates back to Gauss in 1801, who discovered a composition law taking in two primitive quadratic forms of discriminant d and producing a new quadratic form of discriminant d, up to a certain equivalence relation [Gau66]. In modern parlance, we recognize this composition law as the multiplication law in the narrow class group of the quadratic algebra of discriminant d. In order to better understand class groups and this composition law, mathematicians were led to wonder how many quadratic forms there are whose nth power under this law is trivial.

Heuristics on the average size of the n-torsion in class group of quadratic fields were originally formulated by Cohen and Lenstra in [CL84a, CL84b]. Since [CL84a, CL84b], these heuristics have been vastly refined, but an important consequence is the following:

Conjecture 1.2.2 (Cohen-Lenstra, [CL84a, §4 c),§5 c)])**.** For n an odd positive integer, the average size of $\mathrm{Cl}(K)[n]$ over imaginary quadratic fields is $\sum_{m|n} 1$. The average size of $\mathrm{Cl}(K)[n]$ over real quadratic fields is $\sum_{m|n} \frac{1}{m}$. More formally,

$$\lim_{Y\to\infty} \frac{\sum_{\substack{\text{imaginary quadratic } K \\ |\operatorname{disc}(K/\mathbb{Q})|<Y}} \#\mathrm{Cl}(K)[n]}{\#\{\text{imaginary quadratic } K : |\operatorname{disc}(K/\mathbb{Q})| < Y\}} = \sum_{m|n} 1 \qquad \text{and}$$

$$\lim_{Y\to\infty} \frac{\sum_{\substack{\text{real quadratic } K \\ |\operatorname{disc}(K/\mathbb{Q})|<Y}} \#\mathrm{Cl}(K)[n]}{\#\{\text{real quadratic } K : |\operatorname{disc}(K/\mathbb{Q})| < Y\}} = \sum_{m|n} \frac{1}{m}.$$

In particular, because the number of imaginary quadratic fields of discriminant at most Y is $O(Y)$,

$$\lim_{Y\to\infty} \sum_{\substack{\text{quadratic } K \\ |\operatorname{disc}(K/\mathbb{Q})|<Y}} \#\mathrm{Cl}(K)[n] = O(Y). \tag{1.2.1}$$

So far, very little is known about Conjecture 1.2.2 and its generalizations, but each solved case has come from tremendous breakthroughs in arithmetic statistics. The case $n = 3$ was proven in the pioneering work of Davenport and Heilbronn [DH71]. In [Bha05], Bhargava determined the average size of 2-torsion in class groups of cubic fields, which was the beginning of the work leading to his Fields Medal. In [FK07], Fouvry and Klüners proved an extension of Conjecture 1.2.2 to the case $n = 4$. Another variant of the $n = 4$ case was demonstrated in work of Altug, Shankar, Varma, and Wilson [ASVW17]. In groundbreaking recent work [Smi17, Theorem 1.4], Smith verified an extension to n an arbitrary power of 2 for imaginary quadratic fields. In a separate direction, over function fields of the form $\mathbb{F}_q(t)$, Ellenberg, Venkatesh, and Westerland established a version of Conjecture 1.2.2 in a $q \to \infty$ limit [EVW16]. This builds on work of Achter [Ach08, Ach06] and Garton [Gar15] which prove results over function fields in a different $q \to \infty$ limit. Little else is known about Conjecture 1.2.2 for odd n with $n > 3$.

It is not too difficult to see the left hand side of (1.2.1) is bounded by $O(Y^{3/2+\varepsilon})$ for any $\varepsilon > 0$. For example, this may be verified using the class number formula and a bound on the value of the corresponding L function. With a little more work, one may even obtain the bound $O(Y^{3/2})$ via a geometry of numbers style argument for counting quadratic forms, following the general averaging method introduced by Bhargava in [Bha05]. For general n, there are only comparably minor improvements to this $O(Y^{3/2})$ bound. For example, Heath-Brown and Pierce [HBP17, Theorem 1.1] give the improvement that when n is a prime, the left hand side of (1.2.1) summed only over imaginary quadratic fields is bounded by $O_n(Y^{3/2-\frac{3}{2n+2}+\varepsilon})$. In the case of real quadratic fields, and for general n, the best bound we are aware of is $O_{n,\varepsilon}(Y^{3/2-\frac{1}{n+2}+\varepsilon})$, due to Frei and Widmer [FW18, Theorem 1.1]. This improves upon prior work of Ellenberg, Pierce, and Wood [EPW17, Corollary 1.1.1] giving a bound of $O_\varepsilon(Y^{3/2-\frac{1}{2n}+\varepsilon})$. As far as we know, this is the best bound proven for general n. In particular, to our knowledge, there is no bound known on the exponent of Y which tends to a value less than $3/2$ as $n \to \infty$.

1.2.3 A simple characterization of when an element of the class group of a quadratic field is n-torsion

Motivated by the difficulty of counting torsion in class groups of quadratic fields, it seems natural to search for simple moduli spaces parameterizing them. We begin by presenting a surprisingly simple characterization of when an element of the class group of a quadratic field is n-torsion. We dub this "the resultant criterion for torsion in class groups of quadratic rings." The statement seems so fundamental one might have expected it to be previously known, but we were unable to find any previous mention of it. For R a rank 2 free $\mathbb{Z}$ algebra, we use $\mathrm{Cl}(R)$ to denote the class group (or equivalently the Picard group) of R and $\mathrm{Cl}(R)[n]$ to denote its n-torsion. In particular, if R is a normal integral domain with fraction field K, $\mathrm{Cl}(R) = \mathrm{Cl}(K)$, where $\mathrm{Cl}(K)$ is the class group of the number field K. For the following statement, we use the standard map from primitive quadratic forms to class groups of orders of quadratic $\mathbb{Z}$-algebras described, for example, in [Woo11a, Theorem 1.5].

Theorem 1.2.4 (Resultant criterion for torsion in quadratic rings)**.** *Let $n \geq 1$ be an integer* *and fix an integral degree* 2 *free* $\mathbb{Z}$*-algebra* R *of discriminant* d*. A primitive quadratic form* $q := ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]$ *of discriminant* d *corresponds to an element in the subgroup* $\mathrm{Cl}(R)[n] \subset \mathrm{Cl}(R)$ *if and only if there exists a polynomial* $\xi := \sum_{i=0}^{n} t_i x^i y^{n-i} \in \mathbb{Z}[x, y]$ *such that the resultant of* q *and* ξ *is either* 1 *or* -1*.*

Later, we will generalize this in Theorem 1.2.5. which not only gives a criterion as above for when q corresponds to an n-torsion element but moreover describes all elements ξ such that $\mathrm{Res}(q, \xi) = \pm 1$. This more general version also works over arbitrary base schemes B, as opposed to just over $\operatorname{Spec} \mathbb{Z}$. From Theorem 1.2.5 below, it is not difficult to deduce Theorem 1.2.4, and we do so in § 3.3.14. However, it turns out that Theorem 1.2.4 also has several simple direct proofs. We give the geometric version in § 1.3.1 and an algebraic translation in § 1.3.2. From an algebraic perspective, the above theorem essentially boils down the following: If I_q is the ideal class associated to q and has generators $\alpha, \beta \in R$, then $\mathrm{Res}(q, \xi) = \pm 1$ if and only if $\xi(\alpha, \beta)$ is a generator for the ideal $\langle \alpha, \beta \rangle^n$.

In order to have a hope of counting torsion elements in quadratic fields, it is useful to have more refined information that that of Theorem 1.2.4. Namely, given a quadratic form q, we will also need a good understanding of the set of possible elements ξ such that the resultant $\mathrm{Res}(q, \xi) = \pm 1$. There will be a certain group acting on a space, and two such pairs (q, ξ) will correspond to the same element of a class group when they are related by this group action. This type of description of a set of arithmetic objects if sometimes referred to as an "orbit parameterization." Geometrically, it is a description of the relevant moduli space as a relative simple global quotient stack.

We now introduce notation to give the description of these possible elements ξ in terms of orbits of a certain group action over $\operatorname{Spec} \mathbb{Z}$. Let V_n denote the $3 + (n + 1)$ dimensional affine space defined in Definition 2.1.1 parameterizing coefficients of pairs of polynomials (q, ξ). Let $V_n^{\mathrm{Res} \in \mathbb{G}_m} \subset V_n$ denote the open subscheme defined in Definition 3.3.4 parameterizing pairs (q, ξ) whose resultant is a unit. Let G_n denote the algebraic group defined in Definition 2.1.2, which is generated by GL_2 acting on the x and y coordinates, $\mathbb{G}_m$ diagonally scaling q and ξ, and $\mathbb{G}_a^{n-1}$ adding multiples of q to ξ. Also recall the definition of the n-Selmer group of a number field K, $\mathrm{Sel}_n(K)$, as defined in Remark 4.2.9. We note that $\mathrm{Sel}_n(\mathbb{Q})$ is $\mathbb{Z}/2\mathbb{Z}$ if n is even and trivial if n is odd.

Theorem 1.2.5. *Let* K *be a quadratic number field of discriminant* d*. There is a bijection from orbits* (q, ξ) *in* $V_n^{\mathrm{Res} \in \mathbb{G}_m}(\mathbb{Z})/G_n(\mathbb{Z})$ *satisfying* $\mathrm{disc}(q) = d$ *to the set quotient of* $\operatorname{coker}(\mathrm{Sel}_n(\mathbb{Q}) \to \mathrm{Sel}_n(K))$ *by the action of inversion coming from the nontrivial automorphism of* K *over* $\mathbb{Q}$*.*

Theorem 1.2.5 follows fairly immediately from Theorem 1.2.6 by combining it with the more or less self-contained Lemma 4.2.8. In fact, the significantly more general parameterization Theorem 1.2.6 works over an arbitrary normal integral base scheme. To state this more general result, let Π_n denote the natural map from $\left[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n\right]$ to the stack of degree 2 finite locally free covers. Locally, Π_n is given by sending (q, ξ) to the vanishing locus $V(q)$, viewed as a degree 2 finite locally free cover. Here and throughout, we use cohomology with abelian sheaf coefficients to mean flat cohomology.

Theorem 1.2.6. *Let B be a normal integral scheme and $n \geq 3$ an integer. Fix a degree 2 locally free generically étale cover $g : X \to B$. There is an injection from orbits $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B)$ such that $V(q) \simeq X$ to $\Pi_n^{-1}([X]) \subset [V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n](B)$. In turn, $\Pi_n^{-1}([X])$ is identified bijectively with elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{X/B}(B)$. The above injection is a bijection if $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.*

We prove Theorem 1.2.6 and a further statement identifying corresponding stabilizers in Theorem 3.3.7. See § 1.5 for the geometric construction which gives the idea behind the proof of Theorem 1.2.6.

1.2.7 Remarks on the main results

We make a number of remarks investigating the relation of our results above to previous results arithmetic statistics.

Remark 1.2.8. For this remark, we refer to an open subscheme of affine space whose complement has codimension 1 as a *small open* and an open subscheme of affine space whose complement has codimension at least 2 as a *big open*. A seemingly new aspect of the orbit parameterization given in Theorem 1.2.6 is that it can be understood as the points of the quotient stack $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n]$ where $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ a small open subscheme of affine space.

As far as we are aware, in previous work used for counting objects in arithmetic statistics, the relevant open in affine space has always been a big open. To give a highly incomplete sampling of examples, this is the case for the relevant stacks implicit in [BST13], [Bha05], [Bha10] [BS15], and [ASVW17]. Perhaps the reason these have not appeared is that it appears difficult to count integral points on quotients of small opens by group actions, and we certainly have not been able to do so.

Given the fact that these quotient stacks of small opens have not previously appeared in arithmetic statistics research, one might surmise such quotient stacks of small opens are a rare phenomenon. Surprisingly, we believe this phenomenon is quite ubiquitous, and have found similar such stacks appearing in preliminary investigations of many other problems. To name a few, these seem to naturally appear when investigating n-torsion in cubic fields, higher moments of n-torsion in quadratic fields, and n-torsion in fields associated to binary forms.

We conclude this remark by pointing out a general feature which explains why counting points on quotients stacks of small opens by group actions is typically much more difficult than of big opens. If our small open is of the form $\mathbb{A}^m - H$ for $H \subset \mathbb{A}^m$ a hypersurface, nearly all maps $\operatorname{Spec} \mathbb{Z} \to \mathbb{A}^m$ will meet H nontrivially. However, in a big open, $\operatorname{Spec} \mathbb{Z}$ points will rarely meet the codimension 2 complement, and so may typically be counted by a sieving procedure. In fact, in the stacks associated to small opens that come up, it is typically the case that $\operatorname{Spec} \mathbb{Z}$ points lying in this small open agree with $\operatorname{Spec} \mathbb{Z}$ points of a certain related hypersurface. This stems from the fact that $\mathbb{Z}$ has finitely many units. For example, in this paper, we investigate the small open $V_n^{\mathrm{Res} \in \mathbb{G}_m}$ where the resultant of two polynomials is a unit. Because the only units in $\mathbb{Z}$ are ± 1, the $\operatorname{Spec} \mathbb{Z}$ points of this open agrees with the $\operatorname{Spec} \mathbb{Z}$ points of the hypersurface where the resultant is ± 1. If one could asymptotically count points on this hypersurface, one could make great progress toward proving many of the conjectures of Cohen-Lenstra as in [CL84a, CL84b].

We next discuss some connections to parameter spaces for the class groups in quadratic fields previously appearing in the literature.

Remark 1.2.9. Although the orbit parameterization of Theorem 1.2.6 is new, a different parameterization of nearly the same stack in the case $n = 3$ was given in [Bha04, Theorem 13]. See also [BV16, Corollary 11]. There is a map from our parameterization to that in [Bha04, Theorem 13] given by “taking the inflection subscheme,” see Definition 4.1.3 for the definition of the inflection subscheme and Proposition 6.1.1 for the construction of this map.

It appears to us that this construction has some semblance of the construction of “bigger spaces which separate invariants” in the literature. For example, there is a space W appearing in [BSW16, §2] used in the proof of [BSW16, Theorem 1.5]. The relation between this “bigger space” W and the space of squarefree polynomials seems similar to the relation between $V_n^{\mathrm{Res} \in \mathbb{G}_m}$ and the space parameterizing these inflection subschemes. This space parameterizing inflection subschemes turns out to be a big open in $\mathbb{A}^4$ when $n = 3$ but for higher n, it is a variety of dimension 4 embedded in $\mathbb{A}^{n+1}$. This explains why the same counting procedure that works when $n = 3$ does not immediately apply for higher n.

When working over function fields and taking the Weil restriction along $\mathbb{A}^1_{\mathbb{F}_q} \to \operatorname{Spec} \mathbb{F}_q$, related Hurwitz stacks have appeared in [EVW16]. However, the Hurwitz stacks there parameterize $\mathbb{Z}/n\mathbb{Z}$ torsors over degree 2 covers and so correspond to order *n quotients* of the class group. On the other hand, the stacks appearing in this paper parameterizing μ_n torsors over degree 2 covers, and so yield order *n subgroups* of the class group (together with data relating to the units). Of course, the number of order n subgroups and quotients of a finite abelian group have the same cardinality, but the relevant moduli stacks are different.

Remark 1.2.10. In Chapter 6, we see how our structure theorem for torsion line bundles on degree 2 covers relates them to points on the secant variety of a rational normal curve. Under this correspondence, counting 3-torsion in class groups of quadratic fields translates to points on the secant variety of the twisted cubic (the rational normal curve in $\mathbb{P}^3$) missing the twisted cubic. However, as was known to the classical algebraic geometers, this secant variety is all of $\mathbb{P}^3$, which explains why it was tractable to count such points.

Another, similar, but less obvious relation is the following. Counting 2-torsion elements in cubic fields relates to counting lines contained in 3-secant planes to the 2-Veronese surface in $\mathbb{P}^5$, which miss the Veronese surface. Such lines form an 8-dimensional space, and end up birationally sweeping out the full Grassmannian $G(\mathbb{P}^1, \mathbb{P}^5)$ of lines in the $\mathbb{P}^5$ in which the Veronese surface lives.

It would be interesting to flush out the details of the above examples, and also look for other classical coincidences involving secant varieties which may be useful in proving results in arithmetic statistics.

Remark 1.2.11. Theorem 1.2.6 is proven via a geometric perspective which works over an arbitrary base. Some previous results in arithmetic statistics adapting a geometric perspective include [Woo11a, Theorem 2.1], [Woo11b, Theorem 1.1 and 2.1], and [Woo14, Theorem 1.4]. We find this perspective helps clarify the assumptions on the base scheme needed to obtain the desired orbit parameterization. This perspective gives a natural motivation, described in §1.5, for how one might come up with the description of the relevant moduli stack as a global quotient. Specifically, the affine space can be understood as that associated to a certain linear system on a Hirzebruch surface, and group we quotient by is the automorphism group of that Hirzebruch surface.

Remark 1.2.12. Yet another feature of our quotient stack $[V_n^{\mathrm{Res} \in \mathbb{G}_m}/G_n]$ is that the map $V_n^{\mathrm{Res} \in \mathbb{G}_m}(\mathbb{R})/G_n(\mathbb{R}) \to [V_n^{\mathrm{Res} \in \mathbb{G}_m}/G_n](\mathbb{R})$ is not surjective when n is even. In other words, there are real points of $[V_n^{\mathrm{Res} \in \mathbb{G}_m}/G_n]$ not arising from real orbits, see Proposition 5.1.7.. This is related to the fact that $H^1(\operatorname{Spec} \mathbb{R}, \mathrm{PGL}_2) = \mathbb{Z}/2\mathbb{Z}$. We are not aware of any previous examples of arithmetic statistics problems where this failure of bijectivity on real points occurs. For example, the map from real orbits to real points of the relevant stack is a bijection for the stacks parameterizing covers of degrees up to 5 implicitly appearing in [BSW15], as shown in [BSW15, Theorem 6].

Remark 1.2.13 (Relations to heuristics)**.** Let $g : \operatorname{Spec} \mathscr{O}_K \to \operatorname{Spec} \mathbb{Z}$ be the degree d map associated to the spectrum of the ring of integers $\mathscr{O}_K$ in a number field K. We believe that the quotient $\mathrm{coker}(\mathrm{Sel}_n(\mathbb{Q}) \to \mathrm{Sel}_n(K))$ (see Remark 4.2.9 for the definition of the n-Selmer group of a number field) is often a more tractable object to study than $\mathrm{Cl}(K)[n]$. Note that $\mathrm{Sel}_n(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$, and so this cokernel is either isomorphic to $\mathrm{Sel}_n(K)$ or a quotient of $\mathrm{Sel}_n(K)$ by $\mathbb{Z}/2\mathbb{Z}$. Of course, there are many examples in the literature where $\mathrm{Sel}_n(K)$ is studied to understand $\mathrm{Cl}(K)[n]$, though we would like to emphasize that this quotient $\mathrm{coker}(\mathrm{Sel}_n(\mathbb{Q}) \to \mathrm{Sel}_n(K))$ is geometrically more natural than $\mathrm{Sel}_n(K)$. The reason we prefer this quotient is given in Remark 4.2.9, where it is identified with a certain flat hypercohomology group $H^1(\mathrm{Spec}\,\mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$. Elements of this group turn out to describe points of the stack $\mathscr{V}^{\mathrm{smile},(n)}$ parameterizing pairs (q, ξ) of unit resultant, modulo the action of the group G_n. The quotient $\mathrm{coker}(\mathrm{Sel}_n(\mathbb{Q}) \to \mathrm{Sel}_n(K))$ may be thought of as the group of n-coverings of $g_*\mathbb{G}_m/\mathbb{G}_m$.[1]

There have been many conjectures regarding Selmer groups of elliptic curves which are eerily similar to those governing class groups. Note that this n-covering group $\mathrm{coker}(\mathrm{Sel}_n(\mathbb{Q}) \to \mathrm{Sel}_n(K))$, or equivalently $H^1(\mathrm{Spec}\,\mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$, is an intermediary between $\mathrm{Sel}_n((g_{\mathbb{Q}})_*\mathbb{G}_m/\mathbb{G}_m)$ and $\mathrm{Cl}(K)[n]$. Therefore, there may be a way to realize conjectures regarding torsion of class groups (such as those in [CL84a, CL84b]) and n-Selmer groups of abelian varieties (such as those in [PR12, BKL$^+$15]) both as special cases of conjectures regarding n-covering groups of (not necessarily proper) algebraic groups. In particular, conjectures on n-torsion in class groups of quadratic fields and n-Selmer groups of elliptic curves should be special cases of conjectures on n-covering groups of 1-dimensional algebraic groups.

We next discuss some connections to parameter spaces for the class groups in quadratic fields previously appearing in the literature.

Remark 1.2.14. It is our belief that many results in arithmetic statistics can be guided by finding the correct stack to geometrically parameterize the relevant arithmetic data. In many cases, a relevant stack exists and its points make sense over arbitrary bases. We stress that this perspective is nothing fundamentally new, it is merely an alternate viewpoint on ideas already existing in the literature. Many of the results in the literature describe certain orbit parameterizations of relevant arithmetic data, and we encourage the reader to instead think of these as points on stacks, corresponding to isomorphism classes of the arithmetic data. For some examples of stacks related to arithmetic data, see [Woo11a, Theorem 2.1], [Woo11b, Theorem 1.1 and 2.1], and [Woo14, Theorem 1.4]. We hope this geometric perspective may be helpful for researchers pursuing similar arithmetic statistics problems.

[1]The reader may be tempted to think about this as an n-Selmer group, or relative n-Selmer group, but we caution against this viewpoint. Among other issues, there are subtle differences between n-Selmer groups and n-covering groups relating to when the multiplication by n map on the Néron model is not surjective.

One interesting example of where this philosophy has the potential to be applied is in counting and parameterizing number fields of degree up to 5. The current proofs, such as that in [BSW15, §3], rely on more algebraic arguments. It would be extremely interesting if someone were to write a proof from the geometric perspective as given in [CE96] (for degrees 3 and 4) and [Cas96] (for the degree 5), which set up the framework to construct presentations for the stacks parameterizing finite locally free covers of degree at most 5. See also [Poo08] and [Woo11b] for geometric perspectives on some of these moduli stacks.

In the rest of this remark, we expand on a simple example, to see how the above philosophy might work in practice. Once again, we stress that this perspective is nothing fundamentally new, it is merely an alternate viewpoint on ideas already existing in the literature. Let us consider the case of Gorenstein degree 3 covers. These covers are parameterized by the stack $[U/\mathrm{GL}_2]$ where $U \subset \mathbb{A}^4_{a,b,c,d}$ is the open subscheme parameterizing those binary cubics $f := ax^3 + bx^2y + cxy^2 + dy^3$ which are nonzero (in every fiber). The action of GL_2 on U is given by $(g, f) \mapsto \left((x, y) \mapsto \frac{1}{\det g} f((x, y) \cdot g)\right)$ where $(x, y) \cdot g$ is right multiplication of the matrix g on the row vector (x, y).

We can quickly use the moduli theoretic description of this quotient stack to compute the real orbits. For example, the real orbits of this action with nonzero discriminant correspond to degree 3 étale covers of $\operatorname{Spec} \mathbb{R}$, and hence must be either $\operatorname{Spec} \mathbb{R} \times \mathbb{C}$ or $\operatorname{Spec} \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

We can also use the moduli theoretic description to compute the $\mathbb{Q}_p$ orbits corresponding to étale $\mathbb{Q}_p$ algebras for every prime p. We now briefly sketch how one might carry this out, though we note that the level of complexity of this approach is akin to the non-stacky approach in terms of Mass formulae was given in [Bha08, Lemma 20]. We would find it quite interesting if someone were to flesh out the details of the following sketchy suggestion. In any case, such orbits can be lifted from orbits over $\mathbb{Z}/p^k\mathbb{Z}$ for sufficiently high n. We will focus on the case $p > 3$, in which case they can actually be lifted from $\mathbb{Z}/p^2\mathbb{Z}$. The cases $p = 2$ and 3 can be deduced analogously using larger values of k.

To start, we count the total sizes of orbits of cubic forms split up depending on the data of how they are ramified. In our case of degree 3 covers, the ramification type is determined by the discriminant and there are three possibilities: the cover is étale, the cover has discriminant p , and those which have discriminant p^2. In the notation of [BST13, Lemma 18] the étale case corresponds to the union of $(111), (12), (3)$ the discriminant p case corresponds to (1^21), and the discriminant p^2 case corresponds to (1^3). Given one of the three ramification profiles as above let X be the scheme whose reduction of the special fiber consists of copies of only $\mathbb{F}_p$ (so the special fiber is $\mathbb{F}_p^3$ in the first case, $\mathbb{F}_p[\varepsilon]/(\varepsilon^2) \times \mathbb{F}_p$ in the second and $\mathbb{F}_p[\varepsilon])/(\varepsilon^3)$ in the third). Using this stack theoretic perspective, the p-adic density associated each of the possible three ramification profiles above can be computed to be $\frac{\#\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})\#[\mathbb{Z}/p^2\mathbb{Z}/\mathrm{Aut}_{X/\mathbb{Z}/p^2\mathbb{Z}}](\mathbb{Z}/p^2\mathbb{Z})}{\#\mathbb{A}^4(\mathbb{Z}/p^2\mathbb{Z})}$. Let us see how formula plays out in the various cases.

First we examine the étale case. In the étale case, the automorphism group is S_3 and the stacky count $\#[\mathbb{Z}/p^2\mathbb{Z}/\mathrm{Aut}_{X/(\mathbb{Z}/p^2\mathbb{Z})}](\mathbb{Z}/p^2\mathbb{Z}) = 1$ so we get $\frac{\#\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})}{\#\mathbb{A}^4(\mathbb{Z}/p^2\mathbb{Z})}$ as the relevant density. This indeed is the sum of the values in [BST13, Lemma 18] associated to the types (111), (12), and (3). One can further compute the exact densities associated to (111), (12), (3) from this information in terms of the automorphisms of each of these schemes. For example, the type (111) has 6 automorphisms, (12) has 2 automorphisms, and (3) has 3 automorphisms, so densities of three orbits come in ratios $\frac{1}{6} : \frac{1}{2} : \frac{1}{3}$.

To conclude, we examine the two ramified cases. The automorphism group of the discriminant p case has $\#[\mathbb{Z}/p^2\mathbb{Z}/\mathrm{Aut}_{X/\mathbb{Z}/p^2\mathbb{Z}}](\mathbb{Z}/p^2\mathbb{Z}) = 1/p$. (There are automorphisms given by scaling ε by square roots of unity, but these "cancel out" when counting in the stacky way above.) Plugging this in our formula gives $\frac{\#\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})}{p\cdot\#\mathbb{A}^4(\mathbb{Z}/p^2\mathbb{Z})}$, which again agrees with the entry for (1^21) in [BST13]. Finally, in the discriminant p^2 case $\#[\mathbb{Z}/p^2\mathbb{Z}/\mathrm{Aut}_{X/\mathbb{Z}/p^2\mathbb{Z}}](\mathbb{Z}/p^2\mathbb{Z}) = 1/p^2$ and the density is $\frac{\#\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})}{p^2\#\mathbb{A}^4(\mathbb{Z}/p^2\mathbb{Z})}$, again agreeing with the entry for (1^3) in [BST13].

Remark 1.2.15 (Relation to Hurwitz stacks of [EVW16])**.** We comment on the relation of the stacks we consider to Hurwitz stacks considered in [EVW16], which are used to prove a function field variant of the Cohen-Lenstra heuristics.

In order to understand the Cohen-Lenstra heuristics, we relate these heuristics to counting $\mathbb{Z}$ points of a certain stack $\mathscr{V}^{\mathrm{smile},(n)}$. An analogous relation holds in the function field setting, though there one counts $\mathbb{A}^1_{\mathbb{F}_q}$ points of $\mathscr{V}^{\mathrm{smile},(n)}$. In fact, one can see that $\mathscr{V}^{\mathrm{smile},(n)}(\mathbb{A}^1_{\mathbb{F}_q})$ has a map the $\mathbb{F}_q$ points of a certain Hurwitz stack parameterizing covers of $\mathbb{A}^1_{\mathbb{F}_q}$ (at least when n is prime to $p := \mathrm{char}(\mathbb{F}_q)$). Hurwitz stacks also appear in [EVW16], but the automorphism groups of their covers in this context are an extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ while in our context the automorphism groups of the covers are an extension of μ_n by $\mathbb{Z}/2\mathbb{Z}$.

To understand this discrepancy, let us first try to understand what sort of data dominant maps $\mathbb{A}^1_{\mathbb{F}_q} \to \mathscr{V}^{\mathrm{smile},(n)}$ parameterize. When p is prime to n, $\mathscr{V}^{\mathrm{smile},(n)}$ has two geometric points: the generic point which corresponds to nodal genus 1 curves in $\mathbb{P}^{n-1}$ and the closed point which corresponds to cuspidal genus 1 curves in $\mathbb{P}^{n-1}$. A dominant map $\mathbb{A}^1_{\mathbb{F}_q} \to \mathscr{V}^{\mathrm{smile},(n)}$ then corresponds to a generically nodal family of singular genus 1 curves over $\mathbb{A}^1_{\mathbb{F}_q}$ together with an embedding into a $\mathbb{P}^{n-1}$ bundle over $\mathbb{A}^1_{\mathbb{F}_q}$. Taking the inflection subscheme H of this family of singular genus 1 curves (see Definition 4.1.3) we obtain a degree n cover of $\mathbb{A}^1_{\mathbb{F}_q}$. Because p is prime to n, one can actually recover the family of genus 1 curves from this inflection subscheme. The cover $H \to \mathbb{A}^1_{\mathbb{F}_q}$ will be branched precisely at points corresponding to cuspidal genus 1 curves. Let $X \to \mathbb{A}^1_{\mathbb{F}_q}$ be the degree 2 finite locally free cover associated to H. (Although we haven't spelled out precisely what X is, $X \to \mathbb{A}^1_{\mathbb{F}_q}$ will be branched over the above mentioned cuspidal points.) Let H' be the normalization of $H \times_{\mathbb{A}^1_{\mathbb{F}_q}} X$. Then H' is a μ_n torsor over X.

So far, the description of $\mathscr{V}^{\text{smile},(n)}(\mathbb{A}^1_{\mathbb{F}_q})$ sounds very similar to the $\mathbb{F}_q$ points of the Hurwitz stacks of [EVW16]. As described above, points of our Hurwitz stack parameterize elements $[H']$ in $H^1(X, \mu_n)$ for varying X. However, the objects parameterized in [EVW16] instead lie in $H^1(X, \mathbb{Z}/n\mathbb{Z})$. Of course, these two groups can be compared using the surjection $H^1(X, \mu_n) \to H^1(X, \mathbb{G}_m)[n] = \mathrm{Pic}^0(X)[n]$, and the relation between $\mathrm{Pic}^0(X)[n]$ and $H^1(X, \mathbb{Z}/n\mathbb{Z})$ coming from class field theory. See [EVW16, Proposition 8.7] for how points of their Hurwitz stacks are related to $H^1(X, \mathbb{Z}/n\mathbb{Z})$ in the case their group A is taken to be $\mathbb{Z}/n\mathbb{Z}$.

From another perspective, the difference between the two types of Hurwitz stacks can be related to generic isotropy subgroups. In our case, the isotropy group of the generic point of $\mathscr{V}^{\text{smile},(n)}$ is identified with the automorphisms of a nodal genus 1 curve preserving its n-torsion subscheme. This is an extension of $\mathbb{Z}/2\mathbb{Z}$ by μ_n. On the other hand, the Hurwitz stacks considered in [EVW16] can be viewed as the $\mathbb{A}^1_{\mathbb{F}_q}$ points of a stack whose generic point is identified with $B(D_{2n})$.

1.2.16 Overview

We now give an overview of the remaining chapters of this part of the thesis. We begin by giving elementary geometric and algebraic proofs of the resultant criterion for torsion in class groups of quadratic fields Theorem 1.2.4 in § 1.3. In § 1.4, we collect notation used throughout; Figure 1.1 may be useful. In our opinion, the most important section for understanding the proof of our main result Theorem 1.2.6 is § 1.5, which describes the main geometric idea for parameterizing n-torsion in class groups. Strictly speaking, the remaining sections are independent of § 1.5, but this seems to be a more intuitive way to understand what is going on than the actual proof.

We now describe the sections used in proving Theorem 1.2.6. We begin with some not very well known background in Chapter 2. In § 2.1, we introduce the relevant group G_n of automorphisms we will be quotienting by and describe its relation to Hirzebruch surfaces. Following this, we collect definitions of stacks used throughout the thesis in § 2.2 and prove basic facts about them. In particular, we introduce the stack of Weierstrass curves, including cuspidal curves, which seems fundamental, but which we had trouble finding a presentation of in the literature. Next, we give numerous equivalent characterizations of the stack parameterizing n-coverings of genus 1 curves in § 2.3.

We prove our main result in Chapter 3. The heart of the algebro-geometric argument occurs in §3.1, where we relate singular genus 1 curves to divisors on on Hirzebruch surfaces. We then explain the connections between singular genus 1 curves and degree 2 covers in §3.2, which will enable us to connect the preceding discussion regarding genus 1 curves to quadratic field extensions. Using the analysis thus far, we deduce Theorem 1.2.6 in §3.3. In §3.4 we explain the rather simple connection between the n-coverings of singular genus 1 curves and n-torsion in class groups of quadratic fields. We also give some examples to illustrate this connection and Theorem 1.2.4.

In the remainder of the first part of this thesis, we describe some beautiful math we have encountered along the way of unsuccessfully attempting to use Theorem 1.2.6 to count n-torsion in quadratic fields. There are certain pairs (q, ξ) which can be naturally interpreted to live in "cusps" of the relevant fundamental domain for the action of $G_n(\mathbb{Z})$ on $V_n(\mathbb{Z})$, whose points correspond to n-torsion elements in class groups of quadratic fields via Theorem 1.2.5. These cusps, described in Chapter 4, correspond to when a certain associated inflection subscheme has a section. We discuss the inflection subscheme in §4.1 and bound the number of points in these cusps in §4.2. We also make some preliminary investigations for how one might try to set up the Cohen-Lenstra heuristics as a geometry of numbers question in Chapter 5. Unfortunately, this boils down to being able to count points on a certain "resultant = 1" hypersurface, which we do not know how to approach. We also give an version of Bhargava's averaging lemma which works for non-unimodular groups. After this, we use our above description to relate the Cohen-Lenstra heuristics to counting integral points on the secant variety to the rational normal curve in $\mathbb{P}^n$, which do not meet the rational normal curve in Chapter 6. Finally, we examine the cohomology of the "resultant = 1" hypersurface over finite fields in Chapter 7.

See Figure 1.1 for a schematic depiction of how the proof of Theorem 1.2.4 fits together.

1.3 Two proofs of the resultant criterion

In this section we give a geometric proof of Theorem 1.2.4, followed by an algebraic proof.

1.3.1 Geometric proof of Theorem 1.2.4

Let R and q be as in Theorem 1.2.4. We wish to show q is n-torsion if and only if there exists $\xi \in \mathbb{Z}[x, y]$ with $\operatorname{Res}(q, \xi) = \pm 1$. We accomplish this by the following geometric construction, which is visualized in Figure 1.2. Let $X := \operatorname{Spec} R$, so that $X \simeq \operatorname{Proj} \mathbb{Z}[x, y]/(q)$. Then, q determines an embedding $i : X \to \mathbb{P}^1_{\mathbb{Z}}$ corresponding to an

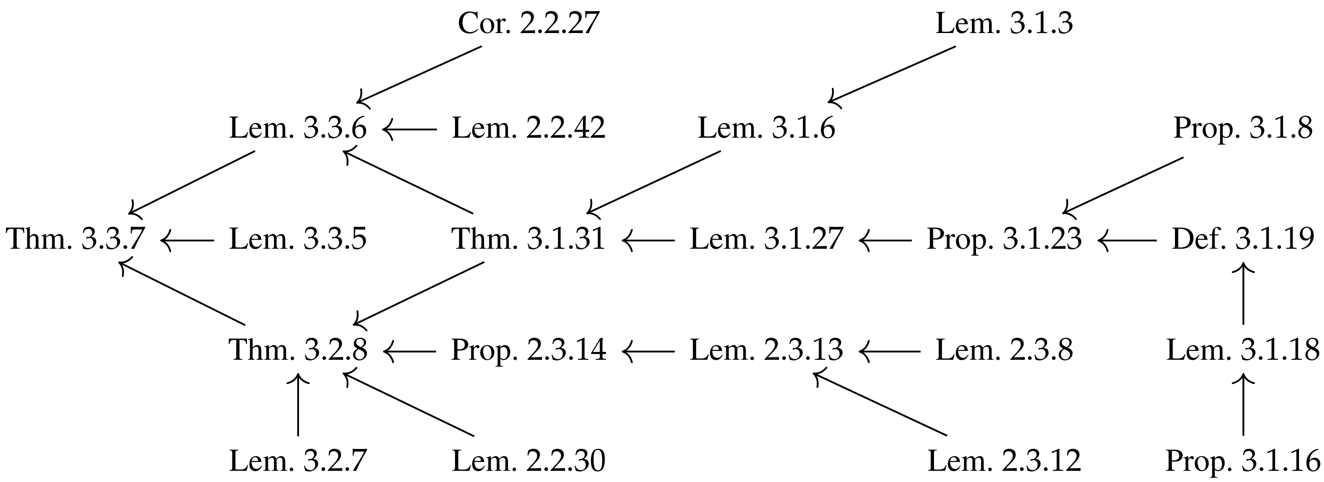


Figure 1.1: A schematic diagram depicting the structure of the proof of Theorem 3.3.7, a slightly stronger form of Theorem 1.2.6.

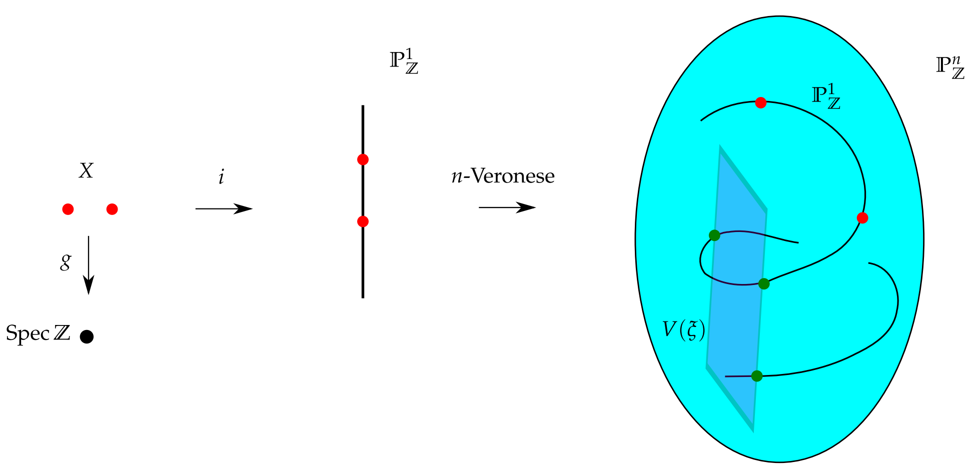


Figure 1.2: A visualization of why the existence of ξ with $\mathrm{Res}(q, \xi) = \pm 1$ forces q to be n-torsion.

invertible sheaf $\mathscr{L}_q := i^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ on X. See the proof of [Woo11a, Theorem 1.4] in [Woo11a, §3] for further description of how this bijection works. Let $g : X \to \operatorname{Spec}\mathbb{Z}$ denote the structure map. Though not strictly needed for the proof, the invertible sheaf $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n)$ yields the n-Veronese map $\mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^n_{\mathbb{Z}}$, which determines an embedding $X \xrightarrow{i} \mathbb{P}^1_{\mathbb{Z}} \xrightarrow{\nu_n} \mathbb{P}^n_{\mathbb{Z}}$ with $\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$ restricting to $\mathscr{L}_q^{\otimes n}$ on X. We could now prove both directions simultaneously, but we find it conceptually simpler to prove them separately.

We will start by assuming the existence of ξ with $\operatorname{Res}(q,\xi) = \pm 1$ and show that q corresponds to an n-torsion element in the class group. The polynomial ξ corresponds to a section of $H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1))$ whose vanishing locus is a hyperplane $V(\xi) \subset \mathbb{P}^n_{\mathbb{Z}}$. By construction, $i^*\nu_n^*\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) \simeq i^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \simeq \mathscr{L}_q^{\otimes n}$. Therefore, ξ restricts to a section of $H^0(X, \mathscr{L}_q^{\otimes n})$ which vanishes nowhere on $X = V(q)$ because $\operatorname{Res}(q,\xi) = \pm 1$. Since $\mathscr{L}_q^{\otimes n}$ has a nowhere vanishing section, it must be the trivial invertible sheaf, so q corresponds to an n-torsion element.

For the converse direction, we suppose q is n-torsion and wish to produce a $\xi \in \mathbb{Z}[x,y]$ of homogeneous degree n with $\operatorname{Res}(q,\xi) = \pm 1$. An isomorphism $\phi : \mathscr{O}_X \simeq \mathscr{L}_q^{\otimes n}$ corresponds to a section $s \in H^0(X, \mathscr{L}_q^{\otimes n})$ vanishing nowhere on X. The restriction map $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n)) \to H^0(X, \mathscr{L}_q^{\otimes n})$ is surjective because the cokernel injects into $H^1(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \otimes \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(X)^\vee) \simeq H^1(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)) = 0$. Therefore, we have produced a section ξ so that $V(\xi)$ does not meet $X = V(q)$, which means $\operatorname{Res}(q,\xi) = \pm 1$. □

1.3.2 Algebraic proof of Theorem 1.2.4

Let q, ξ be as in Theorem 1.2.4. Let I_q denote the ideal class corresponding to q. Using the standard correspondence between equivalence classes of quadratic forms and ideal classes in quadratic algebras, we can write $q = \frac{\operatorname{Nm}_{R/\mathbb{Z}}(-\beta x + \alpha y)}{\operatorname{Nm}_{R/\mathbb{Z}}(\langle \alpha, \beta \rangle)}$, for $\alpha, \beta \in \mathscr{O}_K$ and $I_q = \langle \alpha, \beta \rangle$. We wish to show $\operatorname{Res}(q,\xi) = \pm 1$ if and only if $I_q^n = (\xi(\alpha,\beta))$. This will imply the theorem because $I_q^n = \langle \alpha^n, \alpha^{n-1}\beta, \ldots, \beta^n \rangle$ and so if I_q^n is principal, it must be generated by an element of the form $I_q^n = (\xi(\alpha,\beta))$ for some degree n homogeneous $\xi \in \mathbb{Z}[x,y]$.

By multiplicativity of the resultant,

$$\operatorname{Res}(\operatorname{Nm}_{R/\mathbb{Z}}(-\beta x + \alpha y), \xi) = \operatorname{Res}(\operatorname{Nm}_{R/\mathbb{Z}}(\langle \alpha, \beta \rangle) q, \xi) = \operatorname{Nm}_{R/\mathbb{Z}}(\langle \alpha, \beta \rangle)^n \operatorname{Res}(q, \xi).$$

Let σ denote the unique nontrivial automorphism of R over $\mathbb{Z}$. Using basic properties of the resultant, such as [Lan02, Proposition 8.3],

$$\operatorname{Res}(\operatorname{Nm}_{R/\mathbb{Z}}(-\beta x + \alpha y), \xi) = \xi(\alpha,\beta) \cdot \sigma(\xi(\alpha,\beta)) = \operatorname{Nm}_{R/\mathbb{Z}}(\xi(\alpha,\beta)).$$

Hence, $\mathrm{Nm}_{R/\mathbb{Z}}(\xi(\alpha,\beta)) = \mathrm{Nm}_{R/\mathbb{Z}}(\langle\alpha,\beta\rangle)^n \operatorname{Res}(q,\xi)$. Since we always have $\xi(\alpha,\beta) \in \langle\alpha,\beta\rangle^n$, the two ideals $\langle\alpha,\beta\rangle^n$ and $(\xi(\alpha,\beta))$ are equal if and only if $\operatorname{Res}(q,\xi) = \pm 1$. □

One may wish to obtain a more precise description of when two pairs (q,ξ) correspond to the same element of the class group. This can be understood using the relation to Hirzebruch surfaces, which is further described in the motivational section § 1.5. We believe this additional discussion provides much of the insight behind the proof of Theorem 3.3.7 which strengthens Theorem 1.2.4 by describing the set of possible ξ such that $\operatorname{Res}(q,\xi) = \pm 1$.

1.4 Notation

Throughout this work, we will need a substantial amount of notation. We try to collect some of the more pervasive pieces of notation here.

1.4.1 Standing notation

We will work with an integer $n \geq 3$. Objects indexed by this value of n will ultimately be connected to n-torsion in class groups of quadratic fields. Throughout, we use B for a fixed base scheme.

By "ring" we will mean "commutative ring with unit," unless otherwise specified.

When working with a cover $U_i \to B$, we use $U_{ij} := U_i \times_B U_j$, and in general $U_{i_1\cdots i_k} := U_{i_1} \times_B \cdots \times_B U_{i_k}$.

For $X \to Y$ and $Z \to Y$ maps of schemes or algebraic spaces or stacks we use X_Z to denote the fiber product $X \times_Y Z$. In the case $Z = \operatorname{Spec} A$, we may notate this as X_A.

Given a locally finitely presented map $f : X \to B$, we use X^{sm} for the smooth locus of f.

We will often be concerned with elements of a certain cohomology group $H^1(B, G \xrightarrow{\times n} G)$ for G a group scheme over B. We often refer to objects representing these elements as *n-coverings* of G.

For A an abelian group, and X a space, we use $\underline{A}_X$ or just $\underline{A}$ to denote the constant abelian sheaf associated to A on X.

We use derived functor cohomology with abelian sheaf coefficients to mean flat cohomology, as opposed to, for example, étale cohomology.

Notation for hirzebruch surfaces

For $n \geq 3$, let $\mathbb{F}_{n-2}$ denote the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1_{\mathbb{Z}}}\left(\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}} \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)\right)$ over $\operatorname{Spec} \mathbb{Z}$. By construction, we have a factorization $\mathbb{F}_{n-2} \xrightarrow{g} \mathbb{P}^1_{\mathbb{Z}} \xrightarrow{h} \operatorname{Spec} \mathbb{Z}$.

For B a scheme, let $(\mathbb{F}_{n-2})_B$ denote $\mathbb{F}_{n-2} = \mathrm{Proj}_{\mathbb{P}^1_B}(\mathscr{O}_{\mathbb{P}^1_B} \oplus \mathscr{O}_{\mathbb{P}^1_B}(n-2))$. Consider the surjection $\mathscr{O}_{\mathbb{P}^1_B} \oplus \mathscr{O}_{\mathbb{P}^1_B}(n-2) \to \mathscr{O}_{\mathbb{P}^1_B}$ of sheaves on $\mathbb{P}^1_B$. This surjection is unique up to scaling on B, and so defines a distinguished divisor $E \subset (\mathbb{F}_{n-2})_B$, whose class we denote by e. We call this divisor the *directrix*. We also refer to a section $\mathbb{P}^1 \to (\mathbb{F}_{n-2})_B$ not meeting E as a *codirectrix*. See [Bea96, Proposition IV.1] for general background on the Picard group of Hirzebruch surfaces.

Big-O notation

For $f(t), g(t)$ two real valued functions, we say $f(t) = O(g(t))$ if there are constants C, D such that $|f(t)| < Cg(t)$ whenever $t > D$. When the constants C and D depend on auxiliary parameters, we include those parameters in the subscript of the O-notation. So, for example, if the constants depended on a number n, we would write $f(t) = O_n(g(t))$. Often, we will simply write $f = O_n(g(t))$ when the dependence of f on t is understood. Additionally, we will write $O(g(t))$ in locations to indicate that there is a function $f(t)$ which is $O(g(t))$. So, for example, for $g_i(t)$ real valued functions of t, we write $f = \sum_{i=0}^m O(g_i(t))x^i$ to indicate that f is a polynomial in x whose coefficients are functions of t and whose ith coefficient is $O(g_i(t))$.

Resultants

Throughout, we frequently work with two polynomials over a ring R f, g for $f \in H^1(\mathbb{P}^1_R, \mathscr{O}(\alpha)), g \in H^1(\mathbb{P}^1_R, \mathscr{O}(\beta))$. Frequently, we will call these q, ξ with q of degree 2 and ξ of degree n. If we choose a basis x, y for $H^1(\mathbb{P}^1_R, \mathscr{O}(1))$ then the resultant of $f(x, y) := \sum_{i=0}^{\alpha} a_i x^i y^{\alpha-i}$ and $g(x, y) := \sum_{j=0}^{\beta} b_j x^j y^{\beta-j}$ is the usual notion of resultant, defined as the element of R which is the determinant of the matrix

$$\begin{pmatrix}
a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\
a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\
a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\
\vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\
a_\alpha & a_{\alpha-1} & \cdots & \vdots & b_\beta & b_{\beta-1} & \cdots & \vdots \\
0 & a_\alpha & \ddots & \vdots & 0 & b_\beta & \ddots & \vdots \\
\vdots & \vdots & \ddots & a_{\alpha-1} & \vdots & \vdots & \ddots & b_{\beta-1} \\
0 & 0 & \cdots & a_\alpha & 0 & 0 & \cdots & b_\beta
\end{pmatrix}.$$

For basic properties of resultants, we refer the reader to [Lan02, IV,§8]. However, a different choice of basis for $H^0(\mathbb{P}^1_R, \mathscr{O}(1))$ may change the resultant by a unit, and so the resultant of f, g should, strictly speaking, be viewed as an element of R taken modulo the $\mathrm{GL}(H^0(\mathbb{P}^1_R, \mathscr{O}(1)))$ action. The resultant has several nice properties. For example, it is 0 if and only if f and g have a common factor. Further, if we have a map $h : R \to S$ then $h(\mathrm{Res}(f, g)) = \mathrm{Res}(h(f), h(g))$. By abuse of notation, we will often conflate the value of the resultant in R with its value in R modulo the $\mathrm{GL}(H^0(\mathbb{P}^1_R, \mathscr{O}(1)))$ action. Since elements of $\mathrm{GL}(H^0(\mathbb{P}^1_R, \mathscr{O}(1)))$ act invertibly, the condition that a resultant be a unit is well defined.

Table of notation

For the reader's convenience, in Figure 8.1 we collect notation introduced throughout the first part of the thesis, roughly in order of appearance. The descriptions are intended to be terse and not precise.

1.5 The geometric bijection

In §1.3.1, we have already given a direct proof of Theorem 1.2.4. To prove Theorem 3.3.7, we need a more precise bijection, which tells us exactly when two pairs (q, ξ) correspond to the same element of the class group. In this section, we will describe a geometric construction to explain when two pairs (q, ξ) and (q, ξ') as in Theorem 1.2.4 correspond to the same element of the class group. This construction takes as input a degree 2 finite free cover $g : X \to \operatorname{Spec} \mathbb{Z}$, a line bundle $\mathscr{L}$ on X, and an isomorphism $\iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$. The pair $(g : X \to \operatorname{Spec} \mathbb{Z}, \mathscr{L})$ is determined by q while ι is determined by the additional data of ξ. The construction outputs a particular divisor on a Hirzebruch surface, up to isomorphism. Whether (q, ξ) and (q, ξ') correspond to the same element of the class group is closely tied to whether they are related by the action of a certain group G_n. The group G_n can be understood as a certain linearization of the automorphism group of the above mentioned Hirzebruch surface. Because this section serves primarily as motivation we leave some of the details of the ensuing construction to the reader.

Our method for approaching the above question of which pairs (q, ξ) and (q, ξ') yield the same element of the class group will be to interpret it geometrically. That is, we will construct a stack $\mathscr{V}^{\mathrm{smile},(n)}$ whose $\mathbb{Z}$-points approximately parameterize pairs (R, T) where R is an order in a quadratic number field and $T \in \mathrm{Cl}(R)[n]$.[2]

[2]More precisely, in the case $R = \mathscr{O}_K$, T lies in $\operatorname{coker}(\mathrm{Sel}_n(K) \to \mathrm{Sel}_n(\mathbb{Q}))$, where $\mathrm{Sel}_n(K)$ denotes the n-Selmer group of K, see Remark 4.2.9.

Notation	Description	Location defined
$\mathscr{W}$	Stack of Weierstrass curves	Definition 2.2.2
$\overline{\mathscr{E}}$	Universal curve over $\mathscr{W}$	Proposition 2.2.13
$\mathscr{E}$	Smooth locus of $\overline{\mathscr{E}}$ over $\mathscr{W}$	Definition 2.2.15
$\mathscr{W}_{\text{cusp}}$	Substack of cuspidal curves in $\mathscr{W}$	Definition 2.2.29
$\mathscr{W}_{\text{node}}$	Substack of nodal curves in $\mathscr{W}$	Definition 2.2.29
$\mathscr{W}_{\text{sing}}$	Substack of singular curves in $\mathscr{W}$	Definition 2.2.3
$\widetilde{\mathscr{W}}_{\text{sing}}$	Stack of Weierstrass curves with a marked section in the singular locus	Definition 2.2.3
$\widetilde{\mathscr{W}}_{\text{cusp}}$	Substack of $\widetilde{\mathscr{W}}_{\text{sing}}$ parameterizing cuspidal curves	Definition 2.2.6
$\widetilde{\mathscr{W}}_{\text{node}}$	Substack of $\widetilde{\mathscr{W}}_{\text{sing}}$ parameterizing nodal curves	Definition 2.2.6
$\mathscr{H}^{(n)}$	Hilbert scheme of geometrically integral degree n genus 1 curves in $\mathbb{P}^{n-1}$	Definition 2.2.19
$\widetilde{\mathscr{H}}^{(n)}_{\text{sing}}$	Curves in $\mathscr{H}^{(n)}$ with a marked singular point	Definition 2.2.24
$\mathscr{S}^{(n)}$	Stack of genus 1 curves with an n-covering	Definition 2.2.17
$\mathscr{M}_1^{(n)}$	Stack of genus 1 degree n curves	Definition 2.2.19
$\mathscr{M}^{(n)}_{1,\text{node}}$	Substack of $\mathscr{M}_1^{(n)}$ parameterizing nodal curves	Definition 2.2.29
$\mathscr{M}^{(n)}_{1,\text{cusp}}$	Substack of $\mathscr{M}_1^{(n)}$ parameterizing cuspidal curves	Definition 2.2.29
$\mathscr{M}^{(n)}_{1,\text{inflection}}$	Substack of stack of $\mathscr{M}^{(n)}_{1,\text{cusp}}$ where every point is an inflection point	Definition 4.1.13
$\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$	Stack of genus 1 degree n curves with a section in the singular locus	Definition 2.2.21
$\widetilde{\mathscr{M}}^{(n)}_{1,\text{node}}$	Substack of $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ parameterizing nodal curves	Definition 2.2.23
$\widetilde{\mathscr{M}}^{(n)}_{1,\text{cusp}}$	Substack of $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ parameterizing cuspidal curves	Definition 2.2.23
$\mathcal{V}^{\text{smile},(n)}$	Scheme of smooth curves in the linear system $e + nf$ on $\mathbb{F}_{n-2}$	Definition 2.2.35
$\mathscr{V}^{\text{smile},(n)}$	Stack of smooth curves in the linear system $e + nf$ on $(n-2)$-Hirzebruch twists	Definition 2.2.39
V_n	The affine space associated to $H^0(\mathbb{P}^1, \mathscr{O}(2) \oplus \mathscr{O}(n))$	Definition 2.1.1
E_g	The singular genus 1 curve associated to a degree 2 finite locally free map g	Notation 3.2.1
$V_n^{\text{Res}\in\mathbb{G}_m}$	Open locus of V_n for which the resultant is a unit	Definition 3.3.4
G_n	Group of automorphisms acting on V_n	Definition 2.1.2
G'_n	Subgroup of G_n "fixing the base $\mathbb{P}^1$"	Definition 2.1.2
U_n	Unipotent radical of G_n	Definition 2.1.2
$G_n^{\text{reductive}}$	The maximal reductive quotient of G_n	Definition 5.2.1
A_n	Projective quotient of G_n	Definition 2.1.2
A'_n	Projective quotient of G'_n	Definition 2.1.2
$\text{InflectionEquation}(q, \xi)$	Choice of equation defining the inflection scheme associated to $(q, \xi) \in V_n$	Definition 4.1.9
$\text{Inflection}(C, \iota)$	Inflection subscheme of C mapped to $\mathbb{P}^{n-1}$ via ι	Definition 4.1.3

Figure 1.3: Notation introduced

We now give a brief description of $\mathscr{V}^{\mathrm{smile},(n)}$ and try to highlight the idea of how one might come up with it. Begin with the $n+4$ dimensional affine space parameterizing pairs (q, ξ) for $q = \sum_{i=0}^{2} a_i x^i y^{2-i}$ a degree 2 homogeneous polynomial and $\xi = \sum_{i=0}^{n} b_i x^i y^{n-i}$ a degree n homogeneous polynomial. The $n+4$ coordinates of this affine space parameterize the 3 coefficients of q and the $n+1$ coefficients of ξ. Inside this affine space, we take the complement of the hypersurface defined by $\mathrm{Res}(q, \xi) = 0$. Then, $\mathscr{V}^{\mathrm{smile},(n)}$ is the quotient of this open subset by the action of a certain group G_n, which includes symmetries such as linear automorphisms of the 2 dimensional vector space spanned by x and y as well as those adding multiples of q to ξ. The group G_n is then generated by the above mentioned symmetries, together with an additional central copy of $\mathbb{G}_m$, acting by scaling all $n+4$ coordinates. Under this description, the $\mathbb{Z}$-points of $\mathscr{V}^{\mathrm{smile},(n)}$ are pairs (q, ξ) with integral coefficients, modulo the action of the group G_n.

In the remainder of this section, we give the geometric construction relating $(g : X \to \operatorname{Spec} \mathbb{Z}, \mathscr{L}, \iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X)$ to sections on a Hirzebruch surface. The group G_n will naturally appear as a linearization of the automorphisms of this Hirzebruch surface. We first give the construction over a field in § 1.5.1 before describing the relative version in § 1.5.2.

1.5.1 The construction on fibers

We now give the geometric construction relating $(g : X \to \operatorname{Spec} B, \mathscr{L}, \iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X)$, for g a finite locally free map of degree two, to certain smooth sections on a Hirzebruch surface. The group G_n will naturally appear as a linearization of the automorphisms of this Hirzebruch surface. Figure 1.4 may be helpful in visualizing the geometric construction described in this subsection.

To start, we set up notation. Let $X = B_1 \coprod B_2$ be two points with $B_1 \simeq B_2 \simeq B$ and let $g : X \to B$ denote the natural map. Much of what follows easily generalizes to the case B is arbitrary and X is any degree 2 finite locally free cover, but work in the above case to simplify the exposition. Let $\mathscr{L}$ be an invertible sheaf on X (which, in this degenerate case must be isomorphic to $\mathscr{O}_X$) and suppose we are given an isomorphism $\iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$. Recalling the notation for Hirzebruch surfaces from § 1.4.1, we describe how to obtain a section of class $e + nf$ in the Hirzebruch surface $\mathbb{F}_{n-2}$ over B.

We now construct the desired section of class $e + nf$ in a series of steps. With reference to Figure 1.4, the line bundle $\mathscr{L}^{\otimes n}$ gives the map $X \to \mathbb{P}^n$. Saying this more precisely, the sheaf $g_* \mathscr{L}$ corresponds to a rank 2 vector space V over k. There is a natural embedding $\mathbb{P} V \to \mathbb{P} \operatorname{Sym}^n V \simeq \mathbb{P}^n$ realizing $\mathbb{P} V$ as a rational normal curve R in $\mathbb{P}^n$.

Next, we use the trivialization $\iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$ to construct the line L containing the image

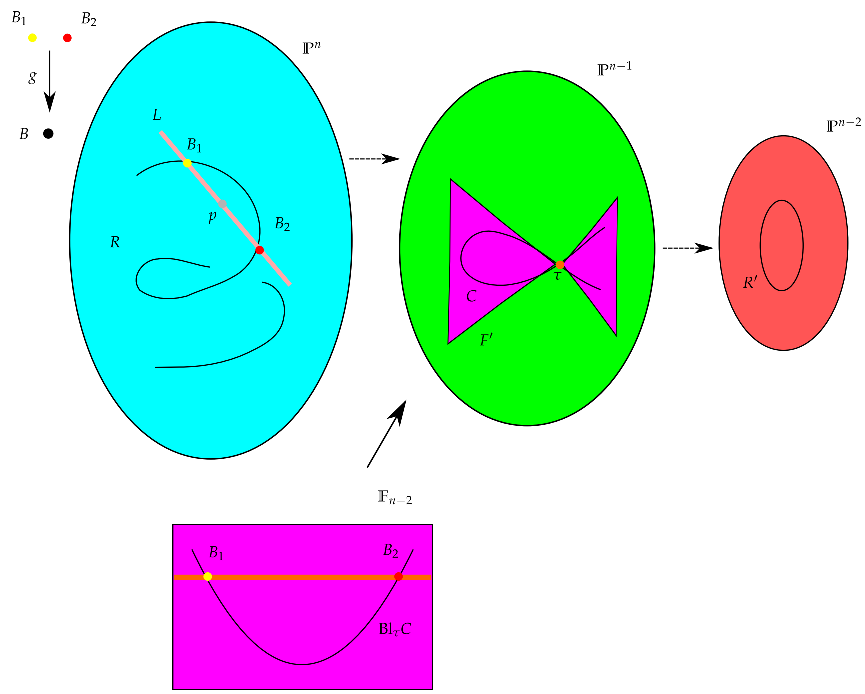


Figure 1.4: A visualization of the bijection of § 1.5.1.

$X \to \mathbb{P}^n$. The surjection $\mathrm{Sym}^n V = \mathrm{Sym}^n(g_*\mathscr{L}) \to g_*(\mathscr{L}^{\otimes n}) \xrightarrow{g_*\iota} g_*\mathscr{O}_X$ gives a line L in $\mathbb{P}^n$. Note that $L \cap R$ consists of two points corresponding to the two further surjections $g_*\mathscr{O}_X \to g_*\mathscr{O}_{B_i} \simeq \mathscr{O}_B$ associated to the inclusions $B_i \to X$ for $i \in \{1, 2\}$.

Having constructed the line L, we now use the structure map $\mathscr{O}_B \to g_*\mathscr{O}_X$ to obtain a point p on L missing X. Let Q denote the cokernel of $\mathscr{O}_B \to g_*\mathscr{O}_X$. We obtain a composite $\mathrm{Sym}^n V \to g_*\mathscr{O}_X \to Q$ which corresponds to a point p on L. We claim this point p is not one of the two intersection points $L \cap R$. Indeed, the two intersection points with R correspond to two idempotent basis vectors e_1 and e_2 associated to the inclusions $B_i \to X$, while the point p corresponds to the diagonal inclusion $k \to ke_1 \oplus ke_2 \simeq W$ sending $1 \mapsto e_1 + e_2$.

We next explain why projecting R from p yields a curve C in $\mathbb{P}^{n-1}$ lying in a cone over a rational normal curve $R' \subset \mathbb{P}^{n-2}$. Project R from the point p to obtain a singular genus 1 curve C in $\mathbb{P}^{n-1}$ from R by gluing the two points B_1 and B_2 of $L \cap R$. Let τ denote the singular point of C. We claim that in fact C lies in the cone over a rational normal curve in $\mathbb{P}^{n-2}$. To see this, note that further projecting C from τ is equivalent to projecting the original curve R from the line L. Since L meets R in two points, this projection is a rational normal curve $R' \subset \mathbb{P}^{n-2}$. Therefore, the projection of C from τ is a rational normal curve, and so C lies in the cone F' over R' passing through τ.

Finally, we blow up C and F' at τ to obtain the desired divisor on a Hirzebruch surface of class $e + nf$. When we blow F' up at τ, we will obtain a Hirzebruch surface isomorphic to $\mathbb{F}_{n-2}$. One can also verify that the blow up of C at τ is then a smooth curve in the linear system $e + nf$ on $\mathbb{F}_{n-2}$. This is the completes our construction.

1.5.2 The Construction over Spec $\mathbb{Z}$

We now generalize the construction of § 1.5.2. To fix ideas, for this subsection the arithmetically minded reader can consider the case $B = \operatorname{Spec}\mathbb{Z}$, though the construction works for arbitrary base schemes B until the last paragraph. We will indicate at which point we use $B = \operatorname{Spec}\mathbb{Z}$ in the final paragraph of this subsection, and really the only thing we are using about Spec $\mathbb{Z}$ is that it has trivial Picard group. The generalization is straightforward, but we hope that spelling it out will be helpful for the reader's understanding.

Given $g : X \to B$ a degree 2 locally free morphism (such as the spectrum of the ring of integers of a quadratic field) and an invertible sheaf $\mathscr{L}$ on X with an isomorphism $\iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$, we describe how to obtain a section of class $e + nf$ in the Hirzebruch surface $\mathbb{F}_{n-2}$.

As a first step, we construct the analogs of the objects R and L in $\mathbb{P}^n$ from § 1.5.1. The invertible sheaf $\mathscr{L}$ on X defines a map $X \to \mathbb{P}(g_*\mathscr{L}) \xrightarrow{\pi} B$ which is an embedding, as can be verified on fibers. Observe that $\mathbb{P}(g_*\mathscr{L})$ comes with an invertible sheaf $\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(1)$ satisfying $\pi_*\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(1) \simeq g_*\mathscr{L}$. The invertible sheaf $\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(n) := \mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(1)^{\otimes n}$ on $\mathbb{P}(g_*\mathscr{L})$ satisfies $\pi_*\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(n) \simeq \operatorname{Sym}^n(\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(1)) \simeq \operatorname{Sym}^n(g_*\mathscr{L})$ and the map coming from adjunction $\pi^*\operatorname{Sym}^n(g_*\mathscr{L}) \simeq \pi^*\pi_*\mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(n) \to \mathscr{O}_{\mathbb{P}(g_*\mathscr{L})}(n)$ defines an embedding $\mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}\operatorname{Sym}^n(g_*\mathscr{L})$. On fibers over B, this realizes $\mathbb{P}(g_*\mathscr{L})$ as a rational normal curve in $\mathbb{P}\operatorname{Sym}^n(g_*\mathscr{L})$. Additionally, we have a canonical surjection

$$\operatorname{Sym}^n(g_*\mathscr{L}) \to g_*(\mathscr{L}^{\otimes n}) \xrightarrow{g_*\iota} g_*\mathscr{O}_X.$$

This defines a linear inclusion of projective bundles $\mathbb{P}(g_*\mathscr{O}_X) \to \mathbb{P}\operatorname{Sym}^n(g_*\mathscr{L})$. There is a canonical inclusion $\mathscr{O}_B \to g_*\mathscr{O}_X$ induced by g. The cokernel $\mathscr{Q}$ is locally free of rank 1, see Lemma 3.2.2

As a second step, we construct the analog of the curve $C \subset \mathbb{P}^{n-1}$ from § 1.5.1. The composition $\operatorname{Sym}^n(g_*\mathscr{L}) \to g_*\mathscr{O}_X \to \mathscr{Q}$ defines a section of $\mathbb{P}\operatorname{Sym}^n(g_*\mathscr{L})$ lying on the $\mathbb{P}^1$-bundle $\mathbb{P}(g_*\mathscr{O}_X) \subset \mathbb{P}\operatorname{Sym}^n(g_*\mathscr{L})$, which on fibers corresponds to a line in $\mathbb{P}^n$. Let $\mathscr{K} := \ker(\operatorname{Sym}^n(g_*\mathscr{L}) \to \mathscr{Q})$ and $\mathscr{K}' := \ker(\operatorname{Sym}^n(g_*\mathscr{L}) \to g_*\mathscr{O}_X)$. Projection away from the section $\mathscr{Q}$ then defines a map $\mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}(g_*\mathscr{K})$. This is not an embedding, and the image is a relative genus 1 curve over B, whose fibers are nodal rational curves over points of X where $B \to X$ is étale and are cuspidal rational curves on fibers where $B \to X$ is ramified. Let $C \subset \mathbb{P}(g_*\mathscr{K})$ denote the scheme theoretic image of $\mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}\,\mathrm{Sym}^n(g_*\mathscr{L}) \to \mathbb{P}(g_*\mathscr{K})$.

As a third step, we construct the analogs of τ and R' from §1.5.1. We now have a surjection $\mathscr{K} \to \mathscr{O}_B$ induced by the fact that composition $\mathscr{K} \to \mathrm{Sym}^n(g_*\mathscr{L}) \to g_*\mathscr{O}_X \to \mathscr{Q}$ is 0, and so $\mathscr{K} \to g_*\mathscr{O}_X$ factors through $\mathscr{O}_B = \ker(g_*\mathscr{O}_X \to \mathscr{Q})$. The map $\mathbb{P}\mathscr{O}_B \to \mathbb{P}\mathscr{K}$ factors through C and induces a section $\tau : B \to C$ lying in the singular locus of $C \to B$. Projection away from this singular section induces a map $\mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}\mathscr{K}'$ which is again an embedding since this could also be obtained from the immediate projection associated to $\mathrm{Sym}^n(g_*\mathscr{L}) \to g_*\mathscr{O}_X$, which defines a map that can be verified to be an embedding on fibers over B, since it arises as the projection of $\mathbb{P}(g_*\mathscr{L}) \subset \mathbb{P}\,\mathrm{Sym}^n(g_*\mathscr{L})$ away from $\mathbb{P}(g_*\mathscr{O}_X)$, which can be thought of as a secant line to $\mathbb{P}(g_*\mathscr{L})$.

Finally, we construct a curve of class $e + nf$ in a relative Hirzebruch surface over B. Using the above description, because the projection of C from $\tau(B)$ is a relative rational normal curve, it follows that C is contained in a cone over the rational normal curve $\mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}\mathscr{K}'$. The cone point is given by the section $\tau : B \to C$ in the singular locus of $C \to B$. Let $\mathbb{F}$ denote the blow up of this cone along the section $\mathbb{P}\mathscr{O}_B \to \mathbb{P}\mathscr{K}$. We can form the proper transform of C in $\mathbb{F}$ which is isomorphic to $\mathbb{P}(g_*\mathscr{L})$.

It is only at this point that we make use of the fact that $B = \operatorname{Spec}\mathbb{Z}$, and really we only use that $\mathbb{Z}$ has trivial Picard group. Because this is so, $g_*\mathscr{L}$ is a rank two locally free sheaf on $\operatorname{Spec}\mathbb{Z}$, hence trivial. It follows that $\mathbb{F}$ is isomorphic to the Hirzebruch surface $\mathbb{F}_{n-2}$. Further, one can check that C is an effective relative Cartier divisor, lying in the linear system $e + nf$ on $\mathbb{F}_{n-2}$. Note that e is the class of the directrix in $\mathbb{F}_{n-2}$, which is given as the exceptional divisor E in $\mathbb{F}_{n-2}$ of the above constructed blow up.

One can then describe sections in the above linear system as pairs (q, ξ), where q is the restriction of to the directrix E and ξ is the restriction to a fixed codirectrix. One may ask when two pairs (q, ξ) correspond to the same element of the class group. This will certainly be the case when they are related by a (linearization of) an automorphism of the Hirzebruch surface. It turns out that this is nearly the only obstruction, as will be seen later in Theorem 3.3.7.

Chapter 2

Advanced Background

In this chapter, we verify a number of facts relating to Hirzebruch surfaces § 2.1, stacks parameterizing genus 1 curves § 2.2, and n-coverings of genus 1 curves § 2.3. The results in this chapter are likely well known to experts, though we had some difficulty finding a suitable reference in the generality we will need them, especially in relation to dealing with singular genus 1 curves which may be cuspidal.

2.1 The automorphism group scheme of a Hirzebruch surface

In this section, we will describe a certain homogeneous space V_n for a group action G_n which will enable us to count n-torsion elements of class groups. The main result of this section identifies the quotient of G_n by a central copy of $\mathbb{G}_m$ with the automorphisms of $\mathbb{F}_{n-2}$.

Fix $n \geq 3$. Let $\mathscr{F}_n := \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}} \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)$ and let $\mathbb{F}_{n-2}$ denote the Hirzebruch surface $\mathbb{P}(\mathscr{F}_n)$ over Spec $\mathbb{Z}$. We use $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}$ to denote the automorphism group scheme of $\mathbb{F}_{n-2}$ over Spec $\mathbb{Z}$. That is, B-points of $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}(B)$ are in natural bijection with automorphisms of $(\mathbb{F}_{n-2})_B$ over B. We define the structure maps $\mathbb{F}_{n-2} \xrightarrow{g} \mathbb{P}^1_B \xrightarrow{h} B$.

We next construct this automorphism scheme explicitly. It turns out that this automorphism scheme is a semidirect product $\mathbb{G}_a^{n-1} \rtimes (\mathrm{GL}_2/\mu_{n-2})$. Instead of describing the semidirect product structure directly, we describe this group as a subgroup of PGL_{n+4}. We will realize it as the quotient of a subgroup of GL_{n+4} by the central copy of $\mathbb{G}_m$. We start by constructing this subgroup $G_n \subset \mathrm{GL}_{n+4}$ which can be understood as a linearization of $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}$.

To describe G_n directly, it is easiest to describe it in terms of its left action on the rank $n+4$ free $\mathbb{Z}$-module $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n))$. This module will be one of the central objects of this paper, and so we give the corresponding scheme a name.

Definition 2.1.1. For $n \geq 3$, define $V_n := \operatorname{Spec}\operatorname{Sym}^\bullet H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n))$.

By abuse of notation, we will often conflate V_n with its underlying vector space. For example, we will use $\mathrm{GL}(V_n)$ and $\mathrm{PGL}(V_n)$ to really mean $\mathrm{GL}(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n)))$ and $\mathrm{PGL}(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n)))$.

Choosing a basis $\mathbb{Z}x \oplus \mathbb{Z}y$ for $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1))$, we can identify points of V_n with pairs $(q, \xi) := (\sum_{i=0}^2 a_i x^i y^{2-j}, \sum_{j=0}^n b_j x^j y^{n-j})$ for $a_i \in \mathbb{Z}, b_j \in \mathbb{Z}$ with $0 \leq i \leq 2, 0 \leq j \leq n$. We next realize actions of $\mathbb{G}_m, \mathbb{G}_a^{n-1}$ and GL_2 on V_n. We can realize the central $\mathbb{G}_m \subset \mathrm{GL}(V_n)$ by

$$\begin{aligned} \mathbb{G}_m \times V_n &\to V_n \\ (\chi, (q, \xi)) &\mapsto (\chi q, \chi \xi). \end{aligned} \tag{2.1.1}$$

There is a subgroup $\mathbb{G}_a^{n-1} \subset \mathrm{GL}\,(V_n)$ defined via the action

$$\begin{aligned} \mathbb{G}_a^{n-1} \times V_n &\to V_n \\ ((\alpha_0, \ldots, \alpha_{n-2}), (q, \xi)) &\mapsto (q, \xi + \sum_{i=0}^{n-2} \alpha_i x^i y^{n-2-i} q). \end{aligned} \tag{2.1.2}$$

Finally, we can realize $\mathrm{GL}_2/\mu_{n-2} \subset \mathrm{GL}(V_n)$ via the image of the twisted action

$$\begin{aligned} &\mathrm{GL}_2 \times V_n \to V_n \\ &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\sum_{i=0}^2 a_i x^i y^{2-j}, \sum_{j=0}^n b_j x^j y^{n-j})\right) \\ &\mapsto \frac{1}{ad - bc}(\sum_{i=0}^2 a_i (ax+by)^i (cx+dy)^{2-j}, \sum_{j=0}^n b_j (ax+by)^j (cx+dy)^{n-j}). \end{aligned} \tag{2.1.3}$$

Definition 2.1.2. Define G_n as the subgroup of $\mathrm{GL}\,(V_n)$ generated the subgroups $\mathbb{G}_m$, $\mathbb{G}_a^{n-1}$, and GL_2/μ_{n-2} defined in (2.1.1), (2.1.2), and (2.1.3), respectively. Define $G'_n \subset G_n$ as the subgroup generated by $\mathbb{G}_m$, $\mathbb{G}_a^{n-1}$, and the central $\mathbb{G}_m$ sitting inside GL_2/μ_{n-2} under the actions (2.1.1), (2.1.2), and (2.1.3). Define $U_n \subset G_n$ as the subgroup isomorphic to $\mathbb{G}_a^{n-1} \subset G_n$ coming from (2.1.2).

Let A_n denote the image of G_n under the map to $\mathrm{GL}(V_n) \to \mathrm{PGL}(V_n)$ and A'_n denote the image of G'_n under the map $\mathrm{GL}(V_n) \to \mathrm{PGL}(V_n)$.

Remark 2.1.3. Taking the twisted action in (2.1.3) is not particularly essential, since if one did not divide by $ad - bc$, one would be able to use the $\mathbb{G}_m$ action of (2.1.1) to accomplish this division. Furthermore, in some ways it is more natural to use the action which is not twisted, because that is the natural action of GL_2 induces on $H^0(\mathbb{P}^1, \mathscr{O}(2))$ and $H^0(\mathbb{P}^1, \mathscr{O}(n))$ via its action on $H^0(\mathbb{P}^1, \mathscr{O}(1))$. However, we chose to take the twisted action as it somewhat simplifies future statements largely due to the fact that the image of GL_2 in $\mathrm{GL}(V_n)$ intersects the central $\mathbb{G}_m$ trivially as opposed to in μ_{n-2}.

So far, it is not clear whether the group G_n, which is generated by U_n, $\mathbb{G}_m$, and GL_2/μ_{n-2}, contains elements which are not products of elements of these three subgroups. The following lemma establishes that that all element of G_n are products of elements from these three subgroups.

Lemma 2.1.4. *For any $n \geq 3$,*

1. $G_n' \simeq U_n \rtimes \mathbb{G}_m^2$,
2. *we have an exact sequence*

$$0 \longrightarrow G_n' \longrightarrow G_n \longrightarrow \mathrm{PGL}_2 \longrightarrow 0 \tag{2.1.4}$$

where the induced map $\mathrm{GL}_2/\mu_{n-2} \to G_n \to \mathrm{PGL}_2$ *is the quotient of* GL_2/μ_{n-2} *by its central* $\mathbb{G}_m \subset \mathrm{GL}_2/\mu_{n-2}$,

3. $G_n \simeq U_n \rtimes (\mathbb{G}_m \times (\mathrm{GL}_2/\mu_{n-2}))$,
4. $A_n \simeq U_n \rtimes \mathrm{GL}_2/\mu_{n-2}$.

Remark 2.1.5. In the exact sequence (2.1.4) PGL_2 may be realized as the automorphisms of $\mathbb{P}^1$ and G_n' may be identified with automorphisms of $\mathscr{F}_n$ over $\mathbb{P}^1$.

Proof. We first prove (1). This will follow from the direct calculation that for any ring R and any $g, g' \in U_n(R)$ and $h, h' \in \mathbb{G}_m^2(R)$ we have $h^{-1}gh \in U_n(R)$. To check this identity, we may work on a flat cover of R, and hence assume the R points $h, h' \in \mathbb{G}_m^2(R) \subset (\mathbb{G}_m \times \mathrm{GL}_2/\mu_{n-2})(R)$ lift to R points of $\mathbb{G}_m \times \mathrm{GL}_2$. Let h be given by $(\chi, \zeta) \in \mathbb{G}_m^2(R) \subset (\mathbb{G}_m \times \mathrm{GL}_2)(R)$ and let g correspond to a tuple $(\alpha_0, \ldots, \alpha_{n-2})$ as in (2.1.2). Define $\alpha := \sum_{i=1}^{n-1} \alpha_i y^i x^{n-2-i}$. Then, for $(q, \xi) \in V_n$, we have

$$h^{-1}gh(q, \xi) = h^{-1}g(\chi q, \chi\zeta^{n-2}\xi) = h^{-1}(\chi q, \chi\zeta^{n-2}\xi + \alpha\chi q) = (q, \xi + \alpha\zeta^{2-n}q).$$

Therefore, $h^{-1}gh = (\zeta^{2-n}) \cdot g \in U_n(R)$.

The above calculation implies that we may find $h'' \in \mathbb{G}_m^2(R)$ so that $(gh)(g'h') = (gg')(h''h')$. Therefore every element in G_n' is a product of an element of U_n and an element of $\mathbb{G}_m^2$. This shows G'_n is an extension of $\mathbb{G}_m^2$ by U_n, and it is in fact a semidirect product by the assumption that $\mathbb{G}_m^2$ embeds in G'_n by construction.

Next, we check (2). As a first step, we verify $G'_n \subset G_n$ is a normal subgroup and every element of G_n can be written as a product of an element of G'_n and an element of GL_2/μ_{n-2}. Analogously to our computation for (1), it is enough to show that for any ring R and any $h \in G'_n(R), g \in \mathrm{GL}_2/\mu_{n-2}(R)$, we have $ghg^{-1} \in G'_n(R)$. Again, to check this identity, we may work on a flat cover of R so as to assume g lifts to a point of GL_2. By construction of G'_n, h acts on the quotient $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))$ of $V_n = H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2)) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n))$, only by scaling via the central copy of $\mathbb{G}_m \subset \mathrm{GL}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)$. Via a direct calculation, the subgroup of GL_2 acting via this central $\mathbb{G}_m$ on $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))$ is precisely the central $\mathbb{G}_m \subset \mathrm{GL}_2$, which already lies in G'_n. Therefore, G'_n is characterized as the subgroup of G_n whose action on $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))$ factors through the central copy of $\mathbb{G}_m \subset \mathrm{GL}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)$. Therefore, G'_n is a normal subgroup of G_n.

Next, we show $G_n/G'_n \simeq \mathrm{PGL}_2$. The quotient is generated by GL_2/μ_{n-2}. As mentioned above, G'_n is characterized as the subgroup of G_n which acts by the central $\mathbb{G}_m$ on $\mathrm{GL}(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2)))$. However, the subgroup of GL_2/μ_{n-2} intersecting the central $\mathbb{G}_m \subset \mathrm{GL}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)$ is the central $\mathbb{G}_m \subset \mathrm{GL}_2/\mu_{n-2}$. Therefore, the quotient G_n/G'_n is identified with $(\mathrm{GL}_2/\mu_{n-2})/\mathbb{G}_m \simeq \mathrm{PGL}_2$, with the induced map $\mathrm{GL}_2 \to G_n \to \mathrm{PGL}_2$ the natural quotient map by the central $\mathbb{G}_m \subset \mathrm{GL}_2$.

Now, we check (3). Since we have already shown U_n is the unipotent radical of G'_n, it is a characteristic subgroup, i.e., it is preserved by automorphisms of G'_n. (Although there may not be a good notion of unipotent radical for general relative group schemes, here we simply mean that U_n is a flat subgroup scheme of G_n which base changes to the unipotent radical on every fiber over $\operatorname{Spec}\mathbb{Z}$.) Since G'_n is normal in G_n, and $U_n \subset G'_n$ is a characteristic subgroup, we obtain U_n is normal in G_n. The quotient of G_n by U_n is then generated by GL_2/μ_{n-2} induced by (2.1.3) together with the $\mathbb{G}_m$ of (2.1.1), which is central in G_n. Because $\mathbb{G}_m \cap \mathrm{GL}_2/\mu_{n-2} = 1$, this quotient G_n/U_n is $\mathbb{G}_m \times \mathrm{GL}_2/\mu_{n-2}$. Since the quotient $G_n \to \mathbb{G}_m \times \mathrm{GL}_2/\mu_{n-2}$ has a section, it follows that $G_n \simeq U_n \rtimes (\mathbb{G}_m \times (\mathrm{GL}_2/\mu_{n-2}))$.

Finally, (4) follows from (3) because, by definition, A_n is the quotient of G_n by its central copy of central $\mathbb{G}_m$. □

Our goal for the remainder of the section is to identify A_n with the automorphisms of $\mathbb{F}_{n-2}$. We first construct the map in Lemma 2.1.6 and then prove it is an isomorphism in Proposition 2.1.7. We have proven this in [Lan21b, Lemma 2.5]. For completeness, we now give a similar and slightly more detailed proof.

Lemma 2.1.6. *There is a group monomorphism of group schemes* $\theta : A_n \to \mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}$.

Proof. We claim we can realize $\mathbb{F}_{n-2}$ as a subscheme of $\operatorname{Proj} V_n$ via the complete linear system $\mathscr{O}_{\mathbb{F}_{n-2}}(e+2f)$. Indeed, the pushforward of $\mathscr{O}_{\mathbb{F}_{n-2}}(e+2f)$ to $\mathbb{P}^1$ is $\mathscr{O}(2) \oplus \mathscr{O}(n)$, and so the pushforward to $\operatorname{Spec} \mathbb{Z}$ has projectivization given by $\mathbb{P}V_n$. We claim this induces an embedding $\mathbb{F}_{n-2} \to \mathbb{P}V_n$. We further claim that under this embedding, $\mathbb{F}_{n-2}$ is realized as the subscheme swept out by lines joining a degree 2 rational normal curve embedded in a 2-plane in $\mathbb{P}V_n$ and a degree n rational normal curve embedded in a complementary n-plane in $\mathbb{P}V_n$. These claims are standard, but for a detailed proof, see [LP16, Proposition 4.4.1]. Admittedly the statement there is over a field, but it generalizes easily.

By construction, A_n is a subgroup of $\operatorname{Aut}(\mathbb{P}V_n)$. In order to construct the desired monomorphism, it is enough to show that the images of U_n and GL_2 under the actions (2.1.2) and (2.1.3) fix $\mathbb{F}_{n-2} \subset \mathbb{P}V_n$.

First, we check the action of GL_2 fixes $\mathbb{F}_{n-2}$. Using the above description of $\mathbb{F}_{n-2}$ as the subscheme swept out by lines joining two rational normal curves, because the action of GL_2 on $\mathbb{P}V_n$ is linear, it is enough to check GL_2 fixes the two rational normal curves. Indeed, the degree 2 rational normal curve lies in the copy of $\mathbb{P}H^0(\mathbb{P}^2, \mathscr{O}(2)) \subset \mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}(2) \oplus \mathscr{O}(n))$ associated to the quotient $H^0(\mathbb{P}^1, \mathscr{O}(2) \oplus \mathscr{O}(n)) \to H^0(\mathbb{P}^2, \mathscr{O}(2))$. Because the action of GL_2 on $H^0(\mathbb{P}^1, \mathscr{O}(2))$ is induced by its action on $H^0(\mathbb{P}^1, \mathscr{O}(1))$ (together with a twist dividing by the determinant that does not effect the action on the projectivization) it preserves the corresponding rational normal curve. The second rational normal curve lies in the copy of $\mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}(n))$ associated to the quotient $H^0(\mathbb{P}^1, \mathscr{O}(2) \oplus \mathscr{O}(n)) \to H^0(\mathbb{P}^1, \mathscr{O}(n))$ and GL_2 similarly preserves the corresponding rational normal curve. Altogether, we obtain a map $\mathrm{GL}_2 \to \mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}$.

It remains to verify that the action of U_n preserves $\mathbb{F}_{n-2}$. In fact, we claim U_n acts directly on $\mathbb{P}\mathscr{F}_n$ preserving the projection to $\mathbb{P}^1$. This action is induced by a subgroup of $\operatorname{Aut}(\mathscr{F}_n)$. To see this, for B a scheme and $\mathscr{F}, \mathscr{G}$ two sheaves on B let $\underline{\operatorname{Hom}}_B(\mathscr{F}, \mathscr{G})$ denote the Hom sheaf for maps $\mathscr{F} \to \mathscr{G}$. Then,

$$\begin{aligned} \underline{\operatorname{Hom}}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathscr{F}_n, \mathscr{F}_n) \simeq & \underline{\operatorname{Hom}}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}) \times_{\mathbb{Z}} \underline{\operatorname{Hom}}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)) \\ & \times_{\mathbb{Z}} \underline{\operatorname{Hom}}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2), \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)). \end{aligned} \tag{2.1.5}$$

The corresponding group of automorphisms has a subgroup $\mathbb{G}_a^{n-1} \simeq \mathrm{id} \oplus \underline{\operatorname{Hom}}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n-2)) \oplus \mathrm{id}$ which is identified with U_n. This induces an action of U_n on $\mathbb{P}\mathscr{F}_n$ which can be identified with the action we described on the image of $\mathbb{P}\mathscr{F}_n$ under the embedding to $\mathbb{P}V_n$. (We parenthetically note that another way to obtain this action of U_n on $\mathbb{P}\mathscr{F}_n$ is described in the final paragraph of the proof of Proposition 2.1.7.) □

Proposition 2.1.7. *For $n \geq 3$, the map θ of Lemma 2.1.6 is an isomorphism of group* *schemes.*

Proof. By Lemma 2.1.6, we have realized A_n as a subgroup scheme of $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}$. Note that GL_n/μ_{n-2} is smooth by [Sta, Tag 05B5]. Hence A_n is also smooth. By the fibral isomorphism criterion [Gro67, 17.9.5], it is enough to verify θ is an isomorphism on every fiber of $\operatorname{Spec}\mathbb{Z}$. Grant, for the moment, that for any algebraically closed field k, the group scheme $\mathrm{Aut}_{\mathbb{F}_{n-2}/k}$ is a smooth connected group scheme of dimension $n+3$. On any given fiber, θ is a monomorphism of group schemes, hence a closed immersion [G70, VI_B, 1.4.2]. Therefore, θ restricted to any fiber is a closed immersion of smooth connected group schemes of the same dimension, hence an isomorphism.

It remains to check that for any algebraically closed field k, $\mathrm{Aut}_{\mathbb{F}_{n-2}/k}$ is a smooth connected group scheme of dimension $n+3$. We know the dimension of $\mathrm{Aut}_{\mathbb{F}_{n-2}/k}$ is at least $n+3$ because we have produced a monomorphism from the $n+3$ dimensional group scheme $(A_n)_k$ in Lemma 2.1.6. To check $\mathrm{Aut}_{\mathbb{F}_{n-2}/k}$ is smooth and $n+3$ dimensional, it therefore suffices to show its tangent space at the identity is smooth and $n+3$ dimensional. The tangent space at the identity is identified, via deformation theory, with $H^0(\mathbb{F}_{n-2}, T_{\mathbb{F}_{n-2}})$ for $T_{\mathbb{F}_{n-2}}$ the tangent sheaf. By a standard but somewhat involved calculation carried out in [LP16, Proposition 5.3.10], this has dimension

$$\begin{aligned} h^0(\mathbb{P}^1, \mathrm{End}(\mathscr{F}_n)) + 2 &= h^0(\mathbb{P}^1, \mathrm{Hom}(\mathscr{O}, \mathscr{O})) + h^0(\mathbb{P}^1, \mathrm{Hom}(\mathscr{O}, \mathscr{O}(n-2))) \\ &\quad + h^0(\mathbb{P}^1, \mathrm{Hom}(\mathscr{O}(n-2), \mathscr{O}(n-2))) + 2 \\ &= 1 + (n-1) + 1 + 2 \\ &= n+3, \end{aligned}$$

as desired.

It remains to verify connectedness of $\mathrm{Aut}_{\mathbb{F}_{n-2}/k}$. To this end, we set up some notation. Consider the map $\phi : \mathbb{F}_{n-2} \to \mathbb{P}^{n-1}$ induced by the complete linear system $e + (n-2)f$ on $\mathbb{F}_{n-2}$. Let $Y := \phi(\mathbb{F}_{n-2})$ denote the image of ϕ. Let p be the image of the directrix and let X be the image of a choice of codirectrix. The map ϕ contracts the directrix, but is an embedding away from the directrix. This fact is standard, and is also explained in the proof of Lemma 3.1.5. Because $\mathbb{F}_{n-2}$ is the blow up of Y at p, automorphisms of $\mathbb{P}^{n-1}$ fixing Y are in bijection with automorphisms of $\mathbb{F}_{n-2}$.

It remains to show automorphisms of Y, $\mathrm{Aut}(Y)$, are given as k-points of A_n for k algebraically closed. Any automorphism of Y induces an automorphism of $\mathbb{P}^{n-1}$ restricting to that automorphism, and so we can identify $\mathrm{Aut}(Y)$ with the subgroup of $\mathrm{PGL}_n(k)$ fixing Y. Inside Y, we have a rational normal curve X which spans an $(n-2)$-plane $P \subset \mathbb{P}^{n-1}$. There is a surjection $\mathrm{Aut}(Y) \to \mathrm{Aut}(X) \simeq \mathrm{PGL}_2(k)$. We also have a surjection $A_n \to \mathrm{PGL}_2(k)$, and by Lemma 2.1.4, the kernel is $\mathbb{G}_a^{n-1} \rtimes \mathbb{G}_m$. The copy of $\mathbb{G}_a^{n-1}$ can be identified with unipotent matrices fixing p and P pointwise, while $\mathbb{G}_m$ can be identified with block diagonal matrices fixing p and P pointwise. Since products of elements of these two subgroups give all elements of $\mathrm{PGL}_n(k)$ fixing P and p, we obtain the desired isomorphism. □

Remark 2.1.8. If one wishes, one can avoid the reference to [LP16, Proposition 5.3.10] in the proof of Proposition 2.1.7 by directly computing the automorphism group of $\mathbb{F}_{n-2}$ over general bases. This can be done by suitably generalizing the argument of the last two paragraphs of the proof.

2.2 Defining various stacks

In this section, we carry out numerous verifications that various objects are stacks. The verifications are of the sort that often seem to be omitted in modern day articles. We hope it will be helpful for the reader to refer to Figure 8.1 for pithy descriptions of the relevant stacks.

2.2.1 Weierstrass stacks

We begin by defining the stack of Weierstrass curves. By this we mean genus 1 geometrically integral curves with a section in the smooth locus. It turns out this is equivalent to the stack of curves with a Weierstrass equation, as we shall see in Proposition 2.2.13. We also define various substacks such as the nodal and cuspidal substacks.

Once again, the content of this section is surely well known. We were able to find many constructions of the dense open subscheme of this stack parameterizing smooth elliptic curves, or pointed genus 1 curves with at worst nodes. However, it will be crucial in the remainder of the paper to also include pointed genus 1 curves with cusps. These cusps will correspond to places where the associated quadratic fields ramify. Surprisingly, we were not able to find a reference carefully constructing the stack of all Weierstrass curves, and so we now carry this out.

Definition 2.2.2. We define *the stack of Weierstrass curves* $\mathscr{W}$ as the fibered category whose points are tuples $(B, f : C \to B, e : B \to C)$ where $f : C \to B$ is a proper flat finitely presented genus 1 curves with geometrically integral fibers and $e : B \to C$ is a section of f lying in the smooth locus of f. By e being a section, we mean $f \circ e = \mathrm{id}_B$. The morphisms $(B, f : C \to B, e : B \to C) \to (B', f' : C' \to B', e' : B' \to C')$ in this fibered category are morphisms $\alpha : B \to B', \beta : C \to C'$ such that

$$\begin{array}{ccc} C & \xrightarrow{\beta} & C' \\ {\scriptstyle e}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle e'}\uparrow\downarrow{\scriptstyle f'} \\ B & \xrightarrow{\alpha} & B' \end{array} \tag{2.2.1}$$

satisfies $\beta \circ e = e' \circ \alpha$, $\alpha \circ f = f' \circ \beta$ and $C \simeq B \times_{B'} C'$.

Next, we introduce the stack of Weierstrass curves with a marked singular point. Throughout, we will indicate the marking of this singular point by including a tilde in the notation, and we will omit the tilde when we do not mark the singular point.

Definition 2.2.3. Let $\widetilde{\mathscr{W}_{\mathrm{sing}}}$ denote the fibered category whose B points are tuples $(B, f : C \to B, e : B \to C, \tau : B \to C)$ where $(B, f : C \to B, e : B \to C) \in \mathscr{W}(B)$ and $\tau : B \to C$ is a morphism such that $f \circ \tau = \mathrm{id}$ and $\mathrm{im}\,\tau$ contained in the singular locus of f. Morphisms $(B, f : C \to B, e : B \to C, \tau : B \to C) \to (B', f' : C' \to B', e' : B' \to C', \tau' : B' \to C')$ consist of maps $\alpha : B \to B', \beta : C \to C'$ such that the square

$$\begin{array}{ccc} C & \xrightarrow{\beta} & C' \\ {\scriptstyle \tau}\ {\scriptstyle e}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle e'}\ {\scriptstyle \tau'} \\ B & \xrightarrow{\alpha} & B' \end{array} \tag{2.2.2}$$

satisfies $C \simeq B \times_{B'} C'$, $\beta \circ e = e' \circ \alpha$, and $\beta \circ \tau = \tau' \circ \alpha$.

Let $\widetilde{\mathscr{W}_{\mathrm{sing}}} \to \mathscr{W}$ denote the natural map forgetting the section τ, and let $\mathscr{W}_{\mathrm{sing}}$ denote the image of $\widetilde{\mathscr{W}_{\mathrm{sing}}}$ in $\mathscr{W}$.

Next, we wish to show the above defined stacks are algebraic. To do so, we will construct them as quotients of certain Hilbert schemes, which we introduce next.

Definition 2.2.4. Let $\mathcal{H}^{1,3t}$ denote the flag Hilbert scheme parameterizing $p \subset X \subset \mathbb{P}^2$ where p is a section and X is a relative plane cubic. Let $\mathcal{H}^{\circ,1,3t}$ denote the locally closed subscheme of $\mathcal{H}^{1,3t}$ parameterizing those $p \subset X \subset \mathbb{P}^2$ such that X is geometrically integral, $p \in X$ lies in the smooth locus of X, and p is a flex point of X (i.e., the tangent line to X at p meets X in a subscheme of degree 3).

Let $\mathcal{H}^{1,1,3t}$ denote the flag Hilbert scheme parameterizing (p, q, X) with $p, q \in \mathbb{P}^2$ two points, X a plane cubic, and $p \in X, q \in X$. Let $\mathcal{H}^{\circ,1,3t}_{\mathrm{sing}}$ denote the locally closed subscheme of $\mathcal{H}^{1,1,3t}$ such that p lies in the singular locus of X, X is geometrically integral, and q is a flex point in the smooth locus of X.

Note that $\mathcal{H}^{\circ,1,3t}$ and $\mathcal{H}^{\circ,1,3t}_{\mathrm{sing}}$ have actions of PGL_3 via the action on the ambient $\mathbb{P}^2$.

Lemma 2.2.5. *We have isomorphisms* $\mathscr{W} \simeq [\mathcal{H}^{\circ,1,3t}/\mathrm{PGL}_3]$ *and* $\widetilde{\mathscr{W}}_{\mathrm{sing}} \simeq [\mathcal{H}^{\circ,1,3t}_{\mathrm{sing}}/\mathrm{PGL}_3]$. *In particular,* $\mathscr{W}$ *and* $\widetilde{\mathscr{W}}_{\mathrm{sing}}$ *are algebraic.*

Proof. First, let $\widetilde{\mathcal{H}^{\circ,1,3t}}$ denote the functor assigning to a scheme B the set of geometrically integral genus 1 curves $f : C \to B$ with a flex point e in the smooth locus together with an isomorphism $f_*\mathscr{O}_C(3e) \simeq \mathscr{O}_B^{\oplus 3}$. Since $\mathcal{H}^{\circ,1,3t} \simeq [\widetilde{\mathcal{H}^{\circ,1,3t}}/\mathbb{G}_m]$ with $\mathbb{G}_m$ scaling $\mathscr{O}_B$, and $\mathscr{W} \simeq [\widetilde{\mathcal{H}^{\circ,1,3t}}/\mathrm{GL}_3]$, it follows that $\mathscr{W} \simeq [\mathcal{H}^{\circ,1,3t}/\mathrm{PGL}_3]$.

The second isomorphism follows similarly because $\mathcal{H}^{\circ,1,3t}_{\mathrm{sing}}$ represents the functor assigning to a scheme B the set of geometrically integral genus 1 curves $f : C \to B$ with a flex point e in the smooth locus and τ in the singular locus, together with an isomorphism $f_*\mathscr{O}_C(3e) \simeq \mathscr{O}_B^{\oplus 3}$, modulo the scaling action of $\mathbb{G}_m$. □

We next define the nodal and cuspidal substacks of $\widetilde{\mathscr{W}}_{\mathrm{sing}}$.

Definition 2.2.6. Let $\widetilde{\mathscr{W}}_{\mathrm{node}}$ be the substack of $\widetilde{\mathscr{W}}_{\mathrm{sing}}$ parameterizing those tuples $(B, f : C \to B, e : B \to C, \tau : B \to C) \in \widetilde{\mathscr{W}}_{\mathrm{sing}}(B)$ such that τ maps B isomorphically to the singular locus of f. Let $\widetilde{\mathscr{W}}_{\mathrm{cusp}}$ denote the substack of $\widetilde{\mathscr{W}}_{\mathrm{sing}}$ defined as the fibered category whose fiber over B is a tuple $(B, f : C \to B, e : B \to C, \tau : B \to C)$ as in (2.2.3) with the following property: Let $X \subset C$ denote the singular locus of $f : C \to B$. Then $\ker(f_*\mathscr{O}_X \to f_*\mathscr{O}_{\tau(B)})$ is not the pushforward of a sheaf from any proper closed subscheme of B.

Remark 2.2.7. The stacks $\widetilde{\mathscr{W}}_{\mathrm{node}}$ and $\widetilde{\mathscr{W}}_{\mathrm{cusp}}$ indeed parameterize families of nodal and cuspidal curves. This follows from Lemma 2.2.8 below.

One can also use the above descriptions to equivalently define $\widetilde{\mathscr{W}}_{\mathrm{cusp}}$ as the scheme theoretic closure of the degree at least 2 part of the flattening stratification associated to the singular locus of the universal curve over $\mathscr{W}$.

Lemma 2.2.8. *The singular locus of a nodal curve* $C \to B$ *maps isomorphically to* B. *The singular locus of a cuspidal curve has relative degree* 3 *in characteristic* 3, *relative degree* 4 *in characteristic* 2, *and relative degree* 2 *in all other characteristics.*

Proof. These claims can be easily verified by direct computation with the complete local models for nodes and cusps. Indeed, a nodal singularity over a field k has complete local ring isomorphic to $k[\![x, y]\!]/(xy)$ and so its singular locus is the section $V(x, y)$. A cuspidal singularity over a field k has complete local ring isomorphic to $k[\![x, y]\!]/(y^2 - x^3)$. Therefore, its singular locus is given by $V(2y, 3x^2, y^2 - x^3)$ which indeed has degree 2 in characteristics other than 2 and 3, degree 3 in characteristic 3, and degree 4 in characteristic 2. □

Lemma 2.2.9. *The substack $\widetilde{\mathscr{W}}_{\text{node}} \subset \widetilde{\mathscr{W}}_{\text{sing}}$ is an open substack and the substack $\widetilde{\mathscr{W}}_{\text{cusp}} \subset \widetilde{\mathscr{W}}_{\text{sing}}$ is a closed substack.*

Proof. To verify openness of $\widetilde{\mathscr{W}}_{\text{node}} \subset \widetilde{\mathscr{W}}_{\text{sing}}$, it suffices to verify that for any $(B, f : C \to B, e : B \to C, \tau : B \to C) \in \widetilde{\mathscr{W}}_{\text{sing}}(B)$, the fiber product $\widetilde{\mathscr{W}}_{\text{node}} \times_{\widetilde{\mathscr{W}}_{\text{sing}}} B$ is an open subscheme of B. To see this, if Z denotes the singular locus of f, observe that $\widetilde{\mathscr{W}}_{\text{node}} \times_{\widetilde{\mathscr{W}}_{\text{sing}}} B$ is the subscheme where f has degree 1. By upper semicontinuity of fiber degree of the $Z \to B$, and because the image of Z in B is schematically dense from the definition of $\widetilde{\mathscr{W}}_{\text{sing}}$, it follows that the subscheme where f has degree 1 is open.

To check $\widetilde{\mathscr{W}}_{\text{cusp}} \subset \widetilde{\mathscr{W}}_{\text{sing}}$ is closed, let $(B, f : C \to B, e : B \to C, \tau : B \to C) \in \widetilde{\mathscr{W}}_{\text{sing}}(B)$, and let $Z \subset C$ denote the singular locus of f. The fiber product $\widetilde{\mathscr{W}}_{\text{cusp}} \times_{\widetilde{\mathscr{W}}_{\text{sing}}} B$ is then closed subscheme of B given by the scheme theoretic support of $f_* \mathscr{O}_Z \to f_* \mathscr{O}_{\tau(B)}$. □

We next describe the relation between the above stack $\mathscr{W}$ and Weierstrass curves. As we will show, $\mathscr{W}$ is actually the quotient of $\mathbb{A}^5$ (and not just a nebulous dense open in $\mathbb{A}^5$) by a certain algebraic group. The $\mathbb{A}^5$ is parameterized by the 5 coordinates in a standard Weierstrass equation, as we now recall.

Definition 2.2.10. Let $E \to \mathbb{A}^5_{a_1,a_2,a_3,a_4,a_6}$ denote the Weierstrass curve defined by

$$y^2 z + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3 \subset \mathbb{P}^2_{x,y,z} \times \mathbb{A}^5_{a_1,a_2,a_3,a_4,a_6}. \tag{2.2.3}$$

Let U denote the 3-dimensional algebraic group of upper triangular matrices, with functor of points

$$U(R) = \left\{ \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix} : b_1, b_3, b_2 \in R \right\}.$$

In the above coordinates, there is an action of U on E induced by

$$U(R) \times \mathbb{P}^2 \to \mathbb{P}^2 \tag{2.2.4}$$

$$\left(\begin{pmatrix} 1 & b_1 & b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}, [y, x, z] \right) \mapsto [y + b_1 x + b_3 z, x + b_2 z, z]. \tag{2.2.5}$$

More precisely, the above action extends to an action of $U(R)$ on $\mathbb{P}^2 \times \mathbb{A}^5$ sending the a_i coordinates on $\mathbb{A}^5$ to what one obtains when one expands (2.2.3) under the inverse change of variables $[y, x, z] \mapsto [y - b_1 x + (b_1 b_2 - b_3)z, x - b_2 z, z]$. The reason for taking the inverse is that the universal curve in $\mathbb{P}^2 \times \mathbb{A}^5$ is given by those (p, f) for which $f(p) = 0$. If we denote the upper triangular matrix of (2.2.4) by B then B should send f to a polynomial f' satisfying $f'(Bp) = 0$ so we can take $f' = f \circ B^{-1}$. Concretely, this action sends (2.2.3) to

$$\begin{aligned}
&y^2 z + (a_1 - 2b_1)xyz + (-a_1 b_2 + a_3 + 2b_1 b_2 - 2b_3)yz^2 \\
&= x^3 + (a_1 b_1 + a_2 - b_1^2 - 3b_2)x^2 z \\
&+ (-2a_1 b_1 b_2 a_1 b_3 - 2a_2 b_2 + a_3 b_1 + a_4 + 2b_1^2 b_2 - 2b_1 b_3 + 3b_2^2)xz^2 \\
&+ (a_1 b_1 b_2^2 - a_1 b_2 b_3 + a_2 b_2^2 - a_3 b_1 b_2 + a_3 b_3 - a_4 b_2 + a_6 - b_1^2 b_2^2 + 2b_1 b_2 b_3 - b_2^3 - b_3^2)z^3.
\end{aligned}$$

Further, we have an action of $\mathbb{G}_m$ on E induced by

$$\begin{aligned}
\mathbb{G}_m(R) \times \mathbb{P}^2 \times \mathbb{A}^5 &\to \mathbb{P}^2 \times \mathbb{A}^5 \\
(\lambda, [y, x, z], (a_1, a_2, a_3, a_4, a_6)) &\mapsto ([\lambda^{-3} y, \lambda^{-2} x, z], (\lambda^{-1} a_1, \lambda^{-2} a_2, \lambda^{-3} a_3, \lambda^{-4} a_4, \lambda^{-6} a_6)).
\end{aligned}$$

Together the above two groups induce an action $G := U \rtimes \mathbb{G}_m$ on E and a compatible action on $\mathbb{A}^5$ via projection to $\mathbb{A}^5$.

Our next goal is to show $[\mathbb{A}^5/G] \simeq \mathscr{W}$. We first construct the map, then show it is surjective, and finally check it is an isomorphism.

Lemma 2.2.11. *There is a map $\mathbb{A}^5 \to \mathscr{W}$ given by sending a Weierstrass equation f to the genus 1 curve $V(f)$ with the marked point* $[0, 1, 0]$.

Proof. First, the map $\mathbb{A}^5 \to \mathscr{W}$ is given by the genus 1 curve with section $x = z = 0$ over $\mathbb{A}^5$ defined by (2.2.3). This family is certainly flat as its Hilbert polynomial is constant. To obtain the desired map, we need to check this family has geometrically integral fibers. Indeed, suppose some fiber were not geometrically integral. Since it is defined by a single equation in $\mathbb{P}^2$, that equation necessarily factors, say as a product $(ax + by + cz)(ix^2 + jy^2 + mz^2 + nxy + pxz + qyz)$. Now, because the coefficient of y^3 in (2.2.3) vanishes, either $b = 0$ or $j = 0$.

We first rule out the possibility that $j \neq 0$. Suppose $j \neq 0$. Then $b = 0$, and because the coefficient of $y^2 x$ vanishes, but $j \neq 0$, this would force $a = 0$, contradicting the assumption that the coefficient of x^3 is 1. Therefore, we must have $j = 0$.

Now, suppose $j = 0$. If $b \neq 0$, the vanishing of the coefficient of $y^2 x$ forces $n = 0$ and then the vanishing of the coefficient of yx^2 forces $i = 0$, again contradicting the assumption that the coefficient of $x^3 = 1$. Hence, we must have $b = j = 0$. However, this contradicts the assumption that the coefficient of $y^2 z$ is 1. □

Lemma 2.2.12. *The map $\mathbb{A}^5 \to \mathscr{W}$ is surjective.*

Proof. This follows the standard procedure of putting an elliptic curve in Weierstrass form, and we now verify it works for geometrically integral genus 1 curves with a smooth point. In other words, we must show that any genus 1 geometrically integral curve over an algebraically closed field with a marked point has a representation in the form (2.2.3). Given a genus 1 curve C over an algebraically closed field with a marked point e, the invertible sheaf $\mathscr{O}_C(3e)$ embeds C as a plane cubic in $\mathbb{P}^2$. We may apply an automorphism of C so as to assume e is the point $[0, 1, 0]$ and further that the tangent line at C to e is the line $z = 0$. The first condition forces the coefficients of y^3 to be 0, and the second forces the coefficient of y^2x to be 0. Since the tangent line at e meets C to order 3, we also obtain that the coefficient of yx^2 is 0.

It remains to arrange for the coefficients of y^2z and x^3 to be 1. First, observe that the coefficient of x^3 cannot vanish, or else z would divide the equation defining C, and therefore C would be reducible. Hence, we may rescale the equation to assume the coefficient of x^3 is 1. Next, if the coefficient of y^2z were 0, the curve would be singular at e, contradicting the assumption that e lies in the smooth locus. Therefore, the coefficient of y^2z must be nonzero, and rescaling z by this coefficient allows us to assume the coefficient of y^2z is also 1. Altogether, we have obtained an isomorphism between C and a curve of the form (2.2.3). □

Proposition 2.2.13. *The fibered category $\mathscr{W}$ is isomorphic to the stack quotient $[\mathbb{A}^5/G]$ with proper universal curve $\overline{\mathscr{E}} \to \mathscr{W}$ given by $[E/G]$. The map $\overline{\mathscr{E}} \to \mathscr{W}$ satisfies the property that for $(B, f : C \to B, e : B \to C) \to \mathscr{W}(B)$, we have $C \simeq B \times_{\mathscr{W}} \overline{\mathscr{E}}$.*

Proof. We have constructed the map $\mathbb{A}^5 \to \mathscr{W}$ in Lemma 2.2.11. Letting G be the group defined at the end of Definition 2.2.10, we wish to show this map is a G torsor.

As a first step, we verify the map $G \times \mathbb{A}^5 \to \mathbb{A}^5 \times_{\mathscr{W}} \mathbb{A}^5$ sending $(g, a) \mapsto (ga, a)$ is an isomorphism and moreover that the stabilizer in G of a point of $\mathbb{A}^5$ is identified with the automorphism group scheme of the corresponding elliptic curve over that point. By the fibral isomorphism criterion [Gro67, 17.9.5] applied to the map $G \times \mathbb{A}^5 \to \mathbb{A}^5 \times_{\mathscr{W}} \mathbb{A}^5$ over the projection to the second $\mathbb{A}^5$ factor, it is enough to check the map is an isomorphism on geometric fibers over $\mathbb{A}^5$. In other words, given a Weierstrass equation over a field corresponding to a Weierstrass curve $f_a : E_a \to a$ with identity point e_a associated to the point $a \in \mathbb{A}^5$, we wish to check that any Weierstrass family isomorphic to E_a is related to E_a by the G action, and the kernel of the G action is trivial. We check this on R-valued point of G. Any isomorphism of Weierstrass data $(\operatorname{Spec} R, f : E \to \operatorname{Spec} R, e : \operatorname{Spec} R \to E) \simeq (a, E_a, e_a)$ over R is induced by an isomorphism of the flag $f_*\mathscr{O}_E(e) \subset f_*\mathscr{O}_E(2e) \subset f_*\mathscr{O}_E(3e)$ with the standard flag $1 \cdot \mathscr{O}_B \subset (1, x) \cdot \mathscr{O}_B \subset (1, x, y) \cdot \mathscr{O}_B$ that is compatible with the isomorphisms $f_* \mathscr{O}_E \simeq f_* \mathscr{O}_E(e)$ and $(f_* \mathscr{O}_E(2e)/f_* \mathscr{O}_E(e))^{\otimes 3} \simeq (f_* \mathscr{O}_E(3e)/f_* \mathscr{O}_E(2e))^{\otimes 2}$ specified by the Weierstrass equation. Such isomorphisms are given as the composite of the $\mathbb{G}_m$ action of Definition 2.2.10, which scales the graded pieces of the flag, and an action of the group U from Definition 2.2.10 which preserves the graded pieces of the flag. Note here that we only have $\mathbb{G}_m$ as opposed to $\mathbb{G}_m^3$ preserving the flag since the action has to preserve the above two isomorphisms $f_* \mathscr{O}_E \simeq f_* \mathscr{O}_E(e)$ and $(f_* \mathscr{O}_E(2e)/f_* \mathscr{O}_E(e))^{\otimes 3} \simeq (f_* \mathscr{O}_E(3e)/f_* \mathscr{O}_E(2e))^{\otimes 2}$. Therefore, any such isomorphism is specified by an element of the group $G(R)$.

To conclude this first step, we only need to check $G(R)$ acts simply on the Weierstrass data. In other words, the only element of $G(R)$ acting as the identity on (E_a, e_a) is the identity element. However, we may note that $\mathscr{O}_{E_a}(3e_a)$ is very ample on E, and so only the identity element of PGL_3 can act trivially on E. Hence, viewing G as a subgroup of GL_3, it suffices to show that the intersection of G with the diagonal $\mathbb{G}_m$ in GL_3 is trivial. This intersection is contained in the $\mathbb{G}_m$ described in Definition 2.2.10, which acts trivially on the z coordinate, and therefore has trivial intersection with the central $\mathbb{G}_m \subset \mathrm{GL}_3$.

To conclude that $\mathbb{A}^5 \to \mathscr{W}$ is indeed a G torsor, it only remains to verify it is fppf. Using our above verification that the stabilizers of the action of G on $\mathbb{A}^5$ are identified with the automorphisms of the corresponding genus 1 curve with section, we obtain that the stabilizers are 0-dimensional. Therefore, the fibers of the map $\mathbb{A}^5 \to \mathscr{W}$ are all of dimension equal to $\dim G$. We next claim $\mathscr{W}$ is smooth. To see this, its tangent space at a geometric point $(x, f : E \to x, e : x \to E)$ is identified with $H^1(E, \mathscr{O}_E(-e))$, which is 1-dimensional. To verify smoothness, we only need check $\mathscr{W}$ is 1-dimensional, which follows from the explicit description of Lemma 2.2.5. Because $\mathbb{A}^5$ and $\mathscr{W}$ are smooth, it follows from miracle flatness (whose generalization from schemes to stacks follows from passing to a smooth cover of the target) that $\mathbb{A}^5 \to \mathscr{W}$ is flat. Altogether, this flatness combined with Lemma 2.2.12 shows $\mathbb{A}^5 \to \mathscr{W}$ is fppf, hence a G torsor. □

2.2.14 Stacks of n-coverings

We next define various stacks and schemes associated to n-coverings of Weierstrass curves.

Definition 2.2.15. Define $\mathscr{E}$ as the smooth locus of $\overline{\mathscr{E}} \to \mathscr{W}$.

In fact, it turns out that $\mathscr{E}$ is naturally a group scheme.

Lemma 2.2.16. *The natural map $\mathscr{E} \to \mathrm{Pic}^0_{\overline{\mathscr{E}} \to \mathscr{W}}$ sending a section p to $\mathscr{O}_{\overline{\mathscr{E}}}(p - e)$, for e the identity section, is an isomorphism. This gives $\mathscr{E}$ the structure of a commutative group, and in particular endows it with a notion of multiplication by n.*

Proof. This may be verified on geometric fibers by the fibral isomorphism criterion [Gro67, 17.9.5]. In this case one can verify it separately for smooth, nodal, and cuspidal genus 1 curves over algebraically closed fields. For smooth curves, this is just the usual isomorphism of an elliptic curve E over k with $\mathrm{Pic}^0_{E/k}$. When the curve C is nodal with smooth subscheme C^{sm}, we obtain a map $C^{\mathrm{sm}} \to \mathbb{G}_m$. We know C^{sm} is abstractly isomorphic to $\mathbb{G}_m$, but we wish to check this map is actually an isomorphism. However, it is a surjective monomorphism, so it is an open immersion and hence an isomorphism. Analogously we find that when C is cuspidal, the map $C^{\mathrm{sm}} \to \mathbb{G}_a$ is an isomorphism. For more details, see the discussion in [BLR90, §9.2], especially [BLR90, Proposition 3, Proposition 4, Example 8, and Proposition 9], noting that in the above case we can identify Pic^0 with Pic^1 because $\overline{T}$ is generically smooth over an algebraically closed field k and therefore has a k point. □

Definition 2.2.17. Let $\mathscr{E}$ act on itself via the multiplication by n map $\mathscr{E} \xrightarrow{\times n} \mathscr{E}$ via Lemma 2.2.16 relatively over $\mathscr{W}$. Define the *stack of n-coverings of Weierstrass curves* $\mathscr{S}^{(n)} := [\mathscr{E}/n\mathscr{E}]$ as the quotient stack of $\mathscr{E}$ with respect to the above action of $\mathscr{E}$ on itself.

Example 2.2.18. When $n = 1$, we in fact have $\mathscr{S}^{(n)} \simeq \mathscr{W}$.

Later, in Proposition 2.3.15, we will demonstrate that $\mathscr{S}^{(n)}$ is actually isomorphic to a certain quotient of a Hilbert scheme by an action of PGL_n, which we call $\mathscr{M}_1^{(n)}$ and introduce next. This isomorphism takes a bit of setup, but for the moment, we will introduce the relevant Hilbert scheme and stack.

Definition 2.2.19. Let $n \geq 3$ and let $\mathscr{H}^{(n)}$ denote the open subscheme of the Hilbert scheme of subschemes of $\mathbb{P}^{n-1}$ whose geometric points are geometrically integral genus 1 degree n curves. Note this is indeed open by [Gro66, Théoremè 12.2.4(viii)].

We next construct an action of PGL_n on $\mathscr{H}^{(n)}$. induced by the action of PGL_n on $\mathbb{P}^{n-1}$ as automorphisms of $\mathbb{P}^{n-1}$. Namely, for $\mathscr{U}_n \subset \mathscr{H}^{(n)} \times \mathbb{P}^{n-1}$, the universal curve over over $\mathscr{H}^{(n)}$, any R-point of PGL_n induces an automorphism $\phi : \mathbb{P}^{n-1}_R \to \mathbb{P}^{n-1}_R$ and hence an embedding $(\mathscr{U}_n)_R \subset (\mathscr{H}^{(n)})_R \times \mathbb{P}^{n-1}_R \xrightarrow{\mathrm{id} \times \phi} (\mathscr{H}^{(n)})_R \times \mathbb{P}^{n-1}_R$. Since this composite is a subscheme of $\mathscr{H}^{(n)} \times \mathbb{P}^{n-1}$ whose fibers are geometrically integral genus 1 degree n curves, this induces a map $(\mathscr{H}^{(n)})_R \to (\mathscr{H}^{(n)})_R$ by the universal property of the Hilbert scheme. Altogether, this gives an action of PGL_n on $\mathscr{H}^{(n)}$. For $n \geq 3$, define $\mathscr{M}_1^{(n)} := [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ as the resulting stack quotient.

Remark 2.2.20. Let $n \geq 3$. The stack $\mathscr{M}_1^{(n)}$ has the following description as a fibered category: B-points are tuples $(B, f : P \to B, \iota : C \to P)$ such that

1. $f : P \to B$ is a Brauer-Severi scheme of relative dimension $n - 1$ over B,

2. $\iota : C \to P$ is a closed immersion,

3. $f \circ \iota : C \to B$ is a proper flat finitely presented arithmetic genus 1 curve with geometrically integral fibers which has degree n fppf locally on B (or equivalently on geometric fibers over points of B).

A morphism $(B, f : P \to B, \iota : C \to P) \to (B', f' : P' \to B', \iota' : C' \to P')$ is given by maps $\alpha : B \to B', \beta : P \to P', \gamma : C \to C'$ so that all squares in

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ P & \xrightarrow{\beta} & P' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\alpha} & B' \end{array} \tag{2.2.6}$$

are fiber squares.

We next introduce analogs of the above constructions where we mark a section in the singular locus of the genus 1 curve.

Definition 2.2.21. Let $n \in \mathbb{Z}_{\geq 3}$. Define $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ as the fibered category whose B-points are tuples $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ such that

1. $f : P \to B$ is a Brauer-Severi scheme of relative dimension $n - 1$ over B,

2. $\iota : C \to P$ is a closed immersion,

3. $f \circ \iota : C \to B$ is a proper flat finitely presented arithmetic genus 1 curve with geometrically integral fibers which has degree n fppf locally on B (or equivalently on geometric fibers over points of B),

4. $\tau : B \to C$ a morphism so that $f \circ \iota \circ \tau = \mathrm{id}$ with $\mathrm{im}\, \tau$ contained in the singular locus of $f \circ \iota$.

A morphism $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \to (B', f' : P' \to B', \iota' : C' \to P', \tau' : B' \to C')$ is given by maps $\alpha : B \to B', \beta : P \to P', \gamma : C \to C'$ so that all squares in

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ P & \xrightarrow{\beta} & P' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\alpha} & B' \end{array} \quad (\tau : B \to C,\ \tau' : B' \to C') \tag{2.2.7}$$

are fiber squares with $\gamma \circ \tau = \tau' \circ \alpha$.

Remark 2.2.22. Note that the Brauer-Severi scheme P in Definition 2.2.21 possesses a section τ and is therefore trivial, hence isomorphic to a projective bundle over B.

We next define the nodal and cuspidal loci of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$, $\widetilde{\mathscr{M}}_{1,\mathrm{node}}^{(n)}$ and $\widetilde{\mathscr{M}}_{1,\mathrm{cusp}}^{(n)}$. The argument for why these are open an closed substacks of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ is completely analogous to the argument given in Lemma 2.2.9.

Definition 2.2.23. For $n \geq 3$, let $\widetilde{\mathscr{M}}_{1,\mathrm{node}}^{(n)}$ be the open substack of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ parameterizing those tuples $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ such that τ maps B isomorphically to the singular locus of $f \circ \iota$. Let $\widetilde{\mathscr{M}}_{1,\mathrm{cusp}}^{(n)}$ denote the closed substack of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ defined as the fibered category whose fiber over B is a tuple $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ as in Definition 2.2.21 with the following property: Let $X \subset C$ denote the singular locus of $f : C \to B$. Then $\ker(f_* \mathscr{O}_X \to f_* \mathscr{O}_{\tau(B)})$ is not the pushforward of a sheaf from any proper closed subscheme of B.

In order to show $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ is an algebraic stack, we will construct it as the quotient of a certain Hilbert scheme by PGL_n. We now define this Hilbert scheme.

Definition 2.2.24. Let $n \in \mathbb{Z}_{\geq 3}$. Let $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$ denote the functor from schemes to sets whose B-points are $(B, \iota : C \to \mathbb{P}^{n-1}_B, \tau : B \to C)$ defined as follows. Let $f : \mathbb{P}^{n-1}_B \to B$ denote the structure morphism. Then $\iota : C \to \mathbb{P}^{n-1}_B$ is a closed immersion, $f \circ \iota : C \to B$ a proper flat finitely presented arithmetic genus 1 degree n curve with geometrically integral fibers, and $\tau : B \to C$ a morphism so that $f \circ \iota \circ \tau = \mathrm{id}$ with $\mathrm{im}\, \tau$ contained in the singular locus of $f \circ \iota$.

Lemma 2.2.25. *For $n \geq 3$, $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$ is represented by a scheme.*

Remark 2.2.26. Lemma 2.2.25 can be generalized to an analogous statement for $n = 1, 2$ after suitably generalizing the definition of $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$, though we will not need this.

Proof. Indeed, $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$ is a locally closed subscheme of the flag Hilbert scheme parameterizing $\tau \subset X \subset \mathbb{P}^{n-1}$ where X is a genus 1 degree n curve and τ is a section. This has a closed subscheme where τ is contained in the singular locus of X over the base. Further, this has an open subscheme where the geometric fibers of $C \to B$ are integral by openness of the geometrically integral locus [Gro66, Théoremè 12.2.4(viii)]. □

There is a natural map $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)} \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ sending a B-point of $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$, $(B, \iota : C \to \mathbb{P}^{n-1}_B, \tau : B \to C)$, to the tuple $(B, f : \mathbb{P}^{n-1}_B \to B, \iota : C \to \mathbb{P}^{n-1}_B, \tau : B \to C)$, considered as a point over B in the fibered category $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$. Observe that PGL_n acts naturally on $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$ via its action on $\mathbb{P}^{n-1}$ and by definition of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$, the map $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)} \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ is invariant for this action of PGL_n. Therefore, we obtain a map $\phi : [\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}/\mathrm{PGL}_n] \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$.

Corollary 2.2.27. *For $n \geq 3$, the map $\phi : [\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}/\mathrm{PGL}_n] \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ constructed above is an equivalence of fibered categories. In particular, $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ is an algebraic stack.*

Proof. We construct the inverse map. Given a point $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ over B, we need to construct a PGL_n torsor over this point with a PGL_n equivariant map to $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$. Indeed, because P is a Brauer-Severi scheme, we have a PGL_n torsor $T := \mathrm{isom}_B(\mathbb{P}^n_B, P)$ over B [Gro68, 8.1]. By the universal property of isom, we obtain an isomorphism $P_T \simeq \mathbb{P}^{n-1}_T$. Pulling back C to T gives a closed subscheme $C_T \to \mathbb{P}^{n-1}_T$ which has degree n because it had degree n over B fppf locally. Altogether, this yields the desired PGL_n equivariant map $T \to \widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$.

We claim this map is inverse to the map ϕ. To prove this, we use the fact that there is an equivalence of categories between Brauer-Severi schemes of dimension $n-1$ over B and PGL_n torsors over B given by sending a Brauer-Severi scheme $P \to B$ to $\mathrm{isom}_B(\mathbb{P}^{n-1}_B, P)$ [Gro68, 8.1]. By descent for closed embeddings, there is an equivalence between the following two categories. First, the category of PGL_n torsors T over B together with PGL_n equivariant data of $\iota_T : C_T \to \mathbb{P}^{n-1}_T$ and $\tau_T : T \to C_T$. Second, the category of Brauer-Severi schemes $P \to B$ with the closed embeddings $\iota : C \to P$ and $\tau : B \to C$. □

We next introduce various nodal and cuspidal substacks of $\mathscr{M}_1^{(n)}$. The following lemma will be used to define nodal and cuspidal loci without a marked section.

Lemma 2.2.28. *For $n \geq 3$, the maps $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{M}_1^{(n)}$ are finite. Similarly, the map $\widetilde{\mathscr{W}}_{\mathrm{sing}} \to \mathscr{W}$ is finite.*

Proof. The idea will be to use the valuative criterion for properness. We will begin with verifying the statement for $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{M}_1^{(n)}$. Using Corollary 2.2.27 the map $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{M}_1^{(n)}$ is in fact the quotient of a map of finite presentation Hilbert schemes $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)} \to \mathscr{H}^{(n)}$ by the respective PGL_n actions. Therefore, it is enough to check $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)} \to \mathscr{H}^{(n)}$ is finite. We will do so by checking it is proper and quasi-finite.

We first check quasi-finiteness of $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)} \to \mathscr{H}^{(n)}$. Take a geometric point of $\mathscr{H}^{(n)}$ corresponding to a degree n genus 1 geometrically integral curve $E \to \mathbb{P}^n$ over an algebraically closed field. The preimage in $\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$ is by definition the singular locus of E, which is quasi-finite by the assumption that E is geometrically reduced.

It remains to check $\widetilde{\mathscr{H}}^{(n)}_{\text{sing}} \to \mathscr{H}^{(n)}$ is proper by verifying the valuative criterion for properness. We need to show that if we are given a map $\operatorname{Spec} R \to \mathscr{H}^{(n)}$ for R a discrete valuation ring with fraction field K, and a map $\operatorname{Spec} K \to \widetilde{\mathscr{H}}^{(n)}_{\text{sing}}$, there is a unique extension of $\operatorname{Spec} R$ to $\widetilde{\mathscr{H}}^{(n)}_{\text{sing}}$. Indeed, the given data corresponds to a genus 1 degree n curve $E \to \mathbb{P}^n$ over $\operatorname{Spec} R$ and a map $\operatorname{Spec} K \to E$ lying in the singular locus of E. The closure of the image of $\operatorname{Spec} K \to E$ defines a section $\tau : \operatorname{Spec} R \to E$ also lying in the singular locus. By separatedness of $E \to \operatorname{Spec} R$, this closure is the unique such extension. Because the data of a map to $\widetilde{\mathscr{H}}^{(n)}_{\text{sing}}$ is precisely the data of a map to $\mathscr{H}^{(n)}$ together with a section lying in the singular locus, we have verified the valuative criterion for properness.

The proof that $\widetilde{\mathscr{W}}_{\text{sing}} \to \mathscr{W}$ is finite is analogous, where one uses Lemma 2.2.5 and the map of covers $\mathcal{H}^{\circ,1,3t}_{\text{sing}} \to \mathcal{H}^{\circ,1,3t}$ in place of $\widetilde{\mathscr{H}}^{(n)}_{\text{sing}} \to \mathscr{H}^{(n)}$. □

Since the maps $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}} \to \mathscr{M}^{(n)}_1$ and $\widetilde{\mathscr{W}}_{\text{sing}} \to \mathscr{W}$ are finite by Lemma 2.2.28, and in particular proper, we can make sense of the nodal and cuspidal loci inside $\mathscr{M}^{(n)}_1$ and $\mathscr{W}$ as the images of those in $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ and $\widetilde{\mathscr{W}}_{\text{sing}}$. One can alternatively define these loci in terms of their schematic covers by respective Hilbert schemes.

Definition 2.2.29. For $n \geq 3$, let $f_n : \widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}} \to \mathscr{M}^{(n)}_1$ be the natural map forgetting the singular section and define $\mathscr{M}^{(n)}_{1,\text{cusp}}$ as the closed substack of $\mathscr{M}^{(n)}_1$ which is the image $f_n(\widetilde{\mathscr{M}}^{(n)}_{1,\text{cusp}})$. Let $\mathscr{M}^{(n)}_{1,\text{node}} \subset \mathscr{M}^{(n)}_1$ denote the locally closed substack given as $f_n(\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}) - \mathscr{M}^{(n)}_{1,\text{cusp}}$.

Similarly, let $f : \widetilde{\mathscr{W}}_{\text{sing}} \to \mathscr{W}$ denote the projection forgetting the singular section, let $\mathscr{W}_{\text{cusp}}$ denote the closed substack of $\mathscr{W}$ given by $f(\widetilde{\mathscr{W}}_{\text{cusp}})$ and let $\mathscr{W}_{\text{node}}$ denote the locally closed substack of $\mathscr{W}$ given by $f(\widetilde{\mathscr{W}}_{\text{sing}}) - \mathscr{W}_{\text{cusp}}$.

We informally say a point of $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ or $\mathscr{W}$ lies in the nodal locus if it factors through $\mathscr{M}^{(n)}_{1,\text{node}}$ or $\mathscr{W}_{\text{node}}$ and lies in the cuspidal locus if it factors through $\mathscr{M}^{(n)}_{1,\text{cusp}}$ or $\mathscr{W}_{\text{cusp}}$.

To conclude our discussion of $\mathscr{M}^{(n)}_1$ for the moment, we note the following lemma, which will allow us to lift certain points of $\mathscr{M}^{(n)}_1$ to points of $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$.

Lemma 2.2.30. *Let B be a normal integral scheme with generic point η and let $n \geq 3$. Then if $\phi : B \to \mathscr{M}^{(n)}_1$ is a map such that η factors through $\mathscr{M}^{(n)}_{1,\text{node}}$, then ϕ factors through $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$.*

Remark 2.2.31. In fact, it turns out (using Lemma 2.2.28 and also Theorem 3.1.31) that $\widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ is the normalization of its image in $\mathscr{M}^{(n)}_1$, and the map induces an isomorphism $\widetilde{\mathscr{M}}^{(n)}_{1,\text{node}} \to \mathscr{M}^{(n)}_{1,\text{node}}$, which yields an alternate proof of Lemma 2.2.30.

Proof. The map $B \to \mathscr{M}_1^{(n)}$ corresponds to a proper flat family of genus 1 curves $C \to B$ with a closed embedding $\iota : C \to P$ for $f : P \to B$ an $(n-1)$-dimensional Brauer-Severi scheme over B. Further, by assumption the generic fiber is a nodal curve.

By definition of $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$, we only need produce a section $\tau : B \to P$ contained in the singular locus of $C \to B$. Because the generic fiber of $f \circ \iota$ is nodal by assumption, it follows from Lemma 2.2.8 that the singular locus over the generic point of B maps isomorphically to the generic point of B.

Let $Z \subset C$ denote the singular locus of the map $f \circ \iota$. Let $\widetilde{Z}$ denote the normalization of Z. Then, $\widetilde{Z} \to B$ is a finite birational map of normal integral schemes, hence an isomorphism. By inverting this isomorphism and composing with the map $\widetilde{Z} \to Z \to C$, we obtain the desired section $\tau : B \to C$ such that $f \circ \iota \circ \tau = \mathrm{id}$. Note that $\tau(B)$ factors through the singular locus of C because $\tau(B)$ is closed by properness of τ and the generic point of B maps to the singular locus of $f \circ \iota$ by assumption. □

2.2.32 The stack of Hirzebruch surface sections

We now define various stacks and schemes relating to sections on Hirzebruch surfaces. One of the main tasks of this paper is that they are closely related to the above stack of n-coverings, $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$. The key result we will show to this effect is Theorem 3.1.31.

Recall the notation for Hirzebruch surfaces from § 1.4.1. Letting $(\mathbb{F}_{n-2})_B \xrightarrow{g} \mathbb{P}^1_B \xrightarrow{h} B$ denote the structure morphisms, $\sigma : B \to \mathbb{P}^1_B$ a section of h, and f denote the class of the divisor $h^{-1}(\sigma)$ for $g : (\mathbb{F}_{n-2})_B \to \mathbb{P}^1_B$ the structure map.

Notation 2.2.33. The Hirzebruch surface $\mathbb{F}_{n-2}$ has an invertible sheaf $\mathscr{N} := \mathscr{O}_{\mathbb{F}_{n-2}}(1) \otimes g^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2)$. Let $\mathscr{F} := (h \circ g)_* \mathscr{N}$. By construction, $\mathscr{F} \simeq H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n))$.

Remark 2.2.34. A map $B \to \mathbb{P}\mathscr{F}$ corresponds to a flat finitely presented family $X \to B$ with an embedding $X \to (\mathbb{F}_{n-2})_B$ with each fiber in the linear system associated to $\mathscr{N}$. This yields a description of $\mathbb{P}\mathscr{F}$ as a subscheme of a component of the Hilbert scheme of subschemes of $\mathbb{F}_{n-2}$ over $\operatorname{Spec} \mathbb{Z}$. There is a corresponding universal family $\mathcal{U}^{\mathrm{smile},(n)} \subset \mathbb{P}\mathscr{F} \times \mathbb{F}_{n-2}$ with projection map $\pi : \mathcal{U}^{\mathrm{smile},(n)} \to \mathbb{P}\mathscr{F}$.

We next define $\mathcal{V}^{\mathrm{smile},(n)}$ as the subscheme of $\mathbb{P}\mathscr{F}$ parameterizing smooth members of the linear system associated to $\mathscr{N}$.

Definition 2.2.35. Inside $\mathcal{U}^{\mathrm{smile},(n)}$, as defined in Remark 2.2.34, let Z denote the singular locus of $\pi : \mathcal{U}^{\mathrm{smile},(n)} \to \mathbb{P}\mathscr{F}$ and let $\pi(Z)$ denote the image of Z in $\mathbb{P}\mathscr{F}$. Define $\mathcal{V}^{\mathrm{smile},(n)} := \mathbb{P}\mathscr{F} - Z$

We soon define the stack $\mathscr{V}^{\mathrm{smile},(n)}$ in Definition 2.2.39 which we will later see is $[\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_n/\mathbb{Z}}]$ in Lemma 2.2.42. This stack $\mathscr{V}^{\mathrm{smile},(n)}$ will involve twists of Hirzebruch surfaces, which we now define.

Definition 2.2.36. For $n \in \mathbb{Z}_{\geq 1}$ and B a scheme, we define an *n-Hirzebruch twist* over B as a tuple $(B, h : X \to B, g : F \to X)$ where

1. $h : X \to B$ is a 1-dimensional Brauer-Severi scheme over B
2. $g : F \to X$ is a relative dimension 1 projective bundle over X which is Zariski locally trivial such that there is an fppf cover $B' \to B$ having the property that $B' \times_B F \simeq (\mathbb{F}_n)_{B'}$.

Lemma 2.2.37. *For any $n \geq 1$, any n-Hirzebruch twist $F \xrightarrow{g} X \xrightarrow{g} B$ possesses a relative effective Cartier divisor $E \subset F$ and an invertible sheaf $\mathscr{M}$ satisfying the following property: for $B' \to B$ a cover with $F_{B'} \simeq \mathbb{F}_n$, the pullback of E to B' is the relative directrix on $\mathbb{F}_n$ over B' and the pullback of $\mathscr{M}$ has class $2f$ over B'.*

Proof. To see these exist, for $\mathscr{M}$ we may simply take the sheaf $h^*(\Omega^1_{X/B})^\vee$ on F.

Additionally, we claim that the directrix $E' \subset \mathbb{F}_{n-2}$ is preserved scheme theoretically by automorphisms of $\mathbb{F}_{n-2}$. This holds because E' is the unique section in its divisor class and the divisor class of E' is the unique effective class of negative self intersection. Therefore E' descends to the desired subscheme $E \subset F$. We can check it is a relative effective Cartier divisor after the fppf base change to B', upon which it indeed is the relative effective Cartier divisor E'. □

Notation 2.2.38. In light of Lemma 2.2.37, we will continue to use e to denote the class of a directrix E on an n-Hirzebruch twist, and we use $2f$ to denote the class of $\mathscr{M}$ as in Lemma 2.2.37. Keep in mind there may be no invertible sheaf $\mathscr{N}$ with $\mathscr{N}^{\otimes 2} \simeq \mathscr{M}$, hence no sheaf "of class f" on F.

Definition 2.2.39. Let $n \in \mathbb{Z}_{\geq 3}$. Define $\mathscr{V}^{\mathrm{smile},(n)}$, the *smile stack of volatility smiles* as the fibered category over schemes whose objects over a scheme B are tuples $(B, h : X \to B, g : F \to X, i : Z \to F)$ where $(B, h : X \to B, g : F \to X)$ is an $(n-2)$-Hirzebruch twist over B and $i : Z \to F$ is a closed subscheme which fppf locally on B induces a map to $\mathcal{V}^{\mathrm{smile},(n)}$; in other words there is an fppf cover $B' \to B$ having the property that $B' \times_B F \simeq (\mathbb{F}_{n-2})_{B'}$ and $B' \times_B Z \to (\mathbb{F}_{n-2})_{B'}$ is a subscheme smooth over B' which lies in the linear system associated to $\mathscr{N}$ on $\mathbb{F}_{n-2}$, as defined in Notation 2.2.33.

A morphism $(B, h : X \to B, g : F \to X, i : Z \to F) \to (B', h' : X' \to B', g' : F' \to X', i' : Z' \to F')$ consists of maps $\alpha : B \to B', \beta : X \to X', \gamma : F \to F', \delta : Z \to Z'$ making all squares in the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\delta} & Z' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ F & \xrightarrow{\gamma} & F' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{\beta} & X' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ B & \xrightarrow{\alpha} & B' \end{array} \tag{2.2.8}$$

fiber squares.

Remark 2.2.40 (Reason for the name "volatility smiles")**.** Unlike its financial counterpart, the stack of volatility smiles is not a correction to a market prediction, but rather named after the apparent smile on the face of the corresponding Hirzebruch surface twist, pictured as the curve $\mathrm{Bl}_\tau C \subset \mathbb{F}_{n-2}$ in Figure 1.4. However, it is highly volatile, because to obtain a point of $\mathscr{V}^{\mathrm{smile},(n)}$, *every* single fiber over B must be smooth. In other words, the smile must persist in every fiber, even the slightest volatility in the mood will throw the point out the stack completely.

We next wish to show $\mathscr{V}^{\mathrm{smile},(n)}$ is algebraic for $n \geq 3$. (In fact $\mathscr{V}^{\mathrm{smile},(n)}$ can be analogously defined for $n = 1, 2$ and shown to be algebraic in those cases as well. However, this requires a separate definition and argument which we omit.) To show $\mathscr{V}^{\mathrm{smile},(n)}$ is algebraic, we will construct an equivalence $\mathscr{V}^{\mathrm{smile},(n)} \to [\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}]$. The first step to doing so is to understand the stack of n-Hirzebruch twists. We then bootstrap by equipping these twists with a section of class $e + nf$. The next lemma verifies that the stack of n-Hirzebruch twists is equivalent to the stack quotient $[\mathrm{Spec}\, \mathbb{Z}/\mathrm{Aut}_{\mathbb{F}_n/\mathbb{Z}}]$.

Lemma 2.2.41. *Let B be a scheme. There is an equivalence of categories between* $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ *torsors over B and $(n-2)$-Hirzebruch twists $F \xrightarrow{g} X \xrightarrow{h} B$.*

Proof. Given $F \to X \to B$, we may obtain an associated $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ torsor $\mathrm{isom}_B(\mathbb{F}_{n-2}, F)$. Conversely, given an $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ torsor T, we describe the inverse construction by producing the associated $F \to X \to B$. First, we construct F as the contracted product $T \overset{\mathrm{Aut}_{\mathbb{F}_{n-2}/B}}{\times} \mathbb{F}_{n-2}$, which we recall is the quotient of $T \times \mathbb{F}_{n-2}$ by the functorial action of $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ given by an element g sending $(t, x) \mapsto (tg^{-1}, gx)$. A priori, this quotient is only an algebraic space.

Recall that $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ can be written as an extension of PGL_2 by a certain normal subgroup A'_n, as defined in Definition 2.1.2. Let T/A'_n denote the quotient algebraic space which we note has an action of PGL_2 and define $X := T/A'_n \overset{\mathrm{PGL}_2}{\times} \mathbb{P}^1$. There are maps $T \to T/A'_n$ and $\mathbb{F}_{n-2} \to \mathbb{P}^1$ which induce a map $F \to X$. Since $\mathrm{Aut}_{\mathbb{F}_{n-2}/B}$ is an affine group scheme, T is a scheme by effectivity of descent for affine morphisms. Further T/A'_n is a PGL_2 torsor over B and hence also a scheme. The contracted product X is then a Brauer-Severi scheme, as this contracted product gives the standard way to obtain a Severi schemes from the associated torsor [Gro68, 8.1].

To conclude, we know $F \to X$ is certainly fppf locally a $\mathbb{P}^1$ bundle over X, but we wish to show it is a scheme and even a Zariski $\mathbb{P}^1$ bundle over X. Let E' denote the section of $\mathbb{F}_{n-2} \to \mathbb{P}^1$ corresponding to the surjection $\mathscr{O} \oplus \mathscr{O}(n-2) \to \mathscr{O}$. Then E' has divisor class e. Note that A'_n scheme theoretically preserves E' as follows from Lemma 2.2.37. Therefore, $\mathscr{O}_{\mathbb{F}_{n-2}}(e)$ descends to an invertible sheaf on F, which we also call $\mathscr{O}_F(e)$. This sheaf $\mathscr{O}_F(e)$ is relatively very ample for the map $F \to X$. So, by descent for polarized schemes, we obtain that F is a scheme, and contains a closed subscheme $E \subset F$ which fppf locally becomes the directrix in $\mathbb{F}_{n-2} \to \mathbb{P}^1$. The invertible sheaf $\mathscr{O}_F(E)$ then gives an isomorphism from F to a Zariski $\mathbb{P}^1$ bundle over X, implying F is locally isomorphic to $\mathbb{P}^1$ over X in the Zariski topology. □

Let $n \geq 3$. We are now ready to construct the equivalence $\mathscr{V}^{\mathrm{smile},(n)} \to [\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}]$. To start, we construct the map. A map $B \to \mathscr{V}^{\mathrm{smile},(n)}$ corresponds to the data $(B, h : X \to B, g : F \to X, i : Z \to F)$ Define $I := \mathrm{isom}_B(\mathbb{F}_{n-2}, F)$ and from this we obtain an isomorphism $(\mathbb{F}_{n-2})_I \simeq F_I$ together with a subscheme $X_I \subset F_I \simeq (\mathbb{F}_{n-2})_I$ that induces a map $I \to \mathcal{V}^{\mathrm{smile},(n)}$. Since I is a torsor over B using the equivalence from Lemma 2.2.41, this altogether yields our desired map $\phi : \mathscr{V}^{\mathrm{smile},(n)} \to [\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}]$.

Lemma 2.2.42. *For $n \geq 3$, the map ϕ constructed above is an equivalence of fibered categories. In particular, $\mathscr{V}^{smile,(n)}$ is an algebraic stack.*

Proof. We will construct the inverse map. Given an $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}} \times_{\mathbb{Z}} B$ torsor I over B and a map $I \to \mathcal{V}^{\mathrm{smile},(n)}$ corresponding to a divisor $\widetilde{Z} \subset (\mathbb{F}_{n-2})_I$ which is equivariant for the $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}} \times_{\mathbb{Z}} B$ action, we wish to construct a map $B \to \mathscr{V}^{\mathrm{smile},(n)}$. Using Lemma 2.2.41, we obtain an $(n-2)$-Hirzebruch twist $(B, h : X \to B, g : F \to X)$ which pulls back to $(\mathbb{F}_{n-2})_I \to \mathbb{P}^1_I \to I$ over I. Because the subscheme $\widetilde{Z} \subset (\mathbb{F}_{n-2})_I$ is $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}} \times_{\mathbb{Z}} B$ equivariant, it descends to a closed subscheme $i : Z \to F$ which induces the map $B \to \mathscr{V}^{\mathrm{smile},(n)}$. Note that Z is smooth and has class $e + nf$, as may be verified fppf locally on B.

We claim this map is inverse to the map ϕ. This bijection essentially follows from the bijection established in Lemma 2.2.41, together with uniqueness of descent for the closed immersion $i : Z \to F$. □

2.3 Equivalent descriptions of n-coverings

The main goal of this section is to prove Proposition 2.3.14, which gives several equivalent descriptions of n-coverings of a genus 1 curves. As preparation for proving these equivalent descriptions, we now review some generalities on Čech cohomology and derived functor cohomology of complexes.

2.3.1 Čech cohomology of complexes

In general, if one wishes to compute cohomology of a complex, one can do so in terms of the Čech cohomology of hypercoverings, see [Sta, Tag 01FP], and in particular [Sta, Tag 08BN]. However, for computing the first cohomology of two-term complexes, it turns out that it is enough to work with usual Čech cohomology, as we will demonstrate in Lemma 2.3.3. Again, all this will do is save us the minor annoyance of worrying about hypercoverings, but is not really essential. We encourage the reader to skip ahead to §2.3.7.

In what follows, we will be particularly concerned with understanding the cohomology of a certain two term complex. Specifically, if $X \to B$ is a relative genus 1 curve and $\mathrm{Pic}^0_{X/B} \to B$ is the relative identity component of the Jacobian, then we will be concerned with understanding the first cohomology of the two term complex of flat sheaves $\mathrm{Pic}^0_{X/B} \xrightarrow{\times n} \mathrm{Pic}^0_{X/B}$. Note we may replace $\mathrm{Pic}^0_{X/B}$ by the smooth locus of X using Lemma 2.2.16.

Next, we give a leisurely description of for computing cohomology of a two term complex in terms of Čech cohomology of hypercoverings. Of course, this easily generalizes to longer complexes, but it will somewhat simplify the notation to work with a two term complex. First, recall that in the case of a complex on a scheme X with a single term, say $\mathscr{F}$, the Čech cohomology with respect to a covering $\mathscr{U} = \cup_i U_i$ is computed as the cohomology of the complex

$$\prod_{i_0 \in I} \mathscr{F}(U_{i_0}) \longrightarrow \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0 i_1}) \longrightarrow \prod_{i_0, i_1, i_2 \in I} \mathscr{F}(U_{i_0 i_1 i_2}) \longrightarrow \cdots \qquad (2.3.1)$$

where, as usual, we use $U_{i_0 i_1 \cdots i_k} := U_{i_0} \times_B U_{i_1} \cdots \times_B U_{i_k}$.

Now, we generalize this to computing the cohomology of a two term complex $\mathscr{F} \xrightarrow{\phi} \mathscr{G}$

on X. Again, let $\mathscr{U} = \cup_i U_i$ be an open cover of X. Then, the Čech cohomology of $\mathscr{F} \xrightarrow{\phi} \mathscr{G}$ with respect to $\mathscr{U}$ on X is the cohomology of the complex

$$\begin{array}{ccccccc} \prod_{i_0 \in I} \mathscr{F}(U_{i_0}) & \xrightarrow{\delta_0^{\mathscr{F}}} & \prod_{i_0,i_1 \in I} \mathscr{F}(U_{i_0 i_1}) & \xrightarrow{\delta_1^{\mathscr{F}}} & \prod_{i_0,i_1,i_2 \in I} \mathscr{F}(U_{i_0 i_1 i_2}) & \xrightarrow{\delta_2^{\mathscr{F}}} & \cdots \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \\ \prod_{i_0 \in I} \mathscr{G}(U_{i_0}) & \xrightarrow{-\delta_0^{\mathscr{G}}} & \prod_{i_0,i_1 \in I} \mathscr{G}(U_{i_0 i_1}) & \xrightarrow{-\delta_1^{\mathscr{G}}} & \prod_{i_0,i_1,i_2 \in I} \mathscr{G}(U_{i_0 i_1 i_2}) & \xrightarrow{-\delta_2^{\mathscr{G}}} & \cdots, \end{array} \tag{2.3.2}$$

with $\delta_i^{\mathscr{F}}$ the boundary maps for $\mathscr{F}$ and $\delta_i^{\mathscr{G}}$ the boundary maps for $\mathscr{G}$. To spell this out even more concretely, a *n-cocycle* (i.e., an element of $Z^n(\mathscr{U}, \mathscr{F} \to \mathscr{G})$,) is a pair (α, β) with $\alpha \in \prod_{i_0,\ldots,i_n \in I} \mathscr{F}(U_{i_0 \cdots i_n}), \beta \in \prod_{i_0,\ldots,i_{n-1} \in I} \mathscr{G}(U_{i_0 \cdots i_{n-1}})$ so that the boundary map applied to (α, β) vanishes. In other words, we must have $\delta_n^{\mathscr{F}}(\alpha) = 0$ and $\phi(\alpha) - \delta_i^{\mathscr{G}}(\beta) = 0$. In particular, α is a 1-cocycle for $\mathscr{F}$ with respect to $\mathscr{U}$. We say (α, β) is a *n-coboundary* if it is the image of a pair (μ, ν) with $\mu \in \prod_{i_0,\ldots,i_{n-1} \in I} \mathscr{F}(U_{i_0 \cdots i_{n-1}})$ and $\nu \in \prod_{i_0,\ldots,i_{n-2} \in I} \mathscr{F}(U_{i_0 \cdots i_{n-2}})$ under the boundary map. Concretely, this means $\delta_{n-1}^{\mathscr{F}}(\mu) = \alpha$ and $\phi(\mu) - \delta_{n-2}^{\mathscr{G}}(\nu) = \beta$. The nth Čech cohomology of a 2-term complex is then the quotient of n-cocycles by n-coboundaries. We notate this as $\check{H}^n(X, \mathscr{F} \xrightarrow{\phi} \mathscr{G})$.

Example 2.3.2. Ultimately, we will be interested a certain $\check{H}^1(X, \mathscr{F} \to \mathscr{G})$ group, and so now let us further specialize to spell out what the meaning of an element of $\check{H}^1(X, \mathscr{F} \to \mathscr{G})$ is. Taking $n = 1$ above, we see a 1-cocycle for $\mathscr{F} \to \mathscr{G}$ is with respect to $\mathscr{U}$ an element (α, β) with $\alpha \in \prod_{i,j \in I} \mathscr{F}(U_{ij})$ and $\beta \in \prod_{i \in I} \mathscr{G}(U_i)$ such that $\delta_1^{\mathscr{F}}(\alpha) = 0$ and $\phi(\alpha) = \delta_0^{\mathscr{G}}(\beta)$. Further, 1-coboundaries are described as pairs $(\delta_0^{\mathscr{F}}(\gamma), \phi(\gamma))$ for $\gamma \in \prod_i \mathscr{F}(U_i)$.

The following lemma may be comforting to the reader who wishes to stick with standard Čech cohomology and avoid using hypercoverings. With that said, we mention that Čech and derived cohomology always agree when one uses hypercoverings, see [Sta, Tag 01FP] and also [Sta, Tag 08BN].

Lemma 2.3.3. *The natural map $\check{H}^1(X, \mathscr{F} \to \mathscr{G}) \simeq H^1(X, \mathscr{F} \to \mathscr{G})$ is an isomorphism.*

Proof. The filtration $\mathscr{F} \to (\mathscr{F} \to \mathscr{G}) \to \mathscr{G}$ induces a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{H}^1(X, \mathscr{F}) & \longrightarrow & \check{H}^1(X, \mathscr{F} \to \mathscr{G}) & \longrightarrow & \check{H}^0(X, \mathscr{G}) & \longrightarrow & \check{H}^2(X, \mathscr{F}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(X, \mathscr{F}) & \longrightarrow & H^1(X, \mathscr{F} \to \mathscr{G}) & \longrightarrow & H^0(X, \mathscr{G}) & \longrightarrow & H^2(X, \mathscr{F}). \end{array} \tag{2.3.3}$$

By the five lemma, to show the middle vertical map is an isomorphism, it is enough to show the maps $\check{H}^1(X, \mathscr{F}) \to H^1(X, \mathscr{F})$ and $\check{H}^0(X, \mathscr{G}) \to H^0(X, \mathscr{G})$ are isomorphisms and that the map $\check{H}^2(X, \mathscr{F}) \to H^2(X, \mathscr{F})$ is injective. The isomorphism $\check{H}^0(X, \mathscr{G}) \to H^0(X, \mathscr{G})$ follows from the definition of Čech cohomology, as $\mathscr{G}$ is a sheaf. We can verify that both $\check{H}^1(X, \mathscr{F}) \to H^1(X, \mathscr{F})$ is an isomorphism and $\check{H}^2(X, \mathscr{F}) \to H^2(X, \mathscr{F})$ is an injection using the spectral sequence of [Sta, Tag 01ES] $\check{H}^p(\mathscr{U}, \underline{H}^q(\mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$ for $\mathscr{U}$ a covering of X and $\underline{H}^q(\mathscr{F})$ the cohomology *presheaf* defined as the presheaf sending $U \mapsto H^q(U, \mathscr{F})$. Writing out the exact sequence on low degree terms gives

$$0 \longrightarrow \check{H}^1(\mathscr{U}, \underline{H}^0(\mathscr{F})) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow \check{H}^0(\mathscr{U}, \underline{H}^1(\mathscr{F})) \longrightarrow$$

$$\longrightarrow \check{H}^2(\mathscr{U}, \underline{H}^0(\mathscr{F})) \longrightarrow H^2(X, \mathscr{F}). \tag{2.3.4}$$

Noting that $\mathscr{F} \simeq \underline{H}^0(\mathscr{F})$, in order to obtain both our desired isomorphism and injection, it is enough to show $\check{H}^0(\mathscr{U}, \underline{H}^1(\mathscr{F})) = 0$.

This is a general fact about cohomology, and we conclude by verifying $\check{H}^0(\mathscr{U}, \underline{H}^q(\mathscr{F})) = 0$ for $q > 0$. Observe although $\underline{H}^q(\mathscr{F})$ is a presheaf, its sheafification vanishes for $q > 0$ by local acyclicity of cohomology [Sta, Tag 01ES]. Therefore, for any cover $\mathscr{U}$ of X and element $\alpha \in \check{H}^0(\mathscr{U}, \underline{H}^q(\mathscr{F}))$ there is a refinement $\mathscr{V}$ of $\mathscr{U}$ so that α vanishes in $\check{H}^0(\mathscr{V}, \underline{H}^q(\mathscr{F}))$. Since $\check{H}^0(X, \underline{H}^q(\mathscr{F}))$ is defined as the colimit of $\check{H}^0(\mathscr{U}, \underline{H}^q(\mathscr{F}))$ over all covers $\mathscr{U}$ of X, we obtain that $\check{H}^0(\mathscr{U}, \underline{H}^q(\mathscr{F})) = 0$ when $q > 0$. □

2.3.4 Derived functor cohomology of two-term complexes

We now review some generalities on derived functor cohomology of two term complexes, and associated exact sequences.

Construction 2.3.5. Recall that there are two spectral sequences which compute hypercohomology. In the special case of the hypercohomology of a two term complex, these spectral sequences induce sequences on cohomology appearing as the cohomology associated to a distinguished triangle. That is, given a space X and a complex of sheaves $\mathscr{C} := [\mathscr{F} \xrightarrow{\phi} \mathscr{G}]$ with $\mathscr{F}$ in degree 0 and $\mathscr{G}$ in degree 1, there are two distinguished triangles

$$\mathscr{G}[-1] \longrightarrow \mathscr{C} \longrightarrow \mathscr{F} \to \tag{2.3.5}$$

and

$$\ker \phi \longrightarrow \mathscr{C} \longrightarrow \operatorname{coker} \phi[-1] \to \tag{2.3.6}$$

We further note that the boundary map in the first sequence can be explicitly identified with ϕ because the first distinguished triangle above is a shift of

$$\mathscr{F}[-1] \xrightarrow{\phi[-1]} \mathscr{G}[-1] \longrightarrow \mathscr{C} \tag{2.3.7}$$

with $\mathscr{C}$ the cone of $\mathscr{F}[-1] \xrightarrow{\phi[-1]} \mathscr{G}[-1]$.

We will be most interested in the case that $\mathscr{F} \to \mathscr{G}$ is a specific two-term complex. Now, let us apply Construction 2.3.5 to this setting.

Example 2.3.6. Suppose have a locally free degree 2 map $g : X \to \operatorname{Spec} \mathbb{Z}$ and take $\mathscr{F} = \mathscr{G} = g_* \mathbb{G}_m / \mathbb{G}_m$ and $\phi = \times n$. Then, taking cohomology associated to the two distinguished triangles in Construction 2.3.5, we get exact sequences

$$0 \longrightarrow \frac{H^0(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m)}{nH^0(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m)} \longrightarrow H^1(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \xrightarrow{\rho} H^1(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m)[n] \longrightarrow 0 \tag{2.3.8}$$

and

$$0 \longrightarrow H^1(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m[n]) \xrightarrow{\upsilon} H^1(\operatorname{Spec} \mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \longrightarrow H^0\left(\operatorname{Spec} \mathbb{Z}, \operatorname{coker}\left(g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m\right)\right). \tag{2.3.9}$$

Note that for (2.3.8), we are using that the boundary map in the cohomology sequence associated to (2.3.5) is given by $\times n$, as follows from the above mentioned fact that the first distinguished triangle in (2.3.5) is a shift of (2.3.7).

2.3.7 Relative genus 1 curves

Much of the remainder of this section was inspired by [ASD73, §2], but adapted to work in characteristic dividing n using flat cohomology (in place of étale cohomology used in [ASD73, §2]). Our primary goal for the rest of this section is to prove Proposition 2.3.14, which gives several equivalent characterizations of points of $\mathscr{S}^{(n)}$, and relates it to $\mathscr{M}_1^{(n)}$. One of the main issues in characteristic dividing n is that $\mathbb{G}_a \xrightarrow{\times n} \mathbb{G}_a$ is the 0 map, and hence is not surjective. Let $E \to B$ be a genus 1 curve with integral fibers and smooth locus E^{sm}. In order to deal with the above issue that multiplication by n is not surjective, instead of working with $H^1(B, E^{\mathrm{sm}}[n])$, we work with the hypercohomology group $H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$. First, we give a general lemma relating hypercohomology to sections of a quotient stack. This will be useful in connecting hypercohomology to points of $\mathscr{S}^{(n)}$.

Lemma 2.3.8. *Let $\phi : G \to H$ be a map of smooth commutative group schemes over B. Let $[H/\phi(G)]$ denote the quotient stack of H by the action of G via ϕ. Then $H^1(B, H \xrightarrow{\phi}$*

$G) = H^0(B, [H/\phi(G)])$. *These sets are also in bijection with pairs* $(T, \psi : \phi_* T \to H)$ *up to isomorphism of torsors, where* T *is an* G *torsor and* ψ *is an isomorphism of* H *torsors.*

Proof. Given a G torsor T', let $\phi_* T'$ denote the H torsor given explicitly as the image of $[T'] \in H^1(B, G)$ under the map $H^1(B, G) \to H^1(B, H)$ induced by ϕ.

We will first show elements of $H^0(B, [H/\phi(G)])$, i.e., maps $B \to [H/\phi(G)]$, are in bijection with pairs (T, ψ) where $T \to B$ is a G torsor and $\psi : \phi_* T \to H$ is an isomorphism of H-torsors. By definition, an element of $H^0(B, [H/\phi(G)])$ corresponds to a G torsor $T \to B$ (possibly an algebraic space which is not a scheme) and a G-equivariant map $\alpha : T \to H$ (where G acts on H via ϕ) fitting in the commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & H \\ \downarrow & & \downarrow \\ B & \longrightarrow & [H/\phi(G)]\,. \end{array} \tag{2.3.10}$$

Two such diagrams are considered isomorphic data if they are related by the G action.

We next claim that G-equivariant maps $\chi : T \to H$ are in bijection with H equivariant maps $\psi : \phi_* T \to H$. To verify this claim, first observe that the map $\phi : G \to H$ induces a map $\theta : T \to \phi_* T$, as may be constructed directly via cocycles. Therefore, given ψ, we may send it to the G equivariant map $\psi \circ \theta$. Conversely, we claim that there is a unique H-equivariant map $\psi : \phi_* T \to H$, whose composition with θ yields χ. To verify this claim, by fppf descent, we may pass to a cover of B on which T is trivialized. In this case, $\phi_* T$ is also trivialized. The claim then follows because given $\chi : G \to H$, there is a unique map of H torsors $\psi : \phi_* G \to H$ so that the composition $G \xrightarrow{\alpha} \phi_* G \to H$ yields χ: namely, ψ is the unique map of trivial H torsors that sends $\alpha(e_G)$ to e_H, for e_G the identity of G and e_H the identity of H.

It remains to show that pairs (T, ψ) for T a G-torsor and ψ a trivialization of $\phi_* T$ correspond to elements of $H^1(B, G \xrightarrow{\phi} H)$. However, using Lemma 2.3.3 and Example 2.3.2, we can describe elements of $H^1(B, G \xrightarrow{\phi} H)$ in terms of Cech cohomology by a pair (α, β) where α is a 1-cocycle for G and β is a 0-cochain for H with $\phi(\alpha_{ij}) = \beta_i - \beta_j$, taken with respect to a trivializing cover U_i of B. The α_{ij} specify the torsor T while the equality $\phi(\alpha_{ij}) = \beta_i - \beta_j$, gives a trivialization of the torsor $\phi_* T$. Conversely, given a pair (T, ϕ), we can, as above, describe T in terms of a 1-cocycle α_{ij} and a trivialization $\phi_* T \simeq H$ can be described by a 0-cochain β_i. □

At this point the reader may wish to recall the definitions of $\mathscr{W}, \overline{\mathscr{E}}, \mathscr{E}$ and $\mathscr{S}^{(n)}$ from Definition 2.2.2, Definition 2.2.10, Definition 2.2.15, and Definition 2.2.17. Using Lemma 2.3.8, we can easily relate a hypercohomology group parameterizing n-coverings to points of $\mathscr{S}^{(n)}$.

Lemma 2.3.9. *Let $g : B \to \mathscr{W}$ be given by a tuple $(B, f : E \to B, e : B \to E)$. Let E^{sm} denote the smooth locus of f and let $\pi_n : \mathscr{S}^{(n)} \to \mathscr{W}$ denote the canonical projection induced by $\mathscr{E} \to \mathscr{W}$. Then, $(B \times_{g,\mathscr{W},\pi_n} \mathscr{S}^{(n)})(B) \simeq H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$.*

Proof. Because of the fiber squares

$$\begin{array}{ccc} E^{\mathrm{sm}} & \longrightarrow & \mathscr{E} \\ \downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} \\ E^{\mathrm{sm}} & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & \mathscr{W} \end{array} \tag{2.3.11}$$

we obtain a fiber square

$$\begin{array}{ccc} [E^{\mathrm{sm}}/nE^{\mathrm{sm}}] & \longrightarrow & \mathscr{S}^{(n)} \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & \mathscr{W}, \end{array} \tag{2.3.12}$$

where $[E^{\mathrm{sm}}/nE^{\mathrm{sm}}]$ denotes the quotient stack of E^{sm} by the action of E^{sm} on itself via multiplication by n. Therefore, $(B \times_{g,\mathscr{W},\pi_n} \mathscr{S}^{(n)})(B)$ is identified with $[E^{\mathrm{sm}}/nE^{\mathrm{sm}}](B) = H^0(B, [E^{\mathrm{sm}}/nE^{\mathrm{sm}}])$. By Lemma 2.3.8, we can then identify $H^0(B, [E^{\mathrm{sm}}/nE^{\mathrm{sm}}]) \simeq H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$. □

Recall that for $G \to B$ a group scheme and T a G-torsor over B and $n \in \mathbb{Z}_{\geq 1}$, we use n_*T to denote the G torsor corresponding to the image of $[T] \in H^1(B, G)$ under the map $H^1(B, G) \to H^1(B, G)$ induced by $\times n : G \to G$. Our upcoming lemmas show that torsors for the smooth locus of a genus 1 curve always have natural compactifications. We first show this compactification exists as an algebraic space in Lemma 2.3.10, and then upgrade this in Lemma 2.3.12 to show it even exists as a scheme.

Lemma 2.3.10. *Given a map $g : B \to \mathscr{W}$ corresponding to $(B, f : E \to B, e : B \to E)$, and E^{sm} the smooth locus of f, let $h : T \to B$ be an E^{sm} torsor with n_*T the trivial E^{sm} torsor. Then, there exists a flat proper algebraic space relative curve $\overline{h} : \overline{T} \to B$ of genus 1 with geometrically integral fibers, such that T is identified with the smooth locus of $\overline{h}$.*

Further, $\overline{T}$ is unique in the following sense: given any other such flat proper algebraic space relative genus 1 curve $\overline{h}' : \overline{T}' \to B$ with geometrically integral fibers so that T is identified with the smooth locus of $\overline{h}'$ there is a unique isomorphism $\sigma : \overline{T} \to \overline{T}'$ over B so that the composition $T \to \overline{T} \xrightarrow{\sigma} \overline{T}'$ is the given inclusion $T \to \overline{T}'$.

Proof. We construct $\overline{T}$ as an algebraic space using cocycles Since an E^{sm} torsor T with a trivialization of n_*T corresponds to an element of $H^0(B, [E^{\mathrm{sm}}/nE^{\mathrm{sm}}])$, we may use

Lemma 2.3.8 and Lemma 2.3.3 to describe such a torsor as an element of $\check{H}^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$. Choose an fppf cover $U_i \to B$ and a 1-cocycle $s_{ij} \in H^0(U_{ij}, E^{\mathrm{sm}}|_{U_{ij}})$ where $U_{ij} = U_i \times_B U_j$. By Example 2.3.2, this can be described as a 0-chain t_i so that $n \cdot s_{ij} = t_i - t_j$. Here,

$$n \cdot s_{ij} = \underbrace{s_{ij} +_{E^{\mathrm{sm}}} \cdots +_{E^{\mathrm{sm}}} s_{ij}}_{n \text{ times}},$$

for $+_{E^{\mathrm{sm}}}$ addition in the group law on E^{sm}. The s_{ij} specify cocycle data to glue $E|_{U_i}$ to $E|_{U_j}$ over U_{ij}, and since the s_{ij} are a cocycle, i.e., $s_{ij} + s_{jk} = s_{ik}$ on U_{ijk}, they define descent data so as to construct an algebraic space $\overline{T}$. Since T is constructed using the same cocycle s_{ij}, but via gluing $E^{\mathrm{sm}}|_{U_i} \to E^{\mathrm{sm}}|_{U_j}$, we obtain the desired open embedding $T \to \overline{T}$. Note that here the inclusion $T \to \overline{T}$ can be constructed locally over U_i, and verified to be an open embedding over U_i because $E^{\mathrm{sm}} \to E$ is an open embedding.

Finally, we verify the uniqueness claim on $\overline{h}$. We know any two such compactifications $\overline{T}$ and $\overline{T}'$ are fppf locally isomorphic to E on a suitable fppf cover $U_i \to B$. Further, because each E_{ij} is separated over U_{ij}, we obtain $\overline{T}$ and $\overline{T}'$ are separated over B. Therefore, the desired map $\overline{T} \to \overline{T}'$, if it exists, is unique, as it must carry the inclusion $T \to \overline{T}$ to the given inclusion $T \to \overline{T}'$. The same holds true on any cover of B. Upon choosing a cover U_i for B which trivializes both $\overline{T}$ and $\overline{T}'$, we obtain trivializations $E_{U_i} \simeq \overline{T}$ and $E_{U_i} \simeq \overline{T}'$. The composition of the inverse of the second with the first yields an isomorphism $\overline{T}'_{U_i} \simeq \overline{T}_{U_i}$ From the uniqueness of these isomorphism on the overlaps U_{ij}, we obtain descent data for the isomorphism, and hence a unique isomorphism $\overline{T}' \to \overline{T}$ compatible with the given inclusions from T. □

We next wish to show $\overline{T}$ of Lemma 2.3.10 is in fact a scheme. For this, the following general lemma will be useful.

Lemma 2.3.11. *Let B be a scheme and W a PGL_n torsor over B. For $[W] \in H^1(B, \mathrm{PGL}_n)$ the corresponding class, we have that $n[W] \in H^1(B, \mathrm{PGL}_n)$ lifts to a GL_n torsor.*

Proof. To see this, we use the map of exact sequences of group schemes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_n & \longrightarrow & \mathrm{SL}_n & \longrightarrow & \mathrm{PSL}_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 0. \end{array} \tag{2.3.13}$$

Note that $\mathrm{PSL}_n \simeq \mathrm{PGL}_n$ via the snake lemma. Therefore, taking the long exact sequence in cohomology, we obtain a commutative square

$$\begin{array}{ccc} H^1(B, \mathrm{PSL}_n) & \longrightarrow & H^2(B, \mu_n) \\ \downarrow & & \downarrow \\ H^1(B, \mathrm{PGL}_n) & \longrightarrow & H^2(B, \mathbb{G}_m). \end{array} \tag{2.3.14}$$

Via the identification $\mathrm{PSL}_n \simeq \mathrm{PGL}_n$, we find that the left vertical map of (2.3.14) is an isomorphism, showing that the bottom map factors through $H^2(B, \mu_n)$. This implies that the image X of $[W]$ in $H^2(B, \mathbb{G}_m)$ also lies in the image of $H^2(B, \mu_n) \to H^2(B, \mathbb{G}_m)$, and therefore X is killed by multiplication by n. Therefore, $n[W]$ maps to the trivial class in $H^2(B, \mathbb{G}_m)$ and hence lies in the image of $H^1(B, \mathrm{GL}_n)$, as claimed. □

Lemma 2.3.12. *In the setting of Lemma 2.3.10, the algebraic space $\overline{T}$ is in fact a scheme and is projective over B. Additionally, there is Brauer-Severi scheme $P \to B$ of relative dimension $n-1$ and a map $\overline{T} \to P$, which is an embedding if $n \geq 3$.*

Proof. In order to show $\overline{T} \to B$ is projective we will use that descent for polarized schemes is effective. Namely, choose descent data s_{ij} and t_i for T as above, and let $e : B \to E^{\mathrm{sm}}$ denote the given section in the smooth locus associated with the data of the map $B \to \mathscr{W}$. Consider the line bundle $\mathscr{L}_{t_i} := \mathscr{O}_{E_{U_i}}((n-1) \cdot e + t_i)$ on E viewed as a degree n line bundle on E_{U_i}. The global sections of this invertible sheaf induce an embedding $E_{U_i} \to \mathrm{Proj}\, f_* \mathscr{L}_{t_i}$.

We will check next that $(s_{ij})^* \mathscr{L}_{t_i}|_{U_{ij}} \simeq \mathscr{L}_{t_j}|_{U_{ij}}$ and that the $\mathscr{L}_{t_i}^{\otimes n}$ descend to an invertible sheaf on $\overline{T}$. Because $\mathscr{L}_{t_i}$ has degree n, translation by s_{ij} corresponds to tensoring with the degree 0 line bundle $\mathscr{O}_{E_{U_{ij}}}(n \cdot s_{ij} - n \cdot e)$: Indeed, this is the action on $\mathrm{Pic}^n_{E/B}$ induced by the translation action of $\mathscr{O}_{E_{U_{ij}}}(s_{ij} - e)$ on $\mathrm{Pic}^1_{E/B}$. Therefore, $(s_{ij})^* \mathscr{L}_{t_i} \simeq \mathscr{O}_{E_{U_{ij}}}((n-1)s_{ij} + (s_{ij} - e + t_j))$. The condition that $n \cdot s_{ij} = t_i - t_j$ can be written in terms of degree 0 line bundles as $\mathscr{O}_{E_{U_{ij}}}(n \cdot s_{ij} - n \cdot e) \simeq \mathscr{O}_{E_{U_{ij}}}(t_i - t_j)$. This yields the desired isomorphism

$$\mathscr{L}_{t_i}|_{U_{ij}} \simeq \mathscr{O}_{E_{U_{ij}}}((n-1)e + t_i) \simeq \mathscr{O}_{E_{U_{ij}}}(n \cdot s_{ij} + t_j - e) \simeq (s_{ij})^* \mathscr{L}_{t_j}|_{U_{ij}}.$$

If these isomorphisms satisfied the cocycle condition, we would be able to apply descent for polarized schemes, which would verify projectivity. However, although the s_{ij} satisfy the cocycle condition, it is not necessarily the case that the above isomorphisms satisfy the cocycle condition. Nevertheless, we *do* obtain isomorphisms of rank n vector bundles $f_* \mathscr{L}_{t_i} \simeq f_* \mathscr{L}_{t_j}$ over U_{ij} which satisfy the cocycle condition *up to scaling*. In other words, after possibly refining the cover U_i, the associated isomorphisms can be identified with elements of GL_n, and their images in PGL_n do satisfy the cocycle condition. Hence, from this data, we obtain a PGL_n torsor over B. Our goal will be to show that the transition functions associated to $\mathscr{L}_{t_i}^{\otimes n}$ do satisfy the cocycle condition.

From Lemma 2.3.11, it follows that $n[W]$ is a GL_n torsor, and hence corresponds to a vector bundle over B. Although the isomorphisms $\mathscr{L}_{t_i} \simeq (s_{ij})^* \mathscr{L}_{t_j}$ did not necessarily satisfy the cocycle condition, we claim that when we multiply the above isomorphisms by n, we obtain isomorphisms $\mathscr{L}_{t_i}^{\otimes n} \simeq (s_{ij})^* \mathscr{L}_{t_j}^{\otimes n}$, which do satisfy the cocycle condition. To verify this, we can do so after pushing forward via f. That is, we know the induced isomorphisms $f_* \mathscr{L}_{t_i}^{\otimes n} \simeq f_*((s_{ij})^* \mathscr{L}_{t_j}^{\otimes n})$ do satisfy the cocycle condition because we have seen the corresponding PGL_n torsor lifts to a GL_n torsor, which implies the isomorphisms $\mathscr{L}_{t_i}^{\otimes n} \simeq (s_{ij})^* \mathscr{L}_{t_j}^{\otimes n}$ also satisfy the cocycle condition. Therefore, taking the polarization $\mathscr{L}_{t_i}^{\otimes n}$ on E_{U_i}, we have descent data defined by translation by s_{ij}. Effectivity of descent for polarized schemes tells us there is a projective scheme $\overline{T} \to B$ whose base change to U_i is E. Further, we also have descent data for the schemes $P_i := \operatorname{Proj} f_* \mathscr{L}_{t_i}$, with the line bundle $\mathscr{O}_{P_i}(n)$, again induced by translation by s_{ij}. Again, by effectivity of descent for polarized schemes, we obtain a scheme $P \to B$ with $P_{U_i} \simeq P_i$, and so P is a Brauer-Severi scheme. Effectivity of descent for closed embeddings implies that the natural closed embeddings $E_{U_i} \to P_i$ descend to a closed embedding $\overline{T} \to P$, which is the claimed embedding into a Brauer-Severi scheme. □

With notation as in Lemma 2.3.9 we use $\mathrm{Aut}_{(E,e)/B}$ to denote the automorphism scheme of the genus 1 curve E which preserve the given section e lying in the smooth locus.

Lemma 2.3.13. *With notation as in Lemma 2.3.9 the following sets are isomorphic, functorially in B and respecting the action of* $\mathrm{Aut}_{(E,e)/B}(B)$*:*

(1) $(B \times_{g, \mathscr{W}, \pi_n} \mathscr{S}^{(n)})(B)$.

(2) $H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$.

(3) $H^0(B, [E^{\mathrm{sm}}/nE^{\mathrm{sm}}])$.

(4) the set of pairs $(T, \psi : n_* T \to E^{\mathrm{sm}})$ *where* T *is an* E^{sm} *torsor and* ψ *is an isomorphism of* E^{sm} *torsors, up to isomorphism.*

(5) the set of pairs (T, M) *where* T *is an* E^{sm} *torsor and* $M \in \mathrm{Pic}^n_{\overline{T}/B}(B)$ *for* $\overline{T}$ *the flat proper genus* 1 *curve associated to* T *as in Lemma 2.3.12, up to isomorphism. Here* (T, M) *is isomorphic to* (T, M') *if they differ by translation by a point of* $\mathrm{Pic}^0_{\overline{T}/B} \simeq E^{\mathrm{sm}}$.

Proof. The equivalence of (1), (2) were established in Lemma 2.3.9. The equivalence of the (2), (3), and (4) follows from Lemma 2.3.8.

Next, we identify (4) with (5). This is essentially explained in [ASD73, Proposition 1.7] and we now recount their explanation. Given a fixed E^{sm} torsor T, we need to naturally identify the set of isomorphisms of E^{sm} torsors $n_*T \simeq E^{\mathrm{sm}}$ with $\mathrm{Pic}^n_{\overline{T}/B}(B)$.

First note that we can identify $\mathrm{Pic}^n_{\overline{T}/B}$ with $n_* \mathrm{Pic}^1_{\overline{T}/B}$. This follows because there is a surjective map $(\mathrm{Pic}^1_{\overline{T}/B})^n \to \mathrm{Pic}^n_{\overline{T}/B}$, which implies that the transition functions defining the $\mathrm{Pic}^0_{\overline{T}/B}$ torsor $\mathrm{Pic}^n_{\overline{T}/B}$ are n times those defining $\mathrm{Pic}^1_{\overline{T}/B}$, and hence we obtain an isomorphism of torsors $n_* \mathrm{Pic}^1_{\overline{T}/B} \simeq \mathrm{Pic}^n_{\overline{T}/B}$.

Note that $\mathrm{Pic}^1_{\overline{T}/B}$ is represented by a scheme by [FGI+05, Theorem 9.4.8] since the geometric fibers of $\overline{T} \to B$ are integral genus 1 curves and $\overline{T} \to B$ is projective by Lemma 2.3.12. Further, we claim there is a natural isomorphism $\nu : T \to \mathrm{Pic}^1_{\overline{T}/B}$, using that T is the smooth locus of the map $\overline{T}/B$. Indeed, this may be verified fppf locally. Hence, we may assume T is the trivial torsor, in which case it follows from Lemma 2.2.16. Using the isomorphism ν and identification $\mathrm{Pic}^n_{\overline{T}/B} \simeq n_* \mathrm{Pic}^1_{\overline{T}/B}$, the choice of trivialization of n_*T corresponds to a trivialization of $\mathrm{Pic}^n_{\overline{T}/B}$, or in other words, a point $\mathrm{Pic}^n_{\overline{T}/B}(B)$, as we wished to show. This completes the identification of (4) with (5).

To complete the proof, one should, strictly speaking, verify the above constructions are bijective, functorial in B, and compatible with automorphisms of E. This follows directly from the constructions, and we omit this verification. □

We are now ready to combine the above lemmas to verify the main result of this section.

Proposition 2.3.14. *With notation as in Lemma 2.3.9 the following sets are isomorphic, functorially in B:*

(1) $(B \times_{g,\mathscr{W},\pi_n} \mathscr{S}^{(n)})(B)/\mathrm{Aut}_{(E,e)/B}(B)$.

(2) $H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})/\mathrm{Aut}_{(E,e)/B}(B)$.

(3) $H^0(B, [E^{\mathrm{sm}}/nE^{\mathrm{sm}}])/\mathrm{Aut}_{(E,e)/B}(B)$.

(4) the set of pairs $(T, \psi : n_*T \to E^{\mathrm{sm}})$ *where* T *is an* E^{sm} *torsor and* ψ *is an isomorphism of* E^{sm} *torsors, up to isomorphism, modulo the action of* $\mathrm{Aut}_{(E,e)/B}(B)$.

(5) the set of pairs (T, M) *where* T *is an* E^{sm} *torsor and* $M \in \mathrm{Pic}^n_{\overline{T}/B}(B)$ *for* $\overline{T}$ *the flat proper genus* 1 *curve associated to* T *as in Lemma 2.3.12, up to isomorphism, modulo the action of* $\mathrm{Aut}_{(E,e)/B}(B)$.

If further, $n \geq 3$*, the above are also equivalent to*

(6) *the set of tuples* (T, P, ι)*, taken up to automorphism, where* T *is an* E^{sm} *torsor,* P *is an* $n-1$ *dimensional Brauer-Severi scheme over* B *and, for* $\overline{T}$ *the flat proper genus* 1 *curve associated to* T*,* $\iota : \overline{T} \to P$ *is a closed embedding.*

(7) *maps* $B \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ *corresponding to* $(n-1)$*-dimensional Brauer-Severi schemes* $P \to B$ *with closed embeddings* $\overline{T} \to P$ *of genus* 1 *flat projective curves with geometrically integral fibers, such that the smooth locus of* $\overline{T} \to B$ *is an* E^{sm} *torsor.*

Proof. The equivalence of the (1)-(5) follows from the analogous statements in Lemma 2.3.13, as the bijections there are compatible with the actions of $\mathrm{Aut}_{(E,e)/B}(B)$.

We now assume $n \geq 3$. We next show how to construct the data of (6) from (4), and then how to construct the data of (5) from (6). This will be done in a bijective fashion under the identification of (4) and (5) above. Given the data of (4), we obtain the data of (6) from Lemma 2.3.12. Note that if we have T and T' as in (4) which are related by an automorphism of (E, σ), then they will still yield isomorphic tuples (T, P, ι) and (T', P', ι').

Conversely, given the data of (6), we recover the data of (5) as follows. To begin, note that that $\mathrm{Pic}_{P/B} \simeq \underline{\mathbb{Z}}$, for $\underline{\mathbb{Z}}$ the constant group scheme associated to $\mathbb{Z}$ on B. Of course, when P is a nontrivial Brauer-Severi scheme, the unique point of $\mathrm{Pic}_{P/B}(B)$ corresponding to the positive generator of $\mathbb{Z}$ will not correspond to a line bundle on P. Rather, it corresponds to the collection of line bundles $\mathscr{O}_{P_{U_i}}(1)$ for $U_i \to B$ an fppf cover trivializing the Brauer-Severi scheme $P \to B$. There is a natural map $\mathrm{Pic}_{P/B} \to \mathrm{Pic}_{\overline{T}/B}$ given by pullback, and the image of $1 \in \mathbb{Z} \simeq \mathrm{Pic}_{P/B}(B)$ pulls back to an element of $\mathrm{Pic}^n_{\overline{T}/B}(B)$ since on geometric fibers over B, $\overline{T}$ has degree n in P. This is the desired point of $\mathrm{Pic}^n_{\overline{T}/B}(B)$ yielding the data of (5). If we have an automorphism $(T, P, \iota) \to (T', P', \iota')$, it will be induced by an automorphism of E (which possibly does not fix σ), as can be checked fppf locally where T becomes isomorphic to E. Therefore, the above constructed element of (5) is well defined.

It only remains to explain the equivalence of (6) and (7). For B a scheme, a point $[\mathscr{H}^{(n)}/\mathrm{PGL}_n](B)$ corresponds to a PGL_n torsor $R \to B$ and a PGL_n equivariant map $R \to \mathscr{H}^{(n)}$. The latter corresponds to a subscheme $\overline{\overline{T}} \to \mathbb{P}^n_R$ flat over B of degree n whose geometric fibers lie in $\mathscr{H}^{(n)}$. By the equivalence between PGL_n torsors and Brauer-Severi schemes and effectivity of descent for closed subschemes, such data descends to a Brauer-Severi scheme $P \to B$ and a subscheme $\overline{T} \to P$ flat over B of degree n whose geometric fibers lie in $\mathscr{H}^{(n)}$. This shows that such maps $B \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ are in bijection with data as in (6). □

We next note that the above proves an equivalence between $\mathscr{M}_1^{(n)}$ and $\mathscr{S}^{(n)}$. This will not be needed in what follows, but we record it as it is a fairly straightforward consequence of the preceding discussion.

Proposition 2.3.15. *For $n \geq 3$, there is an equivalence of stacks $\mathscr{M}_1^{(n)} \simeq \mathscr{S}^{(n)}$ over $\mathscr{W}$.*

Proof. Given a B point $B \to \mathscr{M}_1^{(n)}$ the construction of the equivalence in Proposition 2.3.14 between Proposition 2.3.14(7) and Proposition 2.3.14(1) yields the desired map of stacks which is compatible with the projections to $\mathscr{W}$. This is a bijection on isomorphism classes of B-points by Proposition 2.3.14.

It remains to check this map induces an isomorphism on isotropy groups at every point. As the map $\mathscr{M}_1^{(n)} \to \mathscr{S}^{(n)}$ is compatible with the projections to $\mathscr{W}$, it is enough to check we have an isomorphism on isotropy groups of the base changes to any point $f : B \to \mathscr{W}$, corresponding to a genus 1 curve (E, e). In this case, the isotropy groups for both $B \times_{f,\mathscr{W}} \mathscr{M}_1^{(n)}$ and $B \times_{f,\mathscr{W}} \mathscr{S}^{(n)}$ are identified with $E[n](B)$ and the map between them is identified with the identity map on $E[n]$. □

Chapter 3

Proof of the main result

In this chapter, we prove our main result, relating n-torsion in class groups of quadratic fields to $\mathbb{Z}$-points of $\mathscr{V}^{\text{smile},(n)}$, which we relate to a certain quotient of an open subscheme $V_n^{\text{Res}\in\mathbb{G}_m} \subset V_n \simeq \mathbb{A}^{n+4}$ by the action of an algebraic group G_n.

In § 3.1, we show the stack $\mathscr{V}^{\text{smile},(n)}$ parameterizing divisors in a certain linear system on a Hirzebruch surface can be identified with $\widetilde{\mathscr{M}}_{1,\text{sing}}^{(n)}$, the stack parameterizing singular genus 1 curves with a marked singular point and a degree n divisor. Next, in § 3.2 we explain how the smooth locus of a singular genus 1 curves can be identified with a torus, and relate n-coverings of this torus to points of $\mathscr{V}^{\text{smile},(n)}$. In § 3.3, we use the above to relate $\mathscr{V}^{\text{smile},(n)}$ to pairs $(q, \xi) \in V_n$ having unit resultant, and prove our main theorem. Finally, we conclude the section in § 3.4 by giving some examples of our main theorem, in the setting of quadratic fields.

3.1 Singular genus 1 curves and Hirzebruch surfaces

This section is perhaps the most technically involved section of the paper, and its goal is to construct an equivalence of stacks $\mathscr{V}^{\text{smile},(n)} \simeq \widetilde{\mathscr{M}}_{1,\text{sing}}^{(n)}$. This equivalence seems to us quite intuitive, with the forward map is given by a certain linear system while the reverse map is given by blowing up the singular section, see § 3.1.7. This was perhaps the key step of the construction in § 1.5. However, to actually define the map of stacks, we are forced to carry out these constructions carefully in families, which unfortunately makes the proof rather long. The forward map is constructed in Lemma 3.1.6. The inverse map is quite a bit more involved, and is constructed in Lemma 3.1.27. Even after constructing these two maps, it is not completely trivial to show they are inverse to one another. However, it should not be surprising that they are inverse, and we verify this in the main result of this section, Theorem 3.1.31.

As a prelude to defining the map $\mathscr{V}^{\text{smile},(n)} \simeq \widetilde{\mathscr{M}}_{1,\text{sing}}^{(n)}$, we briefly review the standard description for Picard groups of Hirzebruch surfaces. For $n \geq 3$, a Hirzebruch surface $\mathbb{F}_{n-2}$ over a field k has Picard group $\mathbb{Z} \oplus \mathbb{Z}$ generated by the class of a fiber f and the class of the directrix e. This generalizes to arbitrary bases as follows.

Lemma 3.1.1. *For B a scheme, we have $\underline{\mathbb{Z}}^2 \to \mathrm{Pic}_{(\mathbb{F}_{n-2})_B/B}$, for $\underline{\mathbb{Z}}_B^2$ the constant group scheme associated to $\mathbb{Z}^2$ over B.*

Proof. There is a map $\underline{\mathbb{Z}}_B^2 \to \mathrm{Pic}_{(\mathbb{F}_{n-2})_B/B}$ which sends the first generator of $\mathbb{Z}$ to the class of the directrix E on $\mathbb{F}_{n-2}$ and seconds the second generator to the class of a fiber of the map $(\mathbb{F}_{n-2})_B \to B$. Since $\underline{\mathbb{Z}}_B^2 \to \mathrm{Pic}_{(\mathbb{F}_{n-2})_B/B}$ is a map from a scheme flat over B the fibral isomorphism criterion [Gro67, 17.9.5] reduces our task to checking it is an isomorphism on fibers.

We next claim that the fibers of $\mathrm{Pic}_{(\mathbb{F}_{n-2})_B/B} \to B$ are etale. This follows from deformation theory because the tangent space to the identity point in the fiber over the spectrum of a field $\mathrm{Spec}\, k$ is identified with $H^1((\mathbb{F}_{n-2})_k, \mathscr{O}_{(\mathbb{F}_{n-2})_k})$. This cohomology group can be verified to be 0 using the Leray spectral sequence associated to the composition $(\mathbb{F}_{n-2})_k \to \mathbb{P}_k^1 \to \mathrm{Spec}\, k$. It follows that $\mathrm{Pic}_{(\mathbb{F}_{n-2})_k/k}$ is etale for any field k.

In order to show $\underline{\mathbb{Z}}_k^2 \to \mathrm{Pic}_{(\mathbb{F}_{n-2})_k/k}$ is an isomorphism, étaleness of $\mathrm{Pic}_{(\mathbb{F}_{n-2})_k/k}$ implies that it is enough to verify it is an isomorphism on $\overline{k}$ points. This follows from the classical description of line bundles on Hirzebruch surfaces, see [Bea96, Proposition IV.1]. □

For B a scheme, let $Z \xrightarrow{i} F \xrightarrow{g} X \xrightarrow{h} B$ denote a B-point of $\mathscr{V}^{\text{smile},(n)}$. Recall from Lemma 2.2.37 that we have divisor classes e and $2f$ on F. Be aware, however, that there may fail to be any divisor D with $2[D] = 2f$, hence there may be "no divisor of class f."

Notation 3.1.2. Recall from Notation 2.2.38 that e denotes the class of the relative directrix on $F \to B$. Observe $Z \to F$ has class $e + nf$. Also, $g(Z \cap E)$ has class $\mathscr{O}_X(2)$ since E restricts to a Cartier on the smooth genus 0 curve Z which has degree 2 since it has degree 2 on fibers. We define the invertible sheaf $\mathscr{L} := \mathscr{O}_F(Z) \otimes \mathscr{O}_F(g^{-1}(g(Z \cap E)))$. This has class $(e + nf) - 2f = e + (n-2)f$.

We next verify the complete linear system associated to the class $e + (n-2)f$ on F defines a morphism from F to a rank $n-1$ projective bundle over B.

Lemma 3.1.3. *With notation as in Notation 3.1.2 $(h \circ g)_* \mathscr{L}$ is a locally free sheaf on B of rank n.*

Proof. We can verify the statement fppf locally on B, and hence we may assume $F \simeq \mathbb{F}_{n-2}$ and $\mathscr{L} \simeq \mathscr{O}_F(e + (n-2)f)$. Once we verify $(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ is locally free of rank n and $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f) = 0$ after base change to any algebraically closed field mapping to B, the statement will follow from cohomology and base change over B.

First, we compute the rank of $(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$. Let C denote a curve in the linear system $e + (n-2)f$. We obtain an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{F}_{n-2}} \longrightarrow \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f) \longrightarrow \mathscr{O}_C(e + (n-2)f) \longrightarrow 0. \quad (3.1.1)$$

We will compute the dimension of the 0th cohomology of the second nonzero term of (3.1.1) by computing the dimension of 0th cohomology of the third nonzero term using Riemann Roch. Since $e \cdot e = -(n-2), e \cdot f = 1, f \cdot f = 0$, we find that $\mathscr{O}_C(e + (n-2)f)$ has degree $(e + (n-2)f)(e + (n-2)f) = n - 2$. Further, we claim the genus of C is 0. One may compute this by adjunction using $K_C = (K_{\mathbb{F}_{n-2}} + C)|_C$ and $K_{\mathbb{F}_{n-2}} = \mathscr{O}_{\mathbb{F}_{n-2}}(-2e - nf)$. Indeed, the above shows $K_C = \mathscr{O}_C(-e - 2f)$, which has degree $(-e - 2f)(e + (n-2)f) = -2$. It follows that C has genus 0 and so $H^0(C, \mathscr{O}_C(e + (n-2)f))$ has dimension $n - 1$ because $\deg \mathscr{O}_C(e + (n-2)f = n - 2$. Therefore, (3.1.1) will imply that $(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ has rank n once we verify $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}} = 0$.

It only remains to verify $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}} = 0$ and $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f) = 0$. By Riemann Roch, we know $H^1(C, \mathscr{O}_C(e + (n-2))f) = 0$, since the sheaf has degree $n - 2$ as computed above. Ergo, the vanishing of $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}$ will imply the vanishing of $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$, by (3.1.1). Therefore, we only need show $R^1(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}} = 0$. This now follows from the Leray spectral sequence associated to the composition $h \circ g$. Namely, $R^1 h_*(f_* \mathscr{O}_{\mathbb{F}_{n-2}}) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 0$ and $h_*(R^1 f_* \mathscr{O}_{\mathbb{F}_{n-2}}) = h_*(0) = 0$. □

Define $P := \mathbb{P}(h \circ g)_* \mathscr{L}$. The surjection $(h \circ g)^*(h \circ g)_* \mathscr{L} \to \mathscr{L}$ defines a map $\phi : F \to P$ because the linear system is basepoint free, as can be verified on fibers.

Lemma 3.1.4. *The map $\phi : F \to P$ defined above sends Z to a curve C over B with a section $\tau : B \to C$ yielding a point $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$.*

Proof. Recall that we begin with a B-point of $\mathscr{V}^{\mathrm{smile},(n)}$ corresponding to $Z \xrightarrow{i} F \xrightarrow{g} X \xrightarrow{h} B$ and we have produced $P := \mathbb{P}(h \circ g)_* \mathscr{L}$ with a map $F \xrightarrow{\phi} P$. We wish to produce a genus 1 curve $C \xrightarrow{\iota} P$ with geometrically integral fibers and a section $\tau : B \to C$ lying in the singular locus of C. We will take C to be the image of $Z \xrightarrow{i} F \xrightarrow{\phi} P$ and τ to be the image of the directrix $E \to F \xrightarrow{\phi} P$. Note that E is contracted to a section under the linear system $(h \circ g)_* \mathscr{L}$, as can be checked fppf locally where $\mathscr{L}$ is isomorphic to $\mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$.

We could directly verify the above claim, but it will somewhat simplify matters to note that $\mathscr{V}^{\mathrm{smile},(n)}$ is reduced, being the quotient of an open in projective space by the action of the smooth group scheme $\mathrm{Aut}_{\mathbb{F}_{n-2}}$ Lemma 2.2.42,. Therefore, any map $B \to \mathscr{V}^{\mathrm{smile},(n)}$

factors through the reduction of B, and so we may assume B is reduced. Since B is reduced, we can identify the reduced set theoretic and scheme theoretic images of $Z \xrightarrow{i} F \xrightarrow{\phi} P$ and $E \to F \xrightarrow{\phi} P$. In particular, the formation of the scheme theoretic image commutes with base change along geometric points of B. The following Lemma 3.1.5 now completes the proof in the case B is a point. We will now prove Lemma 3.1.5 before completing the present proof.

Lemma 3.1.5. *Let k be an algebraically closed field and let $Z \xrightarrow{i} \mathbb{F}_{n-2} \xrightarrow{g} \mathbb{P}^1_k \xrightarrow{h} \operatorname{Spec} k$ correspond to a k-point of $\mathscr{V}^{smile,(n)}$. Let P denote the projective space given as the projectivization of $(h \circ g)_* \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ and let $\phi : \mathbb{F}_{n-2} \to P$ denote the associated map. Then, the image of the associated map $Z \xrightarrow{i} \mathbb{F}_{n-2} \xrightarrow{\phi} P$ is a singular geometrically integral genus 1 curve C and the image of $E \to \mathbb{F}_{n-2}$ is a point in the singular locus of C.*

Proof. First, note that because $e + nf$ is an ample divisor class, Z is geometrically connected. Therefore, smoothness of Z implies Z is geometrically integral, and therefore its image under ϕ is also geometrically integral. Additionally, the curve Z has genus 0, as follows from a standard intersection theory computation, analogous to that carried out in the proof of Lemma 3.1.3 for curves of class $e + (n-2)f$.

We claim that $\phi : (\mathbb{F}_{n-2})_B \to P$ contracts the directrix E (the unique effective curve of class e) but is an embedding over the complement of the image of E. To see E is contracted to a point we only need observe that any section of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ meeting E in fact contains E. This holds by intersection theory on $\mathbb{F}_{n-2}$ because $e \cdot (e + (n-2)f) = -(n-2) + (n-2) = 0$. Additionally, there is a codimension 1 subspace H of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ vanishing on E, whose nonzero elements have vanishing locus given by the union of e and $n-2$ fibers. We now explain why this description in fact implies that ϕ is a closed embedding over the complement of E. To check this, it is enough to show that for any degree 2 subscheme W not contained in E, the subspace of sections in $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ vanishing on W has codimension 2. If W meets E, the description of H above shows that sections vanishing on W is codimension 1 in H, and therefore codimension 2. If W lies in a fiber of the map $g : \mathbb{F}_{n-2} \to \mathbb{P}^1$, the sections vanishing on W is again codimension 2 because this is even true after restricting to the fiber, where $\mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ restricts to $\mathscr{O}(n-2)$ on the fiber. Here we are using that $n \geq 3$. Finally, if W does not meet E and is not contained in a fiber of g, the description of H above shows that there is even a codimension 2 subspace of H which vanishes on W. Hence, the sections of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ vanishing on W also has codimension 2.

We next check that the image of Z has genus 1, and is singular at the image of $Z \cap E$. Since Z does not contain E, being a smooth genus 0 curve of class $e + nf$, the intersection of

Z with E has degree $(e+nf)\cdot e = -(n-2)+n = 2$. Let C denote the image of Z under ϕ. Since, $Z \cap \phi^{-1}(\phi(E))$ has degree 2 it follows that the cokernel of the injection $\mathscr{O}_C \to \phi_*\mathscr{O}_Z$ is a skyscraper sheaf of degree 1 supported at $\phi(E) \in C$. Hence, the long exact sequence associated to

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \phi_*\mathscr{O}_Z \longrightarrow \mathscr{O}_{\phi(E)} \longrightarrow 0 \tag{3.1.2}$$

shows the arithmetic genus of C is 1. But, since Z is the normalization of C, we find C is necessarily singular. (In fact, C will be nodal if there are two geometric points in the preimage of $\phi(E)$ and cuspidal if there is one.) The singular locus of C is then precisely the image of $C \cap E$. □

We now finish the proof of Lemma 3.1.4. Let C denote the scheme theoretic image of $Z \xrightarrow{i} F \xrightarrow{\phi} P$. Indeed, by the assumptions that $i, g,$ and h are of finite presentation, we find that C is finitely presented over B. Further, $C \to B$ is proper by properness of $Z \to B$ and ϕ. By Lemma 3.1.5, every fiber of $C \to B$ has hilbert polynomial $p(t) = nt$, the hilbert polynomial of a genus 1 degree n curve in $\mathbb{P}^{n-1}$. Note that $C \to B$ is flat because it is a finitely presented proper curve over the reduced base B such that all fibers have the same Hilbert polynomial $C \to B$ is flat. It remains to show the image of $E \to F \to P$ is a section contained in the singular locus of C. We have already seen it is a section, and so it is contained in the singular locus because this holds on fibers by Lemma 3.1.5. □

In Lemma 3.1.4, we have constructed a map of sets $\mathscr{V}^{\mathrm{smile},(n)}(B)$ to $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ and we now upgrade this to a map of stacks.

Lemma 3.1.6. *The map on objects from $\mathscr{V}^{smile,(n)}(B)$ to $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ described in Lemma 3.1.4 extends to a map of stacks $\mathscr{V}^{smile,(n)} \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$.*

Proof. The map on objects has been described in Lemma 3.1.4. This map is compatible with pullback, and so it is enough to describe where automorphisms of objects are sent. To that end, fix $(B, h : X \to B, g : F \to X, i : Z \to F) \in \mathscr{V}^{\mathrm{smile},(n)}$. By the equivalence $[\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}] \to \mathscr{V}^{\mathrm{smile},(n)}$ of Lemma 2.2.42 an automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$ is induced by an automorphism $\alpha : F \to F$. Let $(B, f : P \to B, i : C \to P, \tau : B \to C)$ denote the associated object in $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$. Namely, for $\mathscr{L}$ as in Notation 3.1.2, $P = \mathbb{P}(h \circ g)_*\mathscr{L}$ and C is the image of Z under the map $Z \to F \to P$, and $\tau(B)$ is the image of the directrix $E \subset F \to P$.

We now use α to construct an automorphism of $(B, f : P \to B, i : C \to P, \tau : B \to C)$. Note that there is an isomorphism $\mathscr{L} \simeq \alpha^*\mathscr{L}$ on F because Z, E and $Z \cap E$ are all fixed by α. This induces an isomorphism $(h \circ g)_*\mathscr{L} \xrightarrow{\zeta} (h \circ g)_*\alpha^*\mathscr{L}$ of rank n locally free sheaves on B and hence a map $\psi : P \to P$. Note that although we made a choice of a

B-unit in the isomorphism ζ, any two different such choices yield the same isomorphism ψ. By construction, the map $\psi : P \to P$ restricts to α on F. Because Z is preserved by α, and C is the image of Z in P, we find that ψ preserves C. Similarly, because $E \subset F$ is preserved by α, as any automorphism of F preserves the directrix, and $\tau(B)$ is the image of E in P, it follows that ψ preserves $\tau(B)$. Therefore, ψ induces an automorphism of $(B, f : P \to B, i : C \to P, \tau : B \to C)$. □

3.1.7 Constructing the inverse map

We will next construct the inverse map. $\widetilde{\mathscr{M}}_{1,\text{sing}}^{(n)} \to \mathscr{V}^{\text{smile},(n)}$. This construction is significantly more involved, mostly due to the annoyance of having to carry it out in families, but the basic idea behind the construction is the following simple and geometric construction, already described in § 1.5. If we begin with a singular genus 1 curve in $\mathbb{P}^{n-1}$ with a marked singular point, there is a unique surface cone formed by the union of lines joining the singular point and points on the genus 1 curve. Blowing up the curve inside the cone at the singular point yields a rational curve of class $e + (n-2)f$ on the Hirzebruch surface $\mathbb{F}_{n-2}$. The actual construction of the map will be done in a slightly different order. Namely, we will first blow up the curve and use this to construct the Hirzebruch surface as a family of lines over the blow up curve. The image of this Hirzebruch surface under a suitable map to $\mathbb{P}^{n-1}$ will then be the desired surface cone. We complete this construction in Lemma 3.1.27. To start, we need to check that the blow up of C is a smooth genus 0 curve over the base.

Proposition 3.1.8. *Let $n \geq 3 \in \mathbb{Z}$ and let $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\text{sing}}^{(n)}(B)$. Then, the blow up $\mathrm{Bl}_\tau C \to B$ is a smooth proper finitely presented genus 0 curve with geometrically connected fibers. Further, the formation of this blow up commutes with arbitrary base change on B. That is, for any $B' \to B$, if we let $C' := C \times_B B'$ and τ' denote the base change of τ to B', the natural map $\mathrm{Bl}_\tau C \times_B B' \to \mathrm{Bl}_{\tau'} C'$ induced by the universal property of blow ups is an isomorphism.*

Proof. As a first step, we reduce to the case that $f : C \to B$ has a section $\sigma : B \to C$ contained in the smooth locus of f. Observe that the properties of being a smooth proper finitely presented genus 0 curve with geometrically connected fibers can be checked fppf locally on B. Additionally, since blowing up commutes with flat base change, we claim we may freely replace B by an fppf cover. Indeed, the base change map is defined over B', and so to check it is an isomorphism we can check this on any fppf cover. Hence, we may assume that in addition to the section τ, $f : C \to B$ contains a section $\sigma : B \to C$ lying in the smooth locus of f.

We now use the above section to express C as a relative plane curve over B defined by a simple equation. Now, since C is a finitely presented genus 1 curve over B with geometrically integral fibers and a section in the smooth locus, from the definition of $\mathscr{W}$ Definition 2.2.2, we obtain a map $B \to \mathscr{W}$. By Proposition 2.2.13 and Definition 2.2.10, after replacing B by an fppf cover we may assume that $B = \operatorname{Spec} R$ is affine and C is defined by an equation of the form $y^2z + a_1xyz + a_3y = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$ in $\operatorname{Proj} R[x, y, z] = \mathbb{P}^2_B$, with $a_1, a_2, a_3, a_4, a_6 \in R$. After again possibly replacing B by a cover, we may assume that the section τ lying in the singular locus is given by $x = y = 0$. The condition that C passes through τ forces $a_6 = 0$ while the condition that C is singular at τ forces $a_3 = a_4 = 0$. Therefore, C is given by an equation of the form $y^2z + a_1xyz = x^3 + a_2x^2z$.

Thus far, we have reduced the problem to showing that the blow up of $y^2z + a_1xyz = x^3 + a_2x^2z$. at the section $x = y = 0$ is a smooth proper finitely presented genus 0 curve with geometrically connected fibers. To this end, we will compute the blow up in terms of equations in Lemma 3.1.9 and then return to this proof.

Lemma 3.1.9. *Let R be a ring and $C \subset \mathbb{A}^2_R$ be given by Spec $R[x, y]/(y^2 + a_1xy = x^3 + a_2x^2)$. Let $\tau : \operatorname{Spec} R \to \mathbb{P}^2_R$ denote the section given by $x = y = 0$. Then,* $\operatorname{Bl}_\tau C$ *is given by Proj $R[x, y, X, Y]/(xY - yX, Y^2 + a_1YX - xX^2 - a_2X^2)$ where X, Y have degree* 1 *and x, y have degree* 0*.*

Proof. We compute this blow up via the blow up closure lemma [Vak, 22.2.6]. Namely, we know that the blow up $\operatorname{Bl}_\tau \mathbb{A}^2_R$ is given by $\operatorname{Proj} R[x, y, X, Y]/(xY - yX)$ and $\operatorname{Bl}_\tau C$ is given as the closure of $C - V(x, y)$ in $\operatorname{Bl}_\tau \mathbb{A}^2_R$. We now compute this closure. We will compute this separately on the two charts $X \neq 0$ and $Y \neq 0$, and then glue them together.

First, we consider $Y \neq 0$. In that case, we may work with the variable $\alpha = X/Y$, and we wish to compute the closure of $C - V(x, y)$ in $\operatorname{Spec} R[x, y, \alpha]/(x - y\alpha)$. On this chart, the equation defining C can be rewritten as

$$y^2 + a_1y^2\alpha - y^2x\alpha^2 - a_2y^2\alpha^2 = y^2(1 + a_1\alpha - x\alpha^2 - a_2\alpha^2) = y^2(1 + a_1\alpha - (x + a_2)\alpha^2).$$

If $y = 0$, as $x = y\alpha$, we find $x = 0$ as well, and therefore we may assume y is invertible. In particular, the closure of $C - V(x, y)$ on the chart $Y \neq 0$ is contained in D_1, the closure of the vanishing locus of $1 + a_1\alpha - (x + a_2)\alpha^2$. Moreover, we claim the closure of $C - V(x, y)$ is equal to D_1. To see this, on this chart, whenever $y = 0$ we also have $x = 0$. Therefore, we are blowing up at the closed subscheme $y = 0$, implying the closure of $C - V(x, y)$ on this chart is given as the $R[x, y, \alpha]/(x - y\alpha, I)$ where I is the kernel of the localization map at the element y:

$$R[x, y, \alpha]/(x - y\alpha, y^2 + a_1 y^2\alpha - y^2\alpha^3 - a_2 y^2\alpha^2) \\ \to \left(R[x, y, \alpha]/(x - y\alpha, y^2 + a_1 y^2\alpha - y^2\alpha^3 - a_2 y^2\alpha^2)\right)_y .$$

We claim this kernel is generated by $1 + a_1\alpha - (x + a_2)\alpha^2$. Indeed, to see this, note that $A := R[x, y, \alpha]/(x - y\alpha, y^2 + a_1 y^2\alpha - y^2\alpha^3 - a_2 y^2\alpha^2) \simeq R[y, \alpha]/(y^2 + a_1 y^2\alpha - y^2\alpha^3 - a_2 y^2\alpha^2)$ and the condition that $s \in A$ lie in the kernel of the above localization map then amounts to the condition that $y^m s = 0$ for some m. In other words, s must lie in the ideal generated by $1 + a_1\alpha - (y\alpha - a_2)\alpha^2 = 1 + a_1\alpha - (x + a_2)\alpha^2$ as y does not divide this polynomial.

We next perform the analogous computation on the chart $X \neq 0$, in which case we introduce the variable $\beta = Y/X$. On this chart, the equation defining C can be rewritten as

$$x^2\beta^2 + a_1 x^2\beta - x^2 - a_2 x^2 = x^2(\beta^2 + a_1\beta - x - a_2).$$

Since x is invertible on $C - V(x, y)$, we may assume $x \neq 0$. Hence, the closure of $C - V(x, y)$ is contained in D_2, where D_2 is the closure of the vanishing locus of $\beta^2 + a_1\beta - x - a_2$. Similarly to the above case $Y \neq 0$, we find that D_2 is equal to the closure of $C - V(x, y)$ restricted to $X \neq 0$, as this can be computed in terms of a similar kernel of a localization map of rings, this time localizing at the element x.

Now, we may observe that the scheme $D := \operatorname{Proj} R[x, y, X, Y]/(xY - yX, Y^2 + a_1 XY - xX^2 - a_2 X^2)$ is a subscheme of $\operatorname{Bl}_\tau \mathbb{A}^2_R$ whose restriction to the coordinate patch $Y = 0$ agrees with the scheme D_1 defined above and whose restriction to the coordinate patch $X = 0$ agrees with the scheme D_2 defined above. Therefore, $\operatorname{Bl}_\tau C$ is equal to D. □

We now finish the proof of Proposition 3.1.8. First, we observe that Lemma 3.1.9 shows the formation of $\operatorname{Bl}_\tau C$ commutes with arbitrary base change. Indeed, this is certainly true on the complement of τ as the blow up map is an isomorphism there, so it suffices to verify this on a Zariski neighborhood of τ. This then follows by Lemma 3.1.9, which gives an explicit description of the blow up that evidently commutes with arbitrary base change on B.

Note that $\operatorname{Bl}_\tau C$ is finitely presented and proper over $B = \operatorname{Spec} R$ by the construction in Lemma 3.1.9. Next, we verify the genus of each fiber of $\operatorname{Bl}_\tau C$ is 0. Because the formation of $\operatorname{Bl}_\tau C$ commutes with arbitrary base change, we may compute the genus of the fibers of $\operatorname{Bl}_\tau C$ over geometric points. The geometric fibers may be computed as the blow up of a singular genus 1 curve at its singular point over a field, which is indeed geometrically connected and of genus 0.

Hence, to complete the proof, it suffices to show $\operatorname{Bl}_\tau C$ is smooth over B. This is now an explicit computation using the description of Lemma 3.1.9. Namely, we verify $\operatorname{Proj} R[x, y, X, Y]/(xY - yX, Y^2 + a_1XY - xX^2 - a_2X^2)$ is smooth in the two charts given by $Y \neq 0$ and $X \neq 0$.

First we address the chart $Y \neq 0$. Let $\alpha = X/Y$ so that $\mathrm{Bl}_\tau C$ restricts to

$$\operatorname{Spec} R[x, y, \alpha]/(x - y\alpha, 1 + a_1\alpha - \alpha^2(x + a_2)) \simeq \operatorname{Spec} R[y, \alpha]/(1 + a_1\alpha - \alpha^3 y - a_2\alpha^2).$$

By the Jacobian criterion for smoothness, we see the partial with respect to y of $1 + a_1\alpha - \alpha^3 y - a_2\alpha^2$ is α^3, and so any singularity lies on $V(\alpha^3)$. However, $V(\alpha^3) = \emptyset$ in $\operatorname{Spec} R[y, \alpha]/(1 + a_1\alpha - \alpha^3 y - a_2\alpha^2)$ because the underlying set of $V(\alpha^3)$ agrees with the underlying set of $V(\alpha)$ and $1 + a_1\alpha - \alpha^3 y - a_2\alpha^2 \bmod \alpha = 1$. Hence, there are no singularities on the chart $Y \neq 0$.

To conclude, we show there are no singularities on the chart $X \neq 0$. Let $\beta = Y/X$ so that $\mathrm{Bl}_\tau C$ restricts to $\operatorname{Spec}[x, y, \beta]/(\beta x - y, \beta^2 + a_1\beta - x - a_2) = \operatorname{Spec}[x, \beta]/(\beta^2 + a_1\beta - x - a_2)$. The partial of $\beta^2 + a_1\beta - x - a_2$ with respect to x is -1, and hence any singular point lies on $V(-1)$ and therefore this variety is again smooth. □

Having constructed our smooth genus 0 curve, the next step is to construct a relative $(n-2)$-Hirzebruch twist containing $\mathrm{Bl}_\tau C$. We begin by introducing some notation used to define this Hirzebruch twist. Figure 3.1 may be helpful in visualizing some of the objects at play.

Notation 3.1.10. Let $n \geq 3 \in \mathbb{Z}$. Let $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$. Blowing C up at τ, we obtain a map $\nu : \mathrm{Bl}_\tau C \to C$. Let $E_\tau C \subset \mathrm{Bl}_\tau C$ denote the exceptional divisor associated to the blow up of C at τ. From this, we obtain a map

$$\psi := (\mathrm{id}, \iota \circ \nu) \coprod (\mathrm{id}, \iota \circ \tau \circ f \circ \iota \circ \nu) : \mathrm{Bl}_\tau C \coprod \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C \times_B P$$

Let W denote the scheme theoretic image of ψ. We let $i_1 : \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C \coprod \mathrm{Bl}_\tau C$ denote the first inclusion and $i_2 : \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C \coprod \mathrm{Bl}_\tau C$ denote the second inclusion. Let L denote the image of $\psi \circ i_1 : \mathrm{Bl}_\tau C \to W$ and let M denote the image of $\psi \circ i_2 : \mathrm{Bl}_\tau C \to W$. In particular, the composition of $\psi \circ i_1$ with the projection $W \to P$ is $\iota \circ \nu$ while the composition of $\psi \circ i_2$ with the projection $W \to P$ is $\iota \circ \tau \circ f \circ \iota \circ \nu$, the constant map to $\iota \circ \tau(B)$.

Remark 3.1.11. On fibers, over B we will see that W is given as two copies of $\mathrm{Bl}_\tau C$ glued along $E_\tau C$. Upon projecting to P, $\psi \circ i_1(\mathrm{Bl}_\tau C)$ maps to C while $\psi \circ i_2(\mathrm{Bl}_\tau C)$ is contracted to the image of τ. We also note that ψ is a closed embedding when restricted to either copy of $\mathrm{Bl}_\tau C$, since the composition of ψ with the first projection to $\mathrm{Bl}_\tau C$ is an isomorphism.

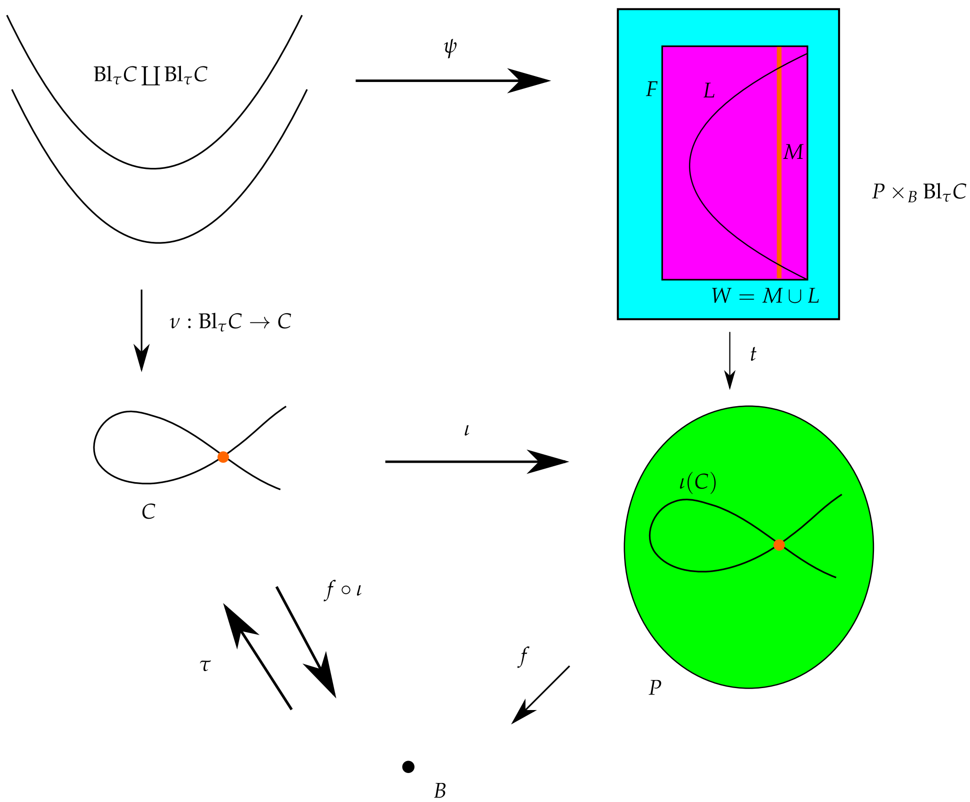


Figure 3.1: A visualization of some of the objects appearing in Notation 3.1.10, Notation 3.1.17, and Definition 3.1.19.

Remark 3.1.12. Note that $i_2(\mathrm{Bl}_\tau C)$ is mapped to P via the constant map at τ, and so its intersection with $\psi \circ i_1(\mathrm{Bl}_\tau C)$ is the preimage of the section τ along $\nu : \mathrm{Bl}_\tau C \to C$. However, by definition of the exceptional divisor of a blow up at τ, this preimage is precisely $E_\tau C$. Therefore, by the universal property of gluing along closed subschemes, the map $\mathrm{Bl}_\tau C \coprod \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C$ induces a map $\rho : \mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C$. Note that the cofiber product $\mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C$ exists as a scheme by [Sta, Tag 0E25].

In order to construct the relative Hirzebruch surface containing $\mathrm{Bl}_\tau C$, we will construct a map from $\mathrm{Bl}_\tau C$ to a Grassmannian of lines. To make this construction, we will need to invoke cohomology and base change. In turn, to apply cohomology and base change, we will need to know $W \to B$ is flat, which we check in Proposition 3.1.16. We now present the unexpectedly tricky verification of this fact. In order to verify flatness, we first need an alternate description of W for which the following generalization of the Chinese remainder theorem will be crucial.

Lemma 3.1.13. *Let R be a ring and I_1, I_2 be two ideals. Then the natural projection*

$$\begin{aligned} \gamma : R/(I_1 \cap I_2) &\to R/I_1 \times_{R/(I_1+I_2)} R/I_2 \\ r + (I_1 \cap I_2) &\mapsto (r + I_1, r + I_2) \end{aligned}$$

is an isomorphism.

Remark 3.1.14. Note that the above specializes to the Chinese remainder theorem in the case $I_1 + I_2 = 1$. Of course, the proof we present for this more general fact specializes to a proof of the Chinese remainder theorem. However, surprisingly, the proof we give appears to be simpler than the standard proof of the Chinese remainder theorem appearing, for example, in [DF04, §7.6, Theorem 17].

Proof. First, the above map γ is well defined because for $r \in R$ its images in R/I_1 and R/I_2 both map to $r + I_1 + I_2$ in $R/(I_1 + I_2)$. The map is injective because if $r = 0 \bmod I_1$ and $r = 0 \bmod I_2$ then $r \in I_1$ and $r \in I_2$, so $r \in I_1 \cap I_2$. Finally, we check γ is surjective. Begin with an element $(x + I_1, y + I_2) \in R/I_1 \times_{R/(I_1+I_2)} R/I_2$. By definition of the fiber product, $x - y \in I_1 + I_2$, so we may find $a \in I_1, b \in I_2$ with $x - y = a + b$. Then, the element $x - a = y + b$ is an element of R whose reduction $\bmod I_1$ is $x - a + I_1 = x + I_1$ and reduction $\bmod I_2$ is $y + b + I_2 = y + I_2$. Therefore, $\gamma(x - a) = (x + I_1, y + I_2)$ and γ is surjective. □

We now present the formerly mentioned alternate description of W.

Lemma 3.1.15. *With notation as in Notation 3.1.10, W is isomorphic to two copies of* $\mathrm{Bl}_\tau C$ *glued along the closed subscheme $E_\tau C$.*

Proof. We notate the cofiber product of two copies of $\mathrm{Bl}_\tau C$ along $E_\tau C$ by $\mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C$. Quasi-compactness of ψ allows us to compute the scheme theoretic image of W affine locally on $\mathrm{Bl}_\tau C \times_B P$. Since ψ is proper and quasi-finite (since it is a closed embedding when restricted to either copy of $\mathrm{Bl}_\tau C$) ψ is finite, hence affine. Therefore, we may compute the image affine locally over an open $\operatorname{Spec} R \subset \mathrm{Bl}_\tau C \times_B P$. The preimage of $\operatorname{Spec} R$ under ψ is an affine open in $\mathrm{Bl}_\tau C \coprod \mathrm{Bl}_\tau C$ and therefore it is the disjoint union of an affine open $\operatorname{Spec} A_1$ in $i_1(\mathrm{Bl}_\tau C)$ and an affine open $\operatorname{Spec} A_2$ in $i_2(\mathrm{Bl}_\tau C)$. We may write this disjoint union as $\operatorname{Spec} A_1 \times A_2$.

The scheme theoretic image is then given locally over $\operatorname{Spec} R$ as $\operatorname{Spec}(R/\ker \psi^\sharp)$ for $\psi^\sharp : R \to A_1 \times A_2$. For $j \in \{1,2\}$, let $i_j^\sharp : R \to A_j$ denote the maps induced by i_j and let $I_j := \ker i_j^\sharp$. To compute the scheme theoretic image of ψ, we wish to compute $\ker \psi^\sharp = I_1 \cap I_2$.

We want to show $\operatorname{Spec} R/(I_1 \cap I_2)$ is given as the restriction of $\mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C$ to $\operatorname{Spec} A_1 \times \operatorname{Spec} A_2$. Observe that $\operatorname{Spec} R/(I_1 + I_2)$ is the restriction of $E_\tau C$ to $\operatorname{Spec} R$. Indeed, since $\operatorname{Spec} R/(I_1 + I_2) = \operatorname{Spec} R/I_1 \cap \operatorname{Spec} R/I_2$, it suffices to show the intersection $\operatorname{im} \psi \circ i_1 \cap \operatorname{im} \psi \circ i_2 = E_\tau C$, which was explained in Remark 3.1.12. Therefore, $R/I_1 \simeq A_1, R/I_2 \simeq A_2$, where we have isomorphisms as opposed to only injections because ψ is a closed immersion when restricted to each copy of $\mathrm{Bl}_\tau C$. Under these identifications, the surjections $R/I_1 \to R/(I_1 + I_2)$ and $R/I_2 \to R/(I_1 + I_2)$ correspond to the two closed immersions $E_\tau C \to i_1(\mathrm{Bl}_\tau C)$ and $E_\tau C \to i_2(\mathrm{Bl}_\tau C)$. Then, under these identifications, the gluing $\mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C$ corresponds locally to $R/I_1 \times_{R/(I_1+I_2)} R/I_2$ since we have shown that $\operatorname{Spec} R/(I_1 + I_2)$ corresponds to the image of $E_\tau C$ in $\mathrm{Bl}_\tau C \times_B P$. It then follows from Lemma 3.1.13 that the natural projection identifies $\operatorname{Spec} R/(I_1 \cap I_2)$, the scheme theoretic image of ϕ restricted to $\operatorname{Spec} R$, with the gluing $\operatorname{Spec} R/I_1 \times_{R/(I_1+I_2)} R/I_2$. Because the above constructions are compatible with localization, it follows that the natural map $\mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C \to W$, coming from the universal property of gluing along closed subschemes, is an isomorphism. □

Using the above description of W, we are ready to verify it is flat.

Proposition 3.1.16. *With notation as in Notation 3.1.10, the natural map $\rho : W \to \mathrm{Bl}_\tau C$, as defined in Remark 3.1.12 is locally free of degree 2. Further, $W \to B$ is flat.*

Proof. By Lemma 3.1.15, we have an isomorphism $W \simeq \mathrm{Bl}_\tau C \coprod_{E_\tau C} \mathrm{Bl}_\tau C$, the coproduct of two copies of $\mathrm{Bl}_\tau C$ along the closed subscheme $E_\tau C$. Since $\mathrm{Bl}_\tau C \to B$ is flat by Proposition 3.1.8, it suffices to show ρ is flat. We may work affine locally on $\mathrm{Bl}_\tau C$, and hence assume it is of the form $\operatorname{Spec} A$. By shrinking $\operatorname{Spec} A$ if necessary, we may assume the preimage of $\operatorname{Spec} A$ in $E_\tau C$ is $\operatorname{Spec} A/f$ for f a non-zerodivisor, using the fact that

$E_\tau C$ is the exceptional divisor of a blow up, hence a Cartier divisor. The preimage of $\operatorname{Spec} A$ in $\operatorname{Bl}_\tau C \coprod_{E_\tau C} \operatorname{Bl}_\tau C$ is then of the form $\operatorname{Spec}(A \times_{A/f} A)$ as follows from the explicit construction of gluings of schemes along closed subscheme.

Hence, we have reduced to the explicit ring theoretic problem of showing that $A \times_{A/(f)} A$ is flat as an A module. Indeed, there is an isomorphism of A-modules $A \times A \to A \times_{A/(f)} A$ given by $(a, b) \mapsto (a, a + fb)$. The map is injective because f is a non-zerodivisor. The map is surjective because any $(x, y) \in A \times_{A/(f)} A$ is the image of (x, z), where z is the unique element of A so that $x - y = fz$. This exists because $x - y \in (f)$ by definition of $A \times_{A/(f)} A$. Therefore, $A \times_{A/(f)} A$ is flat and locally free of degree 2 as an A module. Hence, $W \to \operatorname{Bl}_\tau C$ is flat. □

Having shown $W \to B$ is flat, we next construct the $(n-2)$-Hirzebruch twist. Before embarking on the construction, we describe the idea. Recall that $W \to B$ factors through $\operatorname{Bl}_\tau C$, and the fibers of $W \to \operatorname{Bl}_\tau C$ are degree 2 subschemes of $\mathbb{P}^{n-1}$. There is then a unique line in $\mathbb{P}^{n-1}$ spanned by this degree 2 subscheme, and the union of these lines varying over points of $\operatorname{Bl}_\tau C$ spans the desired Hirzebruch surface. Due to the issue that B may be non-reduced, we need to carry out this construction in families, as we now do.

Notation 3.1.17. Recall that P is a projective bundle, as it is a Brauer-Severi scheme over B with a section $\iota \circ \tau$, and so comes with an invertible sheaf $\mathscr{O}_P(1)$. Define the projections

$$\begin{array}{ccc} & \operatorname{Bl}_\tau C \times_B P & \\ & {}^{s}\swarrow \quad \searrow^{t} & \\ \operatorname{Bl}_\tau C & & P. \end{array} \tag{3.1.3}$$

Let $\mathscr{I}_{W/P\times_B \operatorname{Bl}_\tau C}$ denote the ideal sheaf of W in $P \times_B \operatorname{Bl}_\tau C$.

Pushing forward the ideal sheaf exact sequence twisted by $t^*\mathscr{O}_P(1)$ along s we obtain the exact sequence

$$\begin{aligned} 0 &\longrightarrow s_*\left(\mathscr{I}_{W/P\times_B \operatorname{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow s_*\left(t^*\mathscr{O}_P(1)\right) \longrightarrow s_*\left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow \\ &\longrightarrow R^1 s_*\left(\mathscr{I}_{W/P\times_B \operatorname{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow R^1 s_*\left(t^*\mathscr{O}_P(1)\right). \end{aligned} \tag{3.1.4}$$

In order to define our desired family of lines, we need the following consequence of cohomology and base change. We note that this will crucially use the flatness of W over $\operatorname{Bl}_\tau C$.

Lemma 3.1.18. *We have* $R^1 s_* \left(\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) = 0$ *and the resulting sequence*

$$0 \longrightarrow s_* \left(\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow s_* \left(t^*\mathscr{O}_P(1)\right) \longrightarrow s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow 0 \tag{3.1.5}$$

is an exact sequence of locally free sheaves. The first nonzero term has rank $n-2$*, the second has rank* n*, and the third has rank* 2*. Further, for* $f : P \to B$ *the structure map, we have a natural identification* $s_* \left(t^*\mathscr{O}_P(1)\right) \simeq (f \circ \iota \circ \nu)^* \left(f_*\mathscr{O}_P(1)\right)$.

Proof. Using flatness of $W \to \mathrm{Bl}_\tau C$, as established in Proposition 3.1.16, the sheaf $\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)$ is flat over $\mathrm{Bl}_\tau C$. Since $t^*\mathscr{O}_P(1)$ is flat over $\mathrm{Bl}_\tau C$ being an invertible sheaf on a flat cover, $\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)$ is also flat over $\mathrm{Bl}_\tau C$, being the kernel of a map of flat sheaves. Therefore, cohomology and base change applies to the above three sheaves. To start, we find $R^1 s_* \left(t^*\mathscr{O}_P(1)\right) = 0$ using cohomology and base change because the first derived pushforward of the restriction of $t^*\mathscr{O}_P(1)$ to a fiber over a point x of $\mathrm{Bl}_\tau C$ is $H^1(x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)) = 0$. Note that by cohomology and base change, we find $s_* \left(t^*\mathscr{O}_P(1)\right)$ commutes with base change and is locally free. We may compute the rank of $s_* \left(t^*\mathscr{O}_P(1)\right)$ over any point x of $\mathrm{Bl}_\tau C$, which is given by $h^0(x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)) = n$, as claimed.

Next, we compute the rank of $s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)$ and show it commutes with arbitrary base change. The vanishing of $R^1 s_* \left(t^*\mathscr{O}_P(1)\right)$ implies $R^1 s_* \left(\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right)$ is the cokernel of $s_* \left(t^*\mathscr{O}_P(1)\right) \to s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)$. Because $\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)$ is supported on W which has relative dimension 0 over $\mathrm{Bl}_\tau C$, we find $R^i s_* \mathscr{O}_W \otimes t^*\mathscr{O}_P(1) = 0$ for $i > 0$. Hence, by cohomology and base change, $s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)$ commutes with base change and is locally free. Further, $s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)$ has rank is 2 by Proposition 3.1.16.

The next step is to show $s_* \left(t^*\mathscr{O}_P(1)\right) \to s_* \left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)$ is surjective and that $R^1 s_* \left(\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) = 0$. To check surjectivity, it is enough to verify the cokernel has empty support. Since these two sheaves commute with base change, as shown above, it is enough to check that for each geometric point $x \in \mathrm{Bl}_\tau C$, the base changed map $H^0(\mathbb{P}^{n-1}_x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)) \to H^0(W_x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)|_{W_x})$ is surjective. Since W_x is a degree 2 scheme over the point x, this follows from the fact that the locus of hyperplanes in $\mathbb{P}^{n-1}$ vanishing on any degree 2 subscheme of $\mathbb{P}^{n-1}$ is $n-2$ dimensional. Hence the kernel of $H^0(\mathbb{P}^{n-1}_x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)) \to H^0(W_x, \mathscr{O}_{\mathbb{P}^{n-1}_x}(1)|_{W_x})$ has dimension $n-2$ implying that the above map has image of dimension 2, and is therefore surjective. From this we obtain $R^1 s_* \left(\mathscr{I}_{W/P\times_B \mathrm{Bl}_\tau C} \otimes t^*\mathscr{O}_P(1)\right) = 0$.

To conclude, we only need exhibit the claimed natural identification $s_* \left(t^*\mathscr{O}_P(1)\right) \simeq (f \circ \iota \circ \nu)^* \left(f_*\mathscr{O}_P(1)\right)$. Indeed, this results from flat base change applied to the sheaf $\mathscr{O}_P(1)$

on P in the diagram

$$\begin{array}{ccc} \mathrm{Bl}_\tau C \times_B P & \xrightarrow{t} & P \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Bl}_\tau C & \xrightarrow{f \circ \iota \circ \nu} & B. \end{array}$$

□

We are now prepared to construct the sought $(n-2)$-Hirzebruch twist F. Here is the definition.

Definition 3.1.19. Let $n \geq 3 \in \mathbb{Z}$, let $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ and retain notation from Notation 3.1.17. By Lemma 3.1.18, the exact sequence (3.1.5) furnishes a map $\omega : \mathrm{Bl}_\tau C \to \mathbb{G}(1, P)$, the Grassmannian of lines in P. (As usual, we are using Grothendieck conventions, so a surjection onto a locally free rank 2 sheaf correspond to lines on the source projective bundle.) The universal bundle over $\mathbb{G}(1, P)$ pulls back along ω to a relative family of lines F with an embedding $F \hookrightarrow \mathrm{Bl}_\tau C \times_B P$, where $F \simeq \mathbb{P}\left(s_*\left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right)\right)$ over $\mathrm{Bl}_\tau C$.

Remark 3.1.20. Additionally, we can realize a map $i_W : W \hookrightarrow F$ via the surjection of sheaves $\rho^* s_*\left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right) = \rho^*\rho_*\left((t^*\mathscr{O}_P(1))|_W\right) \to (t^*\mathscr{O}_P(1))|_W$ coming from the natural adjunction.

We will next verify that F as defined in Definition 3.1.19 is an $(n-2)$-Hirzebruch twist. In order to proceed, we will need the following lemma, which will enable us to identify F fppf locally by placing it in an appropriate exact sequence.

Lemma 3.1.21. *Let $B = Spec\, R$ for R a local ring, let $m \in \mathbb{Z}_{\geq 1}$ and suppose $\mathscr{F}$ is a sheaf on $\mathbb{P}^1_B$ fitting in an exact sequence*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_B}(m) \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_{\mathbb{P}^1_B} \longrightarrow 0 \qquad (3.1.6)$$

Then, $\mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^1_B}(m) \oplus \mathscr{O}_{\mathbb{P}^1_B}$

Proof. It is enough to show there are no nontrivial extensions of $\mathscr{O}_{\mathbb{P}^1_B}$ by $\mathscr{O}_{\mathbb{P}^1_B}(m)$. To see why, such extensions can be identified with $Ext^1_{\mathbb{P}^1_B}(\mathscr{O}_{\mathbb{P}^1_B}, \mathscr{O}_{\mathbb{P}^1_B}(m)) = H^1(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(m))$. Letting $\pi : \mathbb{P}^1_B \to B$ denote the structure morphism, we can compute this cohomology group in terms of a spectral sequence, and so it suffices to show $H^0(B, R^1\pi_*\mathscr{O}_{\mathbb{P}^1_B}(m)) = 0$ and $H^1(B, \pi_*\mathscr{O}_{\mathbb{P}^1_B}(m)) = 0$. For the former, it follows from cohomology and base change that $R^1\pi_*\mathscr{O}_{\mathbb{P}^1_B}(m)$ because for any field k, $H^1(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}(m)) = 0$. For the latter, we may note that $\pi_*\mathscr{O}_{\mathbb{P}^1_B}(m)$ is a vector bundle over a local ring, and therefore it is a trivial vector bundle. Hence, we only need verify $H^1(B, \mathscr{O}_B) = 0$. This holds as B is affine, and so higher quasi-coherent cohomology vanishes. □

For the verification that F is an $(n-2)$-Hirzebruch twist, it will also be crucial to know that the exceptional divisor $E_\tau C$ is a Cartier divisor in $\mathrm{Bl}_\tau C$, as we now check.

Lemma 3.1.22. *Retaining notation from Notation 3.1.10, we have an isomorphism $L \cap M \simeq E_\tau C$, for $E_\tau C$ the exceptional divisor of the blow up ν of C at τ. Further, $E_\tau C$ is a degree 2 relative effective Cartier divisor on L over B.*

Proof. First, we show $E_\tau C$ is a relative effective Cartier divisor on $\mathrm{Bl}_\tau C$. Since $E_\tau C$ is identified with the restriction of L to M by Lemma 3.1.15, this will prove $L \cap M \simeq E_\tau C$. By [BLR90, §8.2, Lemma 6], it is enough to know E_τ is a effective Cartier divisor and remains such when restricted to each fiber over B. However, note that E_τ is an effective Cartier divisor as it is the exceptional divisor of a blow up. It remains to check its base change to each fiber is an effective Cartier divisor. This holds as the formation of blow up at τ commutes with arbitrary base change on B by Proposition 3.1.8, and so its fiber over any point is again the exceptional divisor of a blow up, and hence an effective Cartier divisor.

It remains to check $E_\tau C$ has degree 2 over B. We may verify this on fibers over points of B. We can now do explicit computations in the nodal and cuspidal cases to see $E_\tau C$ has degree 2, though instead we choose to give the following uniform argument. For $\nu : \mathrm{Bl}_\tau C \to C$ the natural map, the injection of sheaves $\mathscr{O}_C \to \nu_* \mathscr{O}_{\mathrm{Bl}_\tau C}$ has cokernel which is a skyscraper sheaf supported over τ, call it $\mathscr{S}_\tau$. The resulting exact sequence of sheaves induces a long exact sequence on cohomology

$$\begin{aligned} 0 \longrightarrow H^0(C, \mathscr{O}_C) \longrightarrow H^0(C, \nu_* \mathscr{O}_{\mathrm{Bl}_\tau C}) \longrightarrow H^0(C, \mathscr{S}_\tau) \longrightarrow \\ \longrightarrow H^0(C, \mathscr{O}_C) \longrightarrow H^0(C, \nu_* \mathscr{O}_{\mathrm{Bl}_\tau C}). \end{aligned} \tag{3.1.7}$$

The map $H^0(C, \mathscr{O}_C) \to H^0(C, \nu_* \mathrm{Bl}_\tau C)$ is an isomorphism between 1 dimensional vector spaces. We also have $H^1(C, \nu_* \mathscr{O}_{\mathrm{Bl}_\tau C}) = H^1(\mathrm{Bl}_\tau C, \mathscr{O}_{\mathrm{Bl}_\tau C}) = 0$ since ν is finite and $\mathrm{Bl}_\tau C$ has genus 0. Since $H^1(C, \mathscr{O}_C)$ is 1-dimensional as C has genus 1, we find $\mathscr{S}_\tau$ has degree 1. This implies the fiber of $\nu_* \mathscr{O}_{\mathrm{Bl}_\tau C}$ over τ has degree 2, as desired. □

We now show F is an $(n-2)$-Hirzebruch twist.

Proposition 3.1.23. *The map $F \to \mathrm{Bl}_\tau C \to B$ be as in Definition 3.1.19. gives F the structure of an $(n-2)$-Hirzebruch twist over B.*

Proof. By Proposition 3.1.8, $\mathrm{Bl}_\tau C$ is a 1-dimensional Brauer-Severi scheme over B and by definition F is a relative dimension 1 Zariski-locally trivial projective bundle over $\mathrm{Bl}_\tau C$. Therefore, it only remains to show that F is isomorphic to $\mathbb{F}_{n-2}$ fppf locally on B.

Since $\mathrm{Bl}_\tau C$ is a smooth geometrically connected genus 0 curve over B we may replace B by a suitable fppf cover B' so that $(\mathrm{Bl}_\tau C) \times_B B'$ is isomorphic to $\mathbb{P}^1_{B'}$. We now rename B' as B, and hence may assume that $\mathrm{Bl}_\tau C$ is isomorphic to $\mathbb{P}^1_B$. Then, in Definition 3.1.19, we constructed F as the projectivization of the rank 2 vector bundle $s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1))$ on $\mathbb{P}^1_B$.

Next, by replacing B with an affine open cover, we may assume $B = \operatorname{Spec} R$ is affine. Further, via spreading out it suffices to exhibit the desired isomorphism $F \simeq \mathbb{F}_{n-2}$ over the local ring at each point $b \in B$. Indeed, in this case, the isomorphism extends to some affine open in B, and we can choose finitely many such affine opens to cover B by quasi-compactness of the affine scheme B.

Hence, we have reduced to showing that the free rank 2 vector bundle $s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1))$ on $\mathbb{P}^1_B$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1_B} \oplus \mathscr{O}_{\mathbb{P}^1_B}(n-2)$ in the case B a local scheme, i.e., B is the spectrum of a local ring. Next, recall that the group of line bundles on $\mathbb{P}^1_B$ for B a local scheme is $\mathbb{Z}$, with representatives given by $\mathscr{O}_{\mathbb{P}^1_B}(m)$ for $m \in \mathbb{Z}$. One way to see this is that we have a map of group schemes $\zeta : \underline{\mathbb{Z}}_B \to \mathrm{Pic}_{\mathbb{P}^1_B/B}$ where the source is flat and the map is an isomorphism over every point of B (since the Picard group scheme of $\mathbb{P}^1$ over a field k is smooth as $H^1(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}) = 0$). Therefore, by the fibral isomorphism criterion [Gro67, 17.9.5], ζ is an isomorphism.

We next claim there is an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_B}(n-2) \longrightarrow s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1)) \longrightarrow \mathscr{O}_{\mathbb{P}^1_B} \longrightarrow 0. \tag{3.1.8}$$

We will verify exactness of this sequence in Lemma 3.1.24. Lemma 3.1.21 then completes the proof. ◻

Lemma 3.1.24. *The sequence* (3.1.8) *is exact and the surjection* $s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1)) \to \mathscr{O}_{\mathbb{P}^1_B}$ *in* (3.1.8) *is identified with the restriction map* $r : s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1)) \to s_* (\mathscr{O}_M \otimes t^* \mathscr{O}_P(1))$ *for* M *as in Notation 3.1.10*

Proof. We start by constructing a surjection $s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1)) \to \mathscr{O}_{\mathbb{P}^1_B}$. Indeed, recall from Notation 3.1.10 that W is constructed as the image of two copies of $\mathrm{Bl}_\tau C \simeq \mathbb{P}^1_B$ in $\mathbb{P}^1_B \times_B P$. Recall the definitions of L and M from Notation 3.1.10. Note that M is mapped to P via the constant map through τ, and therefore $\mathscr{O}_M \otimes t^* \mathscr{O}_P(1) \simeq \mathscr{O}_M$. The restriction map $r : s_* (\mathscr{O}_W \otimes t^* \mathscr{O}_P(1)) \to s_* (\mathscr{O}_M \otimes t^* \mathscr{O}_P(1)) \simeq s_*(\mathscr{O}_M) \simeq \mathscr{O}_{\mathbb{P}^1_B}$ will be our desired surjection.

It remains to show the restriction map r is surjective with kernel isomorphic to $\mathscr{O}_{\mathbb{P}^1_B}(n-2)$. Let $\mathscr{I}_{M/W}$ denote the ideal sheaf of M in W. This is supported on L and its restriction to L is isomorphic to the invertible sheaf $\mathscr{O}_L(-E_\tau C)$ on L, for $E_\tau C$ the exceptional divisor in $\mathrm{Bl}_\tau C \simeq \mathbb{P}^1_B$. Note this sheaf is invertible as $E_\tau C$ is a Cartier divisor by Lemma 3.1.22. Therefore, tensoring the ideal sheaf exact sequence for M in W with $t^*\mathscr{O}_P(1)$ and pushing forward via s, we obtain an exact sequence

$$0 \longrightarrow s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) \longrightarrow s_*(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)) \longrightarrow s_*(\mathscr{O}_M \otimes t^*\mathscr{O}_P(1)) \longrightarrow$$

$$\longrightarrow R^1 s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) \tag{3.1.9}$$

It suffices to verify $R^1 s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) = 0$ and $s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) \simeq \mathscr{O}_L(n-2)$.

By cohomology and base change to show $R^1 s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) = 0$, it is enough to show the first cohomology of the restriction of $\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)$ to any fiber over $\mathbb{P}^1_B$ vanishes. Indeed, this holds because the sheaf $\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)$ is supported on L and L maps isomorphically to $\mathbb{P}^1_B$ via s. Since the restriction of s to L is finite, the higher derived pushforward vanishes.

To conclude, it only remains to identify $s_*(\mathscr{O}_L(-E_\tau C)\otimes t^*\mathscr{O}_P(1)) \simeq \mathscr{O}_L(n-2) \simeq \mathscr{O}_{\mathbb{P}^1_B}(n-2)$. In fact, since s is an isomorphism on L, it is enough to show $s_*\mathscr{O}_L(-E_\tau C) \simeq \mathscr{O}_{\mathbb{P}^1_B}(-2)$ and $s_*t^*\mathscr{O}_P(1) \simeq \mathscr{O}_{\mathbb{P}^1_B}(n)$. These are both invertible sheaves on $\mathbb{P}^1_B$. Since B is a local scheme, we find that the group of line bundles on $\mathbb{P}^1_B$ is isomorphic to $\mathbb{Z}$, and so to check the above claims, it suffices to check they hold after restriction to the closed point of B. Hence, we can assume the base B is the spectrum of a field k.

To conclude, it is enough to show $E_\tau C$ has degree 2 on L and $t^*\mathscr{O}_P(1)$ restricts to a degree n divisor on L. The former holds by Lemma 3.1.22. We now verify $t^*\mathscr{O}_P(1)$ has degree n on L. The composition $\mathrm{Bl}_\tau C \simeq L \to W \to \mathrm{Bl}_\tau C \times_B P \to P$ is, by construction of W, the same as the composition $\mathrm{Bl}_\tau C \xrightarrow{\nu} C \xrightarrow{\iota} P$. By definition of $\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}$, C is a genus 1 curve embedded with degree n in the n-dimensional projective space P, which means that $\mathscr{O}_P(1)$ restricts to a degree n invertible sheaf on C. Therefore, the further pullback along the map ν, which is generically an isomorphism, still has degree n. □

Fix $n \geq 3 \in \mathbb{Z}$. Using our above construction of the $(n-2)$-Hirzebruch twist, there is a map $\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}} \to \mathscr{V}^{\mathrm{smile},(n)}$, which we now describe.

Construction 3.1.25. Given $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}(B)$, this data is mapped to the point of $\mathscr{V}^{\mathrm{smile},(n)}$ described by $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F)$

for $g : F \to \mathrm{Bl}_\tau C$ as in Definition 3.1.19 and L as in Notation 3.1.10. The map i is given as the composition of the maps $\psi \circ i_1 : \mathrm{Bl}_\tau C \to W$ of Notation 3.1.10 and $i_W : W \to F$ of Remark 3.1.20.

From Proposition 3.1.23, we find $F \xrightarrow{g} \mathrm{Bl}_\tau C \xrightarrow{h} B$ is an $(n-2)$-Hirzebruch twist, and hence to construct the map $\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}} \to \mathscr{V}^{\mathrm{smile},(n)}$, on objects, it suffices to verify that L has class $e + nf$ on F. This and more is established in the following lemma.

Lemma 3.1.26. *With notation as in Notation 3.1.10 and Notation 2.2.38 we have*

1. *The divisor $M \to F$ has class e fppf locally on B*
2. *The divisor $L \to F$ has class $e + nf$ fppf locally on B*
3. *The invertible sheaf $t^*\mathscr{O}_P(1)|_F$ on F fppf locally on B has class $e + (n-2)f$.*

Proof. To begin, we may work fppf locally on B so as to assume that $F \simeq \mathbb{F}_{n-2}$ over B and B is a local scheme. Recall that the directrix class on the Hirzebruch surface $\mathbb{F}_{n-2}$ over $\mathbb{P}^1_B$ corresponds to the surjection $\mathscr{O}_{\mathbb{P}^1_B}(n) \oplus \mathscr{O}_{\mathbb{P}^1_B} \to \mathscr{O}_{\mathbb{P}^1_B}$. By Lemma 3.1.24, this is identified with the restriction of $s_*(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1))$ (the sheaf whose projectivization is F) to M. It follows that M has class e on $\mathbb{F}_{n-2}$.

For the last two parts, the relevant divisor on F is an effective Cartier divisor, and so by Lemma 3.1.1 to compute the classes above, it is enough to do so on the fiber over every point of B. Henceforth, we assume B is the spectrum of an algebraically closed field.

We now identify the class of the relative Cartier divisor L as $e + nf$. Because $e + nf$ is the unique effective class that has intersection 1 with f and intersection 2 with e, it is enough to verify the following two claims: First, $L \cap M$ has degree 2 on L. Second, the intersection of L with a fiber of projection $\rho \circ \psi \circ i_1 : L \to W \to \mathrm{Bl}_\tau C$ has degree 1. The first claim follows from Lemma 3.1.22. The second claim follows from the fact that $\rho \circ \psi \circ i_1$ is an isomorphism by construction.

Finally, to identify $t^*\mathscr{O}_P(1)|_F$, it is enough to show that this line bundle restricts to a degree 0 divisor on M and a degree n divisor on L, since $e + (n-2)f$ is the unique effective class whose intersection with e is 0 and whose intersection with $e + nf$ is n. We can choose a section of $\mathscr{O}_P(1)$ missing τ. Since L is the preimage of τ in F, we obtain $\mathscr{O}_P(1)$ restricts to the trivial line bundle on M. On the other hand, the restriction of $\mathscr{O}_P(1)$ to L is the pullback of $\mathscr{O}_P(1)$ along $L \to C \hookrightarrow P$. We note that $L \to C$ is an isomorphism away from the section $\tau(B) \to C$. Again choosing a section of $\mathscr{O}_P(1)$ whose vanishing locus misses $\tau(B)$, we find the degree of $\mathscr{O}_P(1)$ restricted to L agrees with its degree when restricted to C. By definition of $\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}$, the curve C has degree n, meaning the restriction of $\mathscr{O}_P(1)$ to C has degree n. □

We are finally prepared to complete our construction of the map $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{V}^{\mathrm{smile},(n)}$.

Lemma 3.1.27. *There is a map of stacks $\widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{V}^{smile,(n)}$ sending $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ to the tuple $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F) \in \mathscr{V}^{smile,(n)}$ as defined in Construction 3.1.25.*

Proof. Observe that by Proposition 3.1.23, $F \xrightarrow{g} \mathrm{Bl}_\tau C \xrightarrow{h} B$ is an $(n-2)$-Hirzebruch twist. Further, L has class $e + nf$ on F fppf locally. It follows that $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F)$ indeed corresponds to a point of $\mathscr{V}^{\mathrm{smile},(n)}$. To conclude, it only remains to describe where morphisms of the data are sent. Further, since the construction above is compatible with pullback, it is enough to describe where automorphisms are sent.

By Corollary 2.2.27, any automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}(B)$ is induced by an element $\phi \in \mathrm{PGL}_n(B) \simeq \mathrm{Aut}_{P/B}(B)$ over B such that $\phi(C) = C$ and $\phi(\tau(B)) = \tau(B)$.

We use this to construct an automorphism of the associated data $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F) \in \mathscr{V}^{\mathrm{smile},(n)}$. Via the universal property of blow ups, ϕ induces an isomorphism $\alpha : \mathrm{Bl}_\tau C \to \mathrm{Bl}_\tau C$. Via the constructions of L and M as closed subschemes of $\mathrm{Bl}_\tau C \times_B P$, we claim that the automorphism $(\alpha, \phi) : \mathrm{Bl}_\tau C \times_B P \to \mathrm{Bl}_\tau C \times_B P$ restricts to automorphisms on L and M. Indeed, both map isomorphically to $\mathrm{Bl}_\tau C$ via the first projection, while the image of L in P is C, which is fixed by ϕ, and the image of M in P is $\tau(B)$, which is also fixed by ϕ. Finally, since L and M are preserved by (α, ϕ), it follows that F is also preserved by (α, ϕ) via its construction in terms of L and M. Altogether, this yields a map $\gamma = (\alpha, \phi)|_F : F \to F$ which restricts to $(\alpha, \phi)|_L$ on L and projects to α on $\mathrm{Bl}_\tau C$. This is precisely the data of an automorphism of $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F) \in \mathscr{V}^{\mathrm{smile},(n)}$. □

We have now defined maps $\Gamma : \mathscr{V}^{\mathrm{smile},(n)} \to \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ in Lemma 3.1.6 $\Delta : \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \to \mathscr{V}^{\mathrm{smile},(n)}$ in Lemma 3.1.27 We wish to show these two maps define an equivalence of stacks. The fact that they define an equivalence could be termed as "obvious" once one has internalized the construction. However, we choose to spell out the details for completeness. The verification is somewhat lengthy and is completed in Theorem 3.1.31.

Lemma 3.1.28. *Let $n \geq 3$. The composition of functors $\Gamma \circ \Delta$ sends $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$ to an isomorphic tuple $(B, f' : P' \to B, \iota' : C' \to P', \tau' : B \to C') \in \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)}$.*

Proof. Given $B \to \widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}$, we obtain a corresponding tuple $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$. and we need to show the resulting point of $[\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}/\mathrm{PGL}_n]$ is equivalent to the point obtained after applying $\Gamma \circ \Delta$.

Let us analyze $(\Gamma \circ \Delta)(B, f : P \to B, \iota : C \to P, \tau : B \to C)$. The map Δ produces a tuple $(B, h : \mathrm{Bl}_\tau C \to B, g : F \to \mathrm{Bl}_\tau C, i : L \to F)$ together with a directrix $M \subset F$ with notation as in Notation 3.1.10 and Definition 3.1.19. Also, from Lemma 3.1.26 we find that the linear system $\mathscr{O}_P(1)$ pulls back to $e + (n-2)f$ on $(\mathbb{F}_{n-2})_B$.

Let $\mathscr{L}$ denote the invertible sheaf $\mathscr{O}_F(L - g^{-1}(g(L \cap M))$ on F. Let $P' := \mathbb{P}\left((h \circ g)_*\mathscr{L}\right)$, and let $f' : P' \to B$ denote the structure map. The map $(h \circ g)^*(h \circ g)_*\mathscr{L} \to \mathscr{L}$ induces a map $\phi : F \to P'$. Let C' denote the image of $i(L)$ under ϕ. And denote by $\iota' : C' \to P'$ the resulting closed embedding. The image of $M \to F \xrightarrow{\phi} P'$ is the image of a section of f', as may be verified fppf locally on B. This section factors through C' as it is also the image of $L \cap M$, since $L \cap M$ is flat over B by Proposition 3.1.16. We denote the resulting section $\tau' : B \to C'$. Altogether, this furnishes the data $(B, f' : P' \to B, \iota' : C' \to P', \tau' : B \to C')$ which we want to show is isomorphic to our original tuple $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$.

First, we construct the isomorphism $P \simeq P'$. By construction, the invertible sheaf $\mathscr{L}$ is isomorphic to $\mathscr{O}_F(e + (n-2)f)$ fppf locally on B, and the same is true for $\mathscr{O}_P(1)|_F$ by Lemma 3.1.26. It follows from cohomology and base change that $\mathscr{M} := (h \circ g)_*(\mathscr{L} \otimes \mathscr{O}_P(-1)|_F)$ is an invertible sheaf on B. Therefore, by the projection formula, $(h \circ g)^*\mathscr{M} \otimes \mathscr{O}_P(1)|_F \simeq \mathscr{L}$. Hence, $\mathbb{P}(h \circ g)_*\mathscr{O}_P(1)|_F \simeq \mathbb{P}(h \circ g)_*\mathscr{L} = P'$. We claim both are isomorphic to P. Note that $\mathbb{P}(h \circ g)_*\mathscr{L}$ has relative dimension $n-1$ by Lemma 3.1.3, and so $\mathbb{P}(h \circ g)_*\mathscr{O}_P(1)|_F$ does as well. We know $F \to P$ is induced by a rank n subbundle[1] $\mathscr{G} \subset (h \circ g)_*(\mathscr{O}_P(1)|_F)$ because $F \to P$ is not contained in a hyperplane. Since P and P' have relative dimension $n-1$, we must have $\mathscr{G} = (h \circ g)_*(\mathscr{O}_P(1)|_F)$, implying $\mathbb{P}(h \circ g)_*(\mathscr{O}_P(1)|_F) \simeq P$ and hence we obtain an isomorphism $\alpha : P' \simeq P$.

We now show that the map α extends to an isomorphism $(B, f' : P' \to B, \iota' : C' \to P', \tau' : B \to C') \simeq (B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$. Under the identification α, $\phi : F \to P'$ is identified with the given morphism $F \to P$. Since $\tau(B)$ is the image of $M \to P$, α identifies τ with τ'. Since C' is the image of $L \to P$, α identifies $\iota' : C' \to P'$ with $\iota : C \to P$. This verifies the claimed isomorphism. □

Lemma 3.1.29. *Let $n \geq 3$. The composition of functors $\Delta \circ \Gamma$ sends $(B, h : X \to B, g : F \to X, i : Z \to F)$ to an isomorphic tuple $(B, h' : X' \to B, g' : F' \to X', i' : Z' \to F') \in \mathscr{V}^{smile,(n)}$.*

Proof. Define schemes P, C, and maps $f : P \to B, \iota : C \to P, \tau : B \to C$ so that $\Gamma(B, h : X \to B, g : F \to X, i : Z \to F) = (B, f : P \to B, \iota : C \to P, \tau : B \to C)$. Explicitly, if $\mathscr{L}$ denotes the invertible sheaf on F of Notation 3.1.2, then $P = \mathbb{P}(h \circ g)_*\mathscr{L}$,

[1]By *subbundle* we mean that $((h \circ g)_*(\mathscr{O}_P(1)|_F))\,/\mathscr{G}$ is a locally free sheaf.

which induces a map $F \to P$ so that $\tau(B)$ is the image of the exceptional divisor $E \subset F \to P$ and C is the image of $Z \subset F \to P$.

Applying the map Δ, we obtain a relative surface $F' \to B$ and two embeddings $L \hookrightarrow F', M \hookrightarrow F'$, as in Definition 3.1.19 and Notation 3.1.10. We wish to produce an isomorphism $F \simeq F'$ over B which restricts to isomorphisms $Z \simeq L$ and $E \simeq M$. We will actually do this "in reverse" by first producing the isomorphisms $Z \simeq L$ and $E \simeq M$ and using this to produce an isomorphism $F \simeq F'$.

First, we produce the isomorphism $Z \simeq L$. By construction, L is the blow up of C at $\tau(B) = \operatorname{im} E$. Observe that $E \cap Z$ is an effective Cartier divisor on Z, being the restriction of the Cartier divisor E on F to Z. Further, $E \cap Z$ maps to τ because the map $F \to P$ contracts E. The universal property for blow ups induces a map $Z \to \mathrm{Bl}_\tau C$ taking $E \cap Z$ to the exceptional divisor $E_\tau C \subset \mathrm{Bl}_\tau C$. By the fibral isomorphism criterion, [Gro67, 17.9.5], it is enough to check this map is an isomorphism on fibers. However, by Proposition 3.1.8, the formation of the blow up commutes with base change, and so we may reduce to the case that B is the spectrum of a field, in which case the resulting map $Z \to L$ becomes a birational map of smooth genus 0 curves, which is therefore an isomorphism.

Next, we construct the isomorphism $E \simeq M$. This comes from the isomorphism $Z \simeq L$ by composing the isomorphisms $E \simeq X \simeq Z$ with $Z \simeq L$ with $L \simeq \mathrm{Bl}_\tau C \simeq M$.

Finally, we wish to use the isomorphisms $E \simeq M$ and $Z \simeq L$ to produce a compatible isomorphism $F \simeq F'$. We have a map $\beta : F \to P \times_B X$ where $\mathscr{L}$ induces the map to P and h gives the map to X. We claim this is a closed embedding whose image is F', and this will be the desired isomorphism.

We first check β is a closed embedding. Observe that both Z and E map isomorphically to X via the projection. Additionally, the image of Z in P is identified with C while the image of E in P is identified with $\tau(B)$. It follows that Z and E are identified with the images of L and M respectively under the maps $\psi \circ i_1$ and $\psi \circ i_2$ of Notation 3.1.10. Now, let $s : P \times_B X \to X$ and $t : P \times_B X \to P$ denote the projections. To check β is a closed embedding, it is equivalent to show it is a proper monomorphism [Gro67, 18.12.6]. Properness is clear because the source and target are proper, while we may verify the map is a monomorphism on fibers over X as follows from [Gro67, 17.2.6]. On fibers, over X, this verification is straightforward as each fiber of F over X is embedded as a line in P.

It remains to identify F' with F. To check this, let $W := E \cup Z \subset F$. Above, we have identified $W \subset P \times_B X$ with $L \cup M \subset P \times_B X$. The idea is now to show that both F and F' are identified with "the lines over X containing W." More precisely, using the universal property of maps to projective bundles, in order to show F is identified with F', we claim it is enough to verify that in the composition

$$s_*(t^*\mathscr{O}_P(1)) \xrightarrow{\kappa} s_*(\mathscr{O}_F \otimes t^*\mathscr{O}_P(1)) \xrightarrow{\sigma} s_*(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)), \tag{3.1.10}$$

the latter restriction map σ is an isomorphism.

Indeed, κ is the map defining $F \to P \times_B X$. while, $\sigma \circ \kappa$ is identified with the restriction map defining the closed immersion $F' \to P \times_B X$ by construction of F' coming from Definition 3.1.19. This completes the proof, once we check σ is an isomorphism, which we now do in Lemma 3.1.30. □

Lemma 3.1.30. *The map σ in* (3.1.10) *is an isomorphism.*

Proof. To verify σ is an isomorphism, we may work fppf locally on B. Hence we may assume $F \simeq (\mathbb{F}_{n-2})_B$ and $\mathscr{O}_P(1) \simeq \mathscr{O}_F(e + (n-2)f)$ by Lemma 3.1.26. Let $\mathscr{I}_{W/F}$ denote the ideal sheaf of W in F and note that upon identifying $W \simeq Z \cup E \subset (\mathbb{F}_{n-2})_B$, this ideal sheaf is sent to $\mathscr{O}_{(\mathbb{F}_{n-2})_B}(-2e - nf)$. We obtain an exact sequence

$$0 \longrightarrow \mathscr{O}_F(-2e - nf) \longrightarrow \mathscr{O}_F \longrightarrow \mathscr{O}_W \longrightarrow 0. \tag{3.1.11}$$

Tensoring with $t^*\mathscr{O}_P(1)$ and pushing forward via s, we obtain an exact sequence

$$\begin{aligned} &s_*\left(\mathscr{O}_F(-2e - nf) \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow s_*(\mathscr{O}_F \otimes t^*\mathscr{O}_P(1)) \longrightarrow \\ &\longrightarrow s_*\left(\mathscr{O}_W \otimes t^*\mathscr{O}_P(1)\right) \longrightarrow R^1 s_*\left(\mathscr{O}_F(-2e - nf) \otimes t^*\mathscr{O}_P(1)\right). \end{aligned} \tag{3.1.12}$$

Hence, by cohomology and base change it is enough to show

$$s_*\left(\mathscr{O}_F(-2e - nf) \otimes t^*\mathscr{O}_P(1)\right) = R^1 s_*\left(\mathscr{O}_F(-2e - nf) \otimes t^*\mathscr{O}_P(1)\right) = 0$$

in the case that B is the spectrum of a field. Because $t^*\mathscr{O}_P(1)$ restricts to $\mathscr{O}_F(e + (n-2)f)$ on fibers by Lemma 3.1.26, we wish to show $s_*\left(\mathscr{O}_F(-e - 2f)\right) = R^1 s_*\left(\mathscr{O}_F(-e - 2f)\right) = 0$. This indeed holds using cohomology and base change because $H^0(\mathbb{P}^1_B, \mathscr{O}(-1)) = H^1(\mathbb{P}^1_B, \mathscr{O}(-1)) = 0$. □

Theorem 3.1.31. *For $n \geq 3$, the maps $\Gamma : \mathscr{V}^{smile,(n)} \to \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$ of Lemma 3.1.6 and $\Delta : \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}} \to \mathscr{V}^{smile,(n)}$ of Lemma 3.1.27 define an equivalence of algebraic stacks.*

Proof. The bijection on objects was shown in Lemma 3.1.28 and Lemma 3.1.29. To complete the proof, we need to verify these maps are also isomorphisms on automorphisms of the data.

So, suppose we have $(B, h : X \to B, g : F \to X, i : Z \to F) \in \mathscr{V}^{\text{smile},(n)}$ mapping under Γ to $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$. To conclude there is an equivalence of stacks, it suffices to check the maps on automorphism groups are both injective both under Γ and Δ.

As a first step, we claim that any automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$ which induces a nontrivial automorphism of Z maps to a nontrivial automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$. Conversely, we also claim that any automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ inducing a nontrivial automorphism of C maps to a nontrivial automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$. To check both these claims, note that by construction of Γ there is a map $Z \to C$ which is an isomorphism away from $\tau(B)$. Because $C \to B$ and $Z \to B$ are separated, automorphisms of $C \to B$ and $Z \to B$ are determined by their restrictions to the dense opens $Z - E \cap Z$ and $C - \tau(B)$. Therefore, if an automorphism of one is nontrivial, then the induced automorphism of the other must also be nontrivial.

To conclude the proof, it is enough to show that any nontrivial automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$ restricts to a nontrivial automorphism of Z and any nontrivial automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ restricts to a nontrivial automorphism of C. First, because $C \to P$ is not contained in a hyperplane, a nontrivial automorphism of P cannot restrict to the trivial automorphism of C. Because an automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ is determined by an automorphism of P preserving C, a nontrivial automorphism of $(B, f : P \to B, \iota : C \to P, \tau : B \to C)$ must restrict to a nontrivial automorphism of C.

Finally, we wish to show a nontrivial automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$ cannot restrict to the identity on Z. An automorphism of $(B, h : X \to B, g : F \to X, i : Z \to F)$ is determined by an automorphism of F, and so it is enough to show that if an automorphism $\alpha : F \to F$ restricts to the identity on Z, then it is the identity. If α restricts to the identity on Z, it must also induce the identity automorphism of X. Therefore, α induces the identity automorphism of P (as argued above because it induces the identity on C) and of X. Because the map sending $F \to P \times_B X$ is a closed embedding, α must also induce the identity automorphism of F. □

3.2 The genus 1 curve associated to a degree 2 cover

Our main goal of this section is to prove Theorem 3.2.8, which associates to a degree 2 cover a certain relative dimension 1 group scheme, and describes n coverings of that group scheme in terms of maps to $\mathscr{V}^{\text{smile},(n)}$.

Given a finite degree 2 locally free cover $g : X \to B$, we now construct an associated

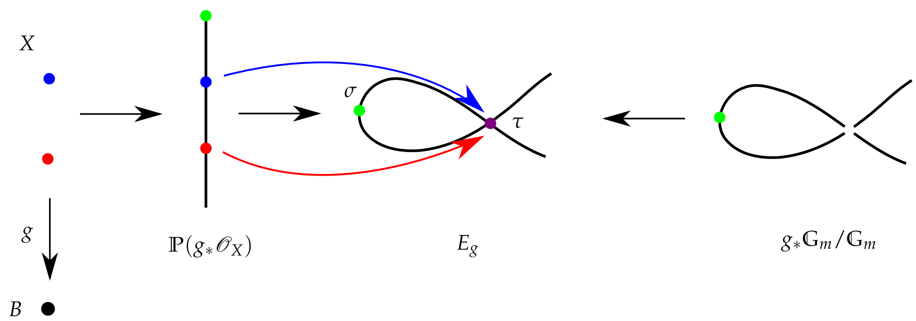


Figure 3.2: A visualization of the singular genus 1 curve associated to a degree 2 cover, as defined in Notation 3.2.1.

genus 1 curve $E_g \to B$ with $E_g^{\mathrm{sm}} \simeq g_*\mathbb{G}_m/\mathbb{G}_m$. See Figure 3.2 for a visualization of E_g.

Notation 3.2.1. Let $g : X \to B$ be a finite locally free degree 2 cover. The surjection $g^*g_*\mathscr{O}_X \to \mathscr{O}_X$ induces a map $\iota : X \to \mathbb{P}(g_*\mathscr{O}_X)$ over B. We have a structure map $\pi : \mathbb{P}(g_*\mathscr{O}_X) \to B$. Additionally, the injective map of sheaves $\mathscr{O}_B \to g_*\mathscr{O}_X$ has cokernel which is an invertible sheaf, as is verified in Lemma 3.2.2. Therefore, the injection $\mathscr{O}_B \to g_*\mathscr{O}_X$ induces a map $\sigma : B \to \mathbb{P}(g_*\mathscr{O}_X)$.

We define the genus 1 curve E_g associated to $g : X \to B$ as the cofiber product

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathbb{P}(g_*\mathscr{O}_X) \\ \downarrow{\scriptstyle g} & & \downarrow \\ B & \xrightarrow{\tau} & E_g \end{array} \tag{3.2.1}$$

Note that E_g exists as a scheme by [Sta, Tag 0E25], whose hypotheses are easily seen to be satisfied.

To make sense of the map σ in Notation 3.2.1, we used the following.

Lemma 3.2.2. *Let $g : X \to B$ be a finite locally free degree 2 cover. Then the injective map of sheaves $\mathscr{O}_B \to g_*\mathscr{O}_X$ has cokernel which is a rank 1 locally free sheaf.*

Proof. We may check this in the case B is an affine scheme, say $B = \operatorname{Spec} R$. Then, we can express the global sections of $g_*\mathscr{O}_X$ as a rank 2 free R-module of the form $R[x]/(x^2 + ax + b)$ for $a, b \in R$. The cokernel of $\mathscr{O}_B \to g_*\mathscr{O}_X$ is then identified with the cokernel $R \to R[x]/(x^2 + ax + b)$, which is a free R module generated by the image of x. Hence, it is a free R module of rank 1. □

Our upcoming goal is to show that E_g, together with σ and τ, defines a point of $\widetilde{\mathscr{W}}_{\text{sing}}$, which we accomplish in Lemma 3.2.5. As a first step, we will need to understand the interaction between the maps ι and σ, so that we will be able to work affine locally away from each of them.

Lemma 3.2.3. *With notation as in Notation 3.2.1, The images $\iota(X)$ and $\sigma(B)$ are disjoint closed subschemes of $\mathbb{P}(g_*\mathscr{O}_X)$.*

Proof. To check $\iota(X)$ is disjoint from $\sigma(B)$, it is equivalent to check the preimage of $\sigma(B)$ in $X \times_B \mathbb{P}(g_*\mathscr{O}_X) = \mathbb{P}(g^*g_*\mathscr{O}_X)$ is disjoint from the section $\iota_X : X \to \mathbb{P}(g^*g_*\mathscr{O}_X)$ induced by the surjection $g^*g_*\mathscr{O}_X \to \mathscr{O}_X$. The preimage of $\sigma(B)$ then corresponds to the exact sequence of vector bundles $g^*\mathscr{O}_B \to g^*g_*\mathscr{O}_X \to g^*g_*\mathscr{O}_X/g_*\mathscr{O}_B$. To show the preimage of $\sigma(B)$ is disjoint from ι_X, it is equivalent to check they are disjoint on each fiber over X, and hence equivalent to check that the composition

$$g^*\mathscr{O}_B \to g^*g_*\mathscr{O}_X \to \mathscr{O}_X \tag{3.2.2}$$

is nonzero on every fiber. Note the first inclusion $g^*\mathscr{O}_B \to g^*g_*\mathscr{O}_X$ is that induced by ι_X and the second surjection $g^*g_*\mathscr{O}_X \to \mathscr{O}_X$ is that corresponding to $\sigma(B)$. The sequence (3.2.2) is adjoint to the sequence $\mathscr{O}_B \to g_*\mathscr{O}_X \xrightarrow{\text{id}} g_*\mathscr{O}_X$, and so to check the composition in (3.2.2) is nonzero on fibers, it is equivalent to check $\mathscr{O}_B \to g_*\mathscr{O}_X \xrightarrow{\text{id}} g_*\mathscr{O}_X$ map is. However, $\mathscr{O}_B \to g_*\mathscr{O}_X \xrightarrow{\text{id}} g_*\mathscr{O}_X$ corresponds the map of structure sheaves associated to $X \to B$, and so it indeed is injective on every fiber. □

We next verify some basic properties of the curve E_g.

Lemma 3.2.4. *For $g : X \to B$ a finite locally free degree 2 cover, E_g is a proper flat which is a finitely presented genus 1 curve over B.*

Proof. By the explicit construction of E_g as a cofiber product, we find $\mathbb{P}(g_*\mathscr{O}_X) \to E_g$ is surjective. Since $\mathbb{P}(g_*\mathscr{O}_X) \to B$ is proper, it follows $E_g \to B$ is proper as well. It follows that $E_g \to B$ is of finite type, being proper.

To conclude, we need only check $E_g \to B$ is in fact locally of finite presentation and flat. Note that once we know it is locally of finite presentation, we can conclude it is also quasi-compact and quasi-separated because it is proper, and so it is of finite presentation. For both of these properties, we may assume B is affine, say $B = \operatorname{Spec} S$. Then X is also affine, so we may assume $X = \operatorname{Spec} R$. In order to facilitate the verification that $E_g \to B$ is locally of finite presentation, we first reduce to the case that B has finite type over $\operatorname{Spec} \mathbb{Z}$. Indeed, we may write B as a limit of finite type affine schemes, and by spreading out for morphisms of finite presentation [Sta, Tag 01ZM], it follows that the cofiber product (3.2.1) is the base change of an analogous diagram where $X, B, \mathbb{P}(g_* \mathscr{O}_X)$ are all finite type over $\operatorname{Spec} \mathbb{Z}$. Therefore, we further assume that R and S are finite type over $\operatorname{Spec} \mathbb{Z}$, and in particular noetherian.

First, we check $E_g \to B$ is quasi-compact and quasi-separated. From Lemma 3.2.3, there is a section $\sigma : B \to \mathbb{P}(g_* \mathscr{O}_X)$ missing X. The complement of this section is Zariski-locally on B isomorphic to $\mathbb{A}^1_B$. We therefore shrink B if necessary so as to assume the complement of σ is $\mathbb{A}^1_B$. The complement of X in $\mathbb{P}(g_* \mathscr{O}_X)$ is also affine, as is the intersection $\mathbb{A}^1_B \cap (\mathbb{P}(g_* \mathscr{O}_X) - \iota(X)) = \mathbb{P}(g_* \mathscr{O}_X) - X - \sigma(B)$. We then obtain that the cofiber product E_g is covered $\mathbb{P}(g_* \mathscr{O}_X) - X$ and the cofiber product E'_g defined by

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & \mathbb{A}^1_B \\ \downarrow & & \downarrow \\ \operatorname{Spec} S & \longrightarrow & E'_g. \end{array} \tag{3.2.3}$$

We have seen above $\mathbb{P}(g_* \mathscr{O}_X) - X$ is affine, and E'_g is also affine by construction as a fiber product of rings. Therefore E_g is covered by two affine schemes with affine intersection, implying it is quasi-compact and quasi-separated.

We next show, assuming $B = \operatorname{Spec} R$ is affine, that $E_g \to B$ has finite presentation. It is enough to show E'_g has finite presentation. Now, E'_g is the spectrum of the fiber product of rings of the form $S[x] \times_R S$. We claim we may Zariski localize S further to assume R is a free S module of rank 2 with an inclusion of rings $S \to R$ and $R = S[x]/(f)$ for $f \in S[x]$ a degree 2 polynomial with invertible leading coefficient on S. Indeed, this is possible due to flatness of R over S, which implies the leading coefficient is invertible as each fiber of $\operatorname{Spec} R$ over $\operatorname{Spec} S$ has dimension 0. We may then express $R = S \oplus S \cdot x$, so R is a free S module of rank 2. After multiplying f by a unit, we may assume $f = x^2 + ax + b$ for $a, b \in S$. Therefore, in order to show E_g is finitely presented, it is enough to show $S[x] \times_R S$ is a free finitely presented S-module.

We now verify $S[x] \times_R S$ is a free finitely presented S-module. This fiber product of rings can equivalently be described as the subring of $S[x]$ consisting of those elements whose reduction mod f has vanishing x coefficient. Note S is noetherian since we have reduced to the case S is a finitely generated $\mathbb{Z}$ algebra. It follows that the noetherian ring $S[x]$ contains $S[x] \times_R S$ as a subring, and $S[x]$ is finitely generated over $S[x] \times_R S$. Namely, 1 and x together generate $S[x]$ as an $S[x] \times_R S$ module. Since $S[x]$ is noetherian due to our initial reductions, it follows $S[x] \times_R S$ is also noetherian, hence of finite presentation over S.

It remains to show $S[x] \times_R S$ is flat over S. It is enough to give a free S basis for $S[x] \times_R S$.

We now construct this desired basis. For each $n \geq 2$, note that x^n can be written uniquely as $g + c_n x + d_n$ where $g \in (f)$ and $c_n, d_n \in S$, essentially via the Euclidean algorithm. We will show that a free basis for $S[x] \times_R S$ is then given by $S \cdot 1$ together with $x^n - c_n x$ for each $n \geq 2$. Observe that these elements of $S[x]$ indeed define elements of $S[x] \times_R S$, viewed as a subring of $S[x]$, since the reduction to $S[x]/(f)$ has vanishing x coefficient. The resulting module is also free as an S-submodule of $S[x]$. To conclude, we only need to show these elements generate all of $S[x] \times_R S \subset S[x]$.

We now check 1 together with $x^n - c_n x$ for $n \geq 2$ generate $S[x] \times_R S \subset S[x]$. Begin with any $g \in S[x] \times_R S$, which we may express as $g = \sum_{i=0}^m a_m x^i$ with the condition that $\overline{g} \in R = S[x]/(f)$ lies in the image of $S \to R$. We wish to show g can be written as a linear combination of 1 and the elements $x^n - c_n x$ for $n \geq 2$. The idea to do so will be to subtract S multiples of the elements $x^n - c_n x$ from g and show we eventually obtain an element of S. We now carry this strategy out. By subtracting $a_m(x^m - c_m x)$ from g we obtain another polynomial of lower degree also lying in $S[x] \times_R S$, and it is enough to show $g - a_m(x^m - c_m x)$ lies in the submodule of $S[x]$ generated by 1 and the elements $x^n - c_n x$ for $n \geq 2$. By repeating this procedure finitely many times, we eventually may assume g has degree 2, and by repeating it one more time, we may assume g has degree 0. Then, g lies in the span of 1, as desired. □

Combining the above, it is fairly straightforward to check E_g together with σ and τ defines a point of $\widetilde{\mathscr{W}}_{\text{sing}}$.

Lemma 3.2.5. *The curve E_g together with the sections $\sigma : B \to E_g$ and $\tau : B \to E_g$ define a point $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \widetilde{\mathscr{W}}_{\text{sing}}(B)$. Further, if $X \to B$ is étale, $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \widetilde{\mathscr{W}}_{\text{node}}(B)$ and so $(B, E_g \to B, \sigma : B \to E_g) \in \mathscr{W}_{\text{node}}(B)$.*

Proof. We first check $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \widetilde{\mathscr{W}}_{\text{sing}}(B)$. Using Lemma 3.2.4, it only remains to show that $\sigma(B)$ lies in the smooth locus of $E_g \to B$, $\tau(B)$ lies in the singular locus, and the fibers are geometrically integral. By Lemma 3.2.3, we find $\sigma(B)$ lies in the smooth locus of $E_g \to B$. Geometric integrality follows from the fact that the cofiber product is compatible with base change on B by construction, and so the geometric fibers are integral as they are obtained by collapsing a degree 2 subscheme of $\mathbb{P}^1$ to a point. Finally, we check that $\tau(B)$ lies in the singular locus of $E_g \to B$. This may be verified after base change to geometric fibers, and so we may assume B is the spectrum of a field. In this case we see that $\mathbb{P}(g_* \mathscr{O}_X) \to E_g$ is a birational map from a smooth curve such that the fiber over B is a degree 2 scheme. This implies that E_g is not normal at $\tau(B)$ and hence is not smooth at $\tau(B)$. Altogether, we obtain $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \widetilde{\mathscr{W}}_{\text{sing}}(B)$.

To conclude, we check that when $g : X \to B$ is étale, the point $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \widetilde{\mathscr{W}}_{\mathrm{sing}}(B)$ factors through the open substack $\widetilde{\mathscr{W}}_{\mathrm{node}} \subset \widetilde{\mathscr{W}}_{\mathrm{sing}}$. It is enough to verify this for each geometric fiber of $E_g \to B$, and so we may assume that B is the spectrum of an algebraically closed field k. In this case, $X \to B$ being étale implies that X consists of two copies of $\operatorname{Spec} k$, implying that the normalization of E_g has two points over $\tau(B)$, and hence cannot be a cusp, and must be a node. This implies the above point of $\widetilde{\mathscr{W}}_{\mathrm{sing}}(B)$ factors through $\widetilde{\mathscr{W}}_{\mathrm{node}}(B)$. □

Having identified nice properties of the curve E_g, we next focus on identifying its smooth locus. We wish to show $E_g^{\mathrm{sm}} \simeq g_*\mathbb{G}_m/\mathbb{G}_m$. To do so, we will need the following description of $g_*\mathbb{G}_m$. Recall that whenever $g : X \to B$ is a finite locally free morphism of schemes, there is a norm map $g_*\mathscr{O}_X \to \mathscr{O}_B$ whose formation commutes with arbitrary base change. [Sta, Tag 0BD2]. This can equivalently be described as a map $\mathrm{Nm}_{X/B} : g_*\mathbb{A}^1 \to \mathbb{A}^1$.

Lemma 3.2.6. *Let $g : X \to B$ denote a finite locally free morphism of schemes. Let $t \in \mathbb{A}^1_B$ denote the section of $\mathscr{O}_B[s]$ corresponding to $s = 0$. Then $g_*\mathbb{G}_m$ is identified with the open subscheme $D(\mathrm{Nm}^{-1}_{X/B}(t)) \subset g_*\mathbb{A}^1$.*

Proof. The norm map $\mathrm{Nm}_{X/B} : g_*\mathbb{A}^1 \to \mathbb{A}^1$ is identified with the norm map $\mathrm{Nm}_{X/B} : g_*\mathscr{O}_X \to \mathscr{O}_B$. We wish to show this restricts to a map $g_*\mathscr{O}_X^\times \to \mathscr{O}_B^\times$. We can verify this locally, and therefore assume that B and X are affine. To check this, note that the norm map sends an element α to the determinant of multiplication by α on $g_*\mathscr{O}_X$. Therefore, we obtain the desired restriction because α is invertible if and only if $\times\alpha$ acts invertibly on $g_*\mathscr{O}_X$, which is further equivalent to the determinant of $\times\alpha$ being invertible. □

Lemma 3.2.7. *We have $E_g^{\mathrm{sm}} \simeq \mathbb{P}(g_*\mathscr{O}_X) - \iota(X)$. Further, $g_*\mathbb{G}_m/\mathbb{G}_m \simeq E_g^{\mathrm{sm}}$.*

Proof. First, note that $\mathbb{P}(g_*\mathscr{O}_X) - \iota(X)$ is mapped via an open immersion to E_g under the map $\mathbb{P}(g_*\mathscr{O}_X) \to E_g$, and $\mathbb{P}(g_*\mathscr{O}_X) - \iota(X)$ is smooth over B. Therefore, we have that $\mathbb{P}(g_*\mathscr{O}_X) - \iota(X)$ is contained in the smooth locus of E_g. Hence, because $\mathbb{P}(g_*\mathscr{O}_X)$ surjects onto E_g, it suffices to show the image of $\iota(X)$ in E_g is completely contained in the singular locus. This was verified in Lemma 3.2.5 because $\iota(X)$ maps to $\tau(B)$ in E_g.

To verify $g_*\mathbb{G}_m/\mathbb{G}_m \simeq E_g^{\mathrm{sm}}$. we will construct this isomorphism affine locally in a fashion compatible with localization so that it glues to give a global isomorphism. In fact, we will construct $g_*\mathbb{G}_m/\mathbb{G}_m$ as the complement of a relative degree 2 subscheme inside a projective bundle over B, and then identify that degree 2 subscheme with $\iota(X)$ in $\mathbb{P}(g_*\mathscr{O}_X)$. Therefore, we assume $B = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$. After possibly localizing further, we may assume R is a free S module of rank 2, generated by 1 and x, and we may hence assume $R = S[x]/(x^2 + ax + b)$. Then, $g_*\mathbb{G}_m$ can be explicitly identified with the open subscheme of $\mathbb{A}^2_B \simeq g_*\mathbb{A}^1_X$ given by those $sx+t \in R$ so that $\mathrm{Nm}_{X/B}(sx+t) \neq 0$, as follows from Lemma 3.2.6. Here we use $\mathrm{Nm}_{X/B}$ to denote the map $g_*\mathscr{O}_X \to \mathscr{O}_B$. Computing this directly yields that

$$\mathrm{Nm}_{X/B}(sx+t) = \det \begin{pmatrix} t & -sb \\ s & -sa+t \end{pmatrix} = t^2 - ast + bs^2.$$

Therefore, $\mathrm{Nm}_{X/B}(sx-t) = t^2 + ast + bs^2$ and so $g_*\mathbb{G}_m$ is the complement of $t^2+ast+bs^2$ in $\mathbb{A}^2_{s,t}$. Hence, when we projectivize $\mathbb{A}^2_{s,t}$, we find $g_*\mathbb{G}_m/\mathbb{G}_m$ is identified with the complement of $V(t^2+ast+bs^2)$ in $\mathbb{P}^1_{s,t} \simeq \mathbb{P}(g_*\mathscr{O}_X)$. We claim that the closed subscheme $V(t^2+ast+bs^2) \subset \mathbb{P}(g_*\mathscr{O}_X)$ is precisely identified with the image $\iota(X)$. This holds because we have chosen basis $1, x$ to trivialize $g_*\mathscr{O}_X$ and under this basis, the image of the closed embedding $\operatorname{Spec} R \to \mathbb{P}(g_*\mathscr{O}_X)$ is identified with the vanishing of the closed subscheme $V(x^2 + a\cdot 1\cdot x + b\cdot 1^2)$. Upon renaming x as t and 1 as s, we obtain the claimed isomorphism.

Hence, thus far, we have identified $g_*\mathbb{G}_m/\mathbb{G}_m$ with $\mathbb{P}(g_*\mathscr{O}_X) - \iota_X = E_g^{\mathrm{sm}}$ over sufficiently small Zariski opens on the target B. Because the above construction is compatible with Zariski localization on the target B, these isomorphism glue and yield an isomorphism $g_*\mathbb{G}_m/\mathbb{G}_m \to \mathbb{P}(g_*\mathscr{O}_X) - \iota(X) = E_g^{\mathrm{sm}}$. □

Combining the above discussion in this section with Proposition 2.3.14, Lemma 2.2.30, and Theorem 3.1.31 gives the following characterization of n-coverings of $g_*\mathbb{G}_m/\mathbb{G}_m$.

Theorem 3.2.8. *Let B be an integral normal scheme and let $n \geq 3$. Fix a degree 2 locally free cover $g : X \to B$ which is generically étale. The composite of the bijection of Proposition 2.3.14 and Theorem 3.1.31 yields a bijection between elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$ and maps $B \to \mathscr{V}^{smile,(n)}$ which map to points $(B, E_g \to B, \sigma : B \to E_g) \in \mathscr{W}_{\mathrm{sing}}(B)$ that generically factor through $\mathscr{W}_{\mathrm{node}}$. Further, this bijection respects automorphisms in the sense that it identifies automorphisms of the objects appearing in Proposition 2.3.14 with the B points of the isotropy group of the corresponding map $B \to \mathscr{V}^{smile,(n)}$.*

Proof. Using Theorem 3.1.31, we have an equivalence $\mathscr{V}^{\mathrm{smile},(n)} \simeq \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$. Composing this with the projection $\widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}} \to \widetilde{\mathscr{W}}_{\mathrm{sing}} \to \mathscr{W}$ yields a bijection between maps $B \to \mathscr{V}^{\mathrm{smile},(n)}$ over a given map $B \to \mathscr{W}$ corresponding to a tuple $(B, E_g, \sigma : B \to E_g) \in \mathscr{W}_{\mathrm{sing}}(B)$ and maps $B \to \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$ over that same map $B \to \mathscr{W}$.

When B is integral, by Lemma 3.2.5, we obtain that that the generic point η of B under the map $B \to \mathscr{W}$ factors through $\widetilde{\mathscr{W}}_{\mathrm{node}} \to \widetilde{\mathscr{W}}_{\mathrm{sing}} \to \mathscr{W}_{\mathrm{sing}} \to \mathscr{W}$. This means the generic fiber of the genus 1 curve corresponding to the map $B \to \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$ is nodal, and so Lemma 2.2.30

implies that all such lifts $B \to \widetilde{\mathscr{M}}^{(n)}_{1,\text{sing}}$ are lifts of maps $B \to \mathscr{M}_1^{(n)}$, compatibly with the projection to $\mathscr{W}$.

By Lemma 3.2.7, the smooth locus of E_g is isomorphic to $g_*\mathbb{G}_m/\mathbb{G}_m$, and so it follows from the equivalence of Proposition 2.3.14(1) and Proposition 2.3.14(2) that maps $B \to \mathscr{M}_1^{(n)}$ mapping to $(B, E_g \to B, \sigma : B \to E_g) \in \mathscr{W}(B)$ are in bijection with elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$.

The final statement regarding automorphisms follows from observing that each of the steps of the above bijection also preserve automorphism data, especially using that Theorem 3.1.31 is an equivalence of stacks. □

3.3 The unit resultant condition

In this section, we prove Theorem 3.3.7, which gives a bijection between pairs $(q, \xi) \in V_n$ of unit resultant and points of $\mathscr{V}^{\text{smile},(n)}$.

Notation 3.3.1. We work over a base scheme B. Keep notation as in § 1.4.1 except let $\mathbb{F}_{n-2}$ denote the Hirzebruch surface over B. For $\mathbb{F}_{n-2} \xrightarrow{g} \mathbb{P}^1_B \xrightarrow{h} B$ the structure maps, recall that $g_*(\mathscr{O}_{\mathbb{F}_{n-2}}(e)) \simeq \mathscr{O}_{\mathbb{P}^1_B}(-n+2) \oplus \mathscr{O}_{\mathbb{P}^1_B}$, as e corresponds to the directrix of the Hirzebruch surface. Using the above fact and the projection formula, we have

$$\begin{aligned} g_*(\mathscr{O}_{\mathbb{F}_{n-2}}(e+nf)) &\simeq g_*(\mathscr{O}_{\mathbb{F}_{n-2}}(e)) \otimes \mathscr{O}_{\mathbb{P}^1_B}(n) \\ &\simeq \left(\mathscr{O}_{\mathbb{P}^1_B} \oplus \mathscr{O}_{\mathbb{P}^1}(-n+2)\right) \otimes \mathscr{O}_{\mathbb{P}^1_B}(n) \\ &\simeq \mathscr{O}_{\mathbb{P}^1_B}(2) \oplus \mathscr{O}_{\mathbb{P}^1_B}(n). \end{aligned} \tag{3.3.1}$$

Under the above isomorphism, the section $s \in H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e+nf))$ can be equivalently described as a pair (q, ξ) for $q \in H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(2))$ and $\xi \in H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(n))$. Given (q, ξ) corresponding to $s \in H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e+nf))$, its vanishing locus defines a subscheme $Z \subset \mathbb{F}_{n-2}$. The complete linear system $\mathscr{O}_{\mathbb{F}_{n-2}}(e+(n-2)f)$ determines a map $\mathbb{F}_{n-2} \to \mathbb{P}^{n-1}_B$ by Lemma 3.1.3, and hence a map $\iota : Z \to \mathbb{P}^{n-1}_B$.

We first describe a basis for the linear system defining the map $\iota : Z \to \mathbb{P}^{n-1}_B$.

Lemma 3.3.2. *Keep notation from Notation 3.3.1 and assume Z is smooth over B, so $Z \simeq \mathbb{P}^1_B$. For $n \geq 3$, the complete linear system $\mathscr{O}_{\mathbb{F}_{n-2}}(e+(n-2)f)$ associated to $\mathbb{F}_{n-2} \to \mathbb{P}^{n-1}_B$ restricts to $\mathscr{O}_{\mathbb{P}^1_B}(n)$ on Z for a suitable choice of isomorphism $Z \simeq \mathbb{P}^1_B$. There is a basis of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e+(n-2)f))$ which restricts to $(\xi, qx^{n-2}, qx^{n-3}y, \ldots, qy^{n-2})$, where x, y form a basis of $H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^n_B}(1))$.*

Proof. First, to see $\mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ restricts to $\mathscr{O}_{\mathbb{P}^1_B}(n)$, note that $e \cdot Z = 2$ on fibers and $f \cdot Z = 1$ on fibers so $e + (n-2)f$ restricts to a divisor on $\mathbb{P}^1_B$ of relative degree $2 + (n-2) \cdot 1 = n$.

Our remaining task is to verify that we can find a basis for $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ which restricts to $(\xi, qx^{n-2}, qx^{n-3}y, \dots, qy^{n-2})$ on Z. Indeed, we now construct this basis explicitly. In order to do so, we need to look back at the definition of (q, ξ). Under the construction of $\xi \in H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(n))$ using (3.3.1), we can view ξ as the restriction of some codirectrix $s_0 \in H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f))$ whose vanishing locus does not meet e. Then, $V(\xi) \subset Z$ is identified with $V(s_0)|_Z$. Note that ξ is uniquely determined from this property, up to a unit $u \in \mathbb{G}_m(B)$. Therefore, the section s_0 restricts to ξ on Z.

We know the complete linear system $\mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ has rank n by Lemma 3.1.3. It remains to construct $n-1$ further independent sections of $\mathscr{O}_{\mathbb{F}_{n-2}}(e + (n-2)f)$ restricting to $qx^{n-2}, qx^{n-3}y, \dots, qy^{n-2}$. For these, reasoning similarly to the above, we note that under the identification $Z \simeq \mathbb{P}^1_B$, the directrix E is the vanishing locus of a nonzero section $s_1 \in H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e))$ that restricts to the section $q \in H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(2))$. This is again determined by its restriction up to an element of $\mathbb{G}_m(B)$. Additionally, choosing x and y as a basis of $H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(1))$, the fiber of g over the point $V(x)$ is a section of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(f))$ restricting to x on Z and the fiber of g over $V(y)$ is a section restricting to y on Z. Therefore, $s_1(g^*x)^{n-2}, s_1(g^*x)^{n-3}(g^*y), \dots, s_1(g^*y)^{n-2}$ span a rank $n-1$ subspace of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + nf))$ which restricts to the sections $qx^{n-2}, qx^{n-3}y, \dots, qy^{n-2}$ on Z. This subspace of $H^0(\mathbb{F}_{n-2}, \mathscr{O}_{\mathbb{F}_{n-2}}(e + nf))$ is independent from the subspace generated by ξ because any member of the above subspace defines a singular section, containing a fiber of g, while ξ is smooth. □

We next connect smoothness of Z to the condition that the resultant is a unit.

Lemma 3.3.3. *Keeping notation as in Notation 3.3.1, a subscheme $W \subset \mathbb{F}_{n-2}$ determined by a section $(q, \xi) \in \left(\mathscr{O}_{\mathbb{P}^1_B}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(n)\right)$ not vanishing on any fibers of $h \circ g$ is smooth if and only the resultant* $\operatorname{Res}(q, \xi)$ *lies in* $\mathbb{G}_m(B)$.

Proof. Because (q, ξ) comes from a fixed linear system and is nowhere zero, its vanishing locus is flat and of finite presentation, so it suffices to show W is smooth over every point p of B if and only if $\operatorname{Res}(q, \xi) \in \mathbb{G}_m(B)$. Let $\overline{q}$ and $\overline{\xi}$ denote the restrictions of q and ξ in the residue field $\kappa(p)$. Because $\operatorname{Res}(\overline{q}, \overline{\xi})$ agrees with the image of $\operatorname{Res}(q, \xi)$ in $\kappa(p)$, it is equivalent to show that W_p is smooth over $\operatorname{Spec} \kappa(p)$ if and only if $\operatorname{Res}(\overline{q}, \overline{\xi}) \neq 0 \in \kappa(p)$.

We next show W_p is singular if and only if it contains a fiber over $\mathbb{P}^1_p$. Note that W_p has class $e + (n-2)f$, and, since we are assuming $(q, \xi) \neq (0, 0)$, the map $g|_{W_p} : W_p \to \mathbb{P}^1_p$ is generically an isomorphism. Further, it is an isomorphism if W_p contains no fibers of g.

Conversely, if W_p does contain a fiber of g, then W_p is singular at any point of intersection of that fiber with another component of W_p. Hence we have shown W_p is singular if and only if it contains a fiber.

To conclude, we show W contains a fiber over $\mathbb{P}^1_p$ if and only if $\operatorname{Res}(\overline{q}, \overline{\xi}) = 0 \in \kappa(p)$. Observe that W contains a fiber over a point of $\mathbb{P}^1_{\kappa(p)}$ if and only both q and ξ vanish over some point in $\mathbb{P}^1_{\kappa(p)}$. This implies $\operatorname{Res}(\overline{q}, \overline{\xi}) = 0 \in \kappa(p)$ as $\overline{q}$ and $\overline{\xi}$ share a common factor corresponding to that fiber. Conversely, if $\operatorname{Res}(\overline{q}, \overline{\xi}) = 0 \in \kappa(p)$ then W contains a fiber at the common root of $\overline{q}$ and $\overline{\xi}$. Altogether, this shows $\operatorname{Res}(\overline{q}, \overline{\xi}) \neq 0 \in \kappa(p)$ if and only if W does not contain a fiber of g if and only if W is smooth. □

Motivated by Lemma 3.3.3, we now define the subscheme of V_n corresponding to the locus where the resultant is a unit.

Definition 3.3.4. The affine space V_n has a resultant map over B to $\mathbb{A}^1_B$ which sends $(q, \xi) \mapsto \operatorname{Res}(q, \xi)$. Let $V_n^{\operatorname{Res}\in\mathbb{G}_m} \subset V_n$ denote the preimage of $\mathbb{G}_m \subset \mathbb{A}^1_B$ under the resultant map.

Concretely, $V_n^{\operatorname{Res}\in\mathbb{G}_m}(B)$ consists of those elements $(q, \xi) \in V_n(B)$ so that $\operatorname{Res}(q, \xi) \in \mathbb{G}_m(B)$.

We are nearly ready to prove our main result, but first we state two preparatory lemmas, which relate various quotient stacks.

Lemma 3.3.5. *Keep notation as in Notation 3.3.1. There is an injective map* $V_n^{\operatorname{Res}\in\mathbb{G}_m}(B)/G_n(B) \hookrightarrow [V_n^{\operatorname{Res}\in\mathbb{G}_m}/G_n](B)$ *which is a bijection if* $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.

Proof. We have a sequence of pointed sets

$$0 \longrightarrow H^0(B, G_n) \longrightarrow H^0(B, V_n^{\operatorname{Res}\in\mathbb{G}_m}) \longrightarrow H^0(B, [V_n^{\operatorname{Res}\in\mathbb{G}_m}/G_n]) \longrightarrow H^1(B, G_n). \tag{3.3.2}$$

This implies the injectivity claim.

We wish to show $H^1(B, G_n) = 0$. This will conclude our proof because then any element of $H^0(B, [V_n^{\operatorname{Res}\in\mathbb{G}_m}/G_n])$ maps to the trivial G_n torsor under the boundary map, and therefore comes from an element of $H^0(B, V_n^{\operatorname{Res}\in\mathbb{G}_m})$.

We now conclude by showing $H^1(B, G_n) = 0$. By Lemma 2.1.4, G_n can be expressed as an iterated extension of PGL_2 by copies of $\mathbb{G}_m$ and $\mathbb{G}_a$. Since we are also assuming $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) H^1(B, \mathbb{G}_a) = 0$ we obtain the desired vanishing of $H^1(B, G_n)$. □

Lemma 3.3.6. *For* $n \geq 3$, *we have equivalences of stacks*

$$[V_n^{\operatorname{Res}\in\mathbb{G}_m}/G_n] \simeq [\mathcal{V}^{smile,(n)}/\operatorname{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}] \simeq \mathscr{V}^{smile,(n)} \simeq \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}} \simeq [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$$

Proof. We have $\mathcal{V}^{\mathrm{smile},(n)} \simeq [V_n^{\mathrm{Res}\in\mathbb{G}_m}/\mathbb{G}_m]$ using Lemma 3.3.3. We also have an isomorphism and $\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}} \simeq A_n \simeq [G_n/\mathbb{G}_m]$, using Proposition 2.1.7. Together, these yield the equivalence $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n] \simeq [\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}]$.

The remaining identifications follow from Lemma 2.2.42, Theorem 3.1.31, and Corollary 2.2.27. which respectively yield the isomorphisms $[\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/\mathbb{Z}}] \simeq \mathscr{V}^{\mathrm{smile},(n)} \simeq \widetilde{\mathscr{M}}_{1,\mathrm{sing}}^{(n)} \simeq [\widetilde{\mathscr{H}}_{\mathrm{sing}}^{(n)}/\mathrm{PGL}_n]$. □

Our above results now combine to give descriptions of elements (q, ξ) with unit resultant as points of various stacks. Given a quadratic form $q \in H^0(\mathbb{P}^1_B, \mathscr{O}_{\mathbb{P}^1_B}(2))$, we use $V(q)$ to denote the subscheme of $\mathbb{P}^1_B$ defined by q. Recall that given a locally free degree 2 cover $X \to B$, we use E_g with section σ to denote the corresponding genus 1 curve as in Notation 3.2.1.

Additionally, note there is a map from $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ to the stack of degree 2 finite locally free covers which sends (q, ξ) to $V(q)$. Since this map is invariant for the action of G_n, we obtain a map Π_n from $\left[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n\right]$ to the stack of degree 2 finite locally free covers.

Theorem 3.3.7. *Keep notation as in Notation 3.3.1 and assume B is a normal integral scheme. Let $n \geq 3$ be an integer. Fix a degree 2 locally free generically étale cover $g : X \to B$. There is an injection from orbits $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B)$ such that $V(q) \simeq X$ to $\Pi_n^{-1}([X]) \subset [V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n](B)$. In turn, $\Pi_n^{-1}([X])$ is identified bijectively with elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$. The above injection is a bijection if $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.*

Further, the above injection identifies the following three groups:

1. *the B points of the isotropy group scheme of the corresponding map $B \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ via the bijection of Proposition 2.3.14*

2. *the B points of the isotropy group scheme of the corresponding map $B \to \mathscr{V}^{\mathrm{smile},(n)}$*

3. *the stabilizer in $G_n(B)$ of (q, ξ).*

Proof. To prove the first part, we wish to produce an injection from orbits $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B)$ to points $B \to \mathscr{V}^{\mathrm{smile},(n)}$ over $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \mathscr{W}_{\mathrm{sing}}(B)$ which is a bijection if $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.

We claim there is a sequence of maps

$$V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B) \to [V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n](B) \to [\mathcal{V}^{\mathrm{smile},(n)}/\mathrm{Aut}_{\mathbb{F}_{n-2}/B}](B) \to \mathscr{V}^{\mathrm{smile},(n)}(B)$$

where all but the first maps are bijections, and the first map is an injection which is a bijection if $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$. Indeed, the statement for the first map is Lemma 3.3.5. The remaining maps are bijections by Lemma 3.3.6.

We next check that the above identifications send $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B)$ with $V(q) \simeq X$ to $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$. Keeping notation as in Notation 3.2.1, by Theorem 3.2.8, there is a bijection between $B \to \mathscr{V}^{\mathrm{smile},(n)}$ lying over $(B, E_g \to B, \sigma : B \to E_g, \tau : B \to E_g) \in \mathscr{W}_{\mathrm{sing}}(B)$ and elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$. Because the above maps of stacks are compatible with the projection to $\mathscr{W}_{\mathrm{sing}}$, it is enough to show that we can recover $V(q)$ from E_g. But indeed, using that B is normal, the normalization of E_g is $\mathbb{P}(g_*\mathscr{O}_X)$, as this is normal and maps birationally to E_g. Then, $V(q)$ can be recovered as the preimage of the singular locus of $E_g \to B$ under the map $\mathbb{P}(g_*\mathscr{O}_X) \to E_g$.

Combining the above identifications, we then obtain that pairs $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)/G_n(B)$ with $V(q) \simeq X$ inject into $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$. Further, we obtain this injection is an isomorphism if $H^1(B, \mathrm{PGL}_2) = H^1(B, \mathbb{G}_m) = H^1(B, \mathbb{G}_a) = 0$.

We next we wish to identify the B-points of the isotropy group scheme of $B \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$, the B points of the isotropy group scheme of $B \to \mathscr{V}^{\mathrm{smile},(n)}$ and the stabilizer of (q, ξ) in $G_n(B)$. The identification of the first two follows from Lemma 3.3.6. The final identification also follows from Lemma 3.3.6 because the stabilizer of a point $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}$ in $G_n(B)$ is the B points of the isotropy group of the map $B \to [V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n]$. □

Remark 3.3.8 (Quotienting by the automorphisms of the elliptic curve)**.** Using Theorem 3.3.7 and preceding results, we have that B points of $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n]$ over a given point $(B, E_g \to B, \sigma : B \to E_g) \in \mathscr{W}(B)$ correspond to elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$. One may find the quotient by $\mathrm{Aut}_{(E_g,\sigma)/B}(B)$ somewhat annoying, and prefer to find a stack which parameterizes $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ as opposed to the above quotient. We note that elements of this cohomology group would correspond to B-points of the fiber product $I_{\mathscr{W}} \times_{\mathscr{W}} \mathscr{S}^{(n)}$, where $I_{\mathscr{W}}$ is the inertia stack associated to $\mathscr{W}$, as opposed to simply points of $\mathscr{S}^{(n)}$. This fiber product rigidifies the data of the automorphism of the elliptic curve.

It would be quite interesting if one could find a presentation of $I_{\mathscr{W}} \times_{\mathscr{W}} \mathscr{S}^{(n)}$ as a global quotient, so that would could parameterize elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$. It seems reasonable one may be able to modify the group action of G_n taking an appropriate twisted action, perhaps similarly to [Woo11a, Theorem 5.2 and Theorem 5.4]. However, we have not worked this out. It may also be the case that the $\mathbb{Z}$-points of $I_{\mathscr{W}} \times_{\mathscr{W}} \mathscr{S}^{(n)}$ do agree with the $\mathbb{Z}$-points of the quotient stack of V_n by the subgroup of G_n generated by U_n and GL_2/μ_n of (2.1.3). At least this seemed to hold in a few examples we tried. However, this seems to be specific to $\operatorname{Spec}\mathbb{Z}$ and a relic of the fact that $\mathbb{G}_m(\mathbb{Z}) = \pm 1$ has size 2 and is therefore identified with $\operatorname{Aut}_{(E_g,\sigma)/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$. Of course for general bases B, $\mathbb{G}_m(B)$ need not have size 2.

For out ultimate purpose of counting torsion in quadratic fields, the quotient by the automorphism group will not pose a problem, as we now explain. The reason is that $\operatorname{Aut}_{(E_g,\sigma)/B}(B)$ will always have order 2, and the nontrivial element will act by inversion. Since we will be able to understand the elements of order 2 in the class group using genus theory (and similarly be able to understand the elements of order 2 in $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$), one can determine the size of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ from the size of its quotient by $\operatorname{Aut}_{(E_g,\sigma)/B}(B)$. Moreover, in this paper, we will only be concerned with asymptotics, and certainly

$$2\cdot\#\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\operatorname{Aut}_{(E_g,\sigma)/B}(B)\right) \geq H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$$
$$\geq \#\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\operatorname{Aut}_{(E_g,\sigma)/B}(B)\right).$$

We next wish to use Theorem 3.3.7 to give a third proof of Theorem 1.2.4. We note that this third proof is quite a bit more complicated that the first proof from § 1.3.1, but we give it here to illustrate how one may deduce it from Theorem 3.3.7. Before doing so, we next review the geometric description of the correspondence between quadratic forms and elements of class groups of quadratic fields, which follows from [Woo16, Theorem 1.2]. The reader familiar with this correspondence should feel free to skip to Example 3.3.11.

Fix a non-square d which is congruent to 0 or 1 modulo 4. Note that the discriminant of any quadratic form satisfies this congruence condition. The proof of Theorem 1.2.4, and even the statement of Theorem 1.2.4, depends on the bijection ν from $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z})$ equivalence classes of primitive quadratic forms $q = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$ having discriminant d and elements of $H^1(X, \mathbb{G}_m)/\operatorname{Aut}_{X/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$ for X the unique degree 2 locally free extension $X \to \operatorname{Spec}\mathbb{Z}$ of discriminant d. We now recall this bijection. The description over general bases is given in [Woo11a, Theorem 1.5]. Because it is fun and simple, we give a short proof of the bijection over $\operatorname{Spec}\mathbb{Z}$. First, we review the action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z})$ on quadratic forms q. Specifically, $\lambda \in \mathrm{GL}_1$ acts on q by sending $q \mapsto \lambda q$ while $g \in \mathrm{GL}_2$ acts by sending $q(x, y) \mapsto q(gx, gy)$. We now define the bijection ν. Given a quadratic form $q \in H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))$ of discriminant d, $V(q)$ defines a closed subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ which also has discriminant d, and is therefore isomorphic to X. Additionally, $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ restricts to an invertible sheaf $\mathscr{L}_q$ on X. This defines a map ν from $H^1(X, \mathbb{G}_m)/\operatorname{Aut}_{X/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$ to the set of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z})$ equivalence classes of primitive binary quadratic forms.

Lemma 3.3.9. *The map ν constructed above is an isomorphism.*

Remark 3.3.10. In fact, the isomorphism ν can be realized as coming from a map on $\mathbb{Z}$ points between two descriptions of the universal Picard stack over the stack of degree 2 locally free covers. One description is the modular one, while the other description is as a quotient of an open in $\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^1, \mathscr{O}(2)))$ by PGL_2. The bijection of Lemma 3.3.9 gives a way to realize two equivalent descriptions $\operatorname{Spec}\mathbb{Z}$ points of the fiber of the universal Picard stack over the degree 2 cover of discriminant d.

Proof. We will construct the inverse map to ν. Given $g : X \to \operatorname{Spec}\mathbb{Z}$ of discriminant d, and $\mathscr{L}$ an invertible sheaf on X up to $\mathrm{Aut}_{X/\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$, we wish to construct a $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z})$ equivalence class of binary quadratic forms. First, observe $g_*\mathscr{L}$ is a locally free rank 2 sheaf on $\operatorname{Spec}\mathbb{Z}$, hence $\mathrm{Proj}(g_*\mathscr{L})$ is isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$. Next, by the universal property of Proj, we claim the adjunction map $g^*g_*\mathscr{L} \to \mathscr{L}$ in fact defines a surjection inducing an embedding $X \to \mathbb{P}^1_{\mathbb{Z}}$. Indeed, this may be verified on a Zariski open cover U_i of $\operatorname{Spec}\mathbb{Z}$ so that $\mathscr{L}|_{g^{-1}(U_i)}$ has a trivialization. Altogether, we have obtained an embedding $X \to \mathbb{P}(g_*\mathscr{L}) \simeq \mathbb{P}^1_{\mathbb{Z}}$ whose composition to $\operatorname{Spec}\mathbb{Z}$ realizes X as a finite degree 2 locally free cover of $\operatorname{Spec}\mathbb{Z}$.

Such an embedding corresponds to a $\operatorname{Spec}\mathbb{Z}$ point of $\mathbb{P}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)/\mathrm{PGL}_2$. Since $H^1(\operatorname{Spec}\mathbb{Z}, \mathrm{PGL}_2) = 0$, we have

$$\left[\mathbb{P}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)/\mathrm{PGL}_2\right](\mathbb{Z}) = \mathbb{P}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)(\operatorname{Spec}\mathbb{Z})/\mathrm{PGL}_2(\operatorname{Spec}\mathbb{Z}).$$

Let $0 \in \mathbb{A}^3_{\mathbb{Z}} \simeq \operatorname{Spec}\mathbb{Z}[x, y, z]$ denote the section defined by $x = y = z = 0$. By the identification $\mathbb{P}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right) \simeq [(\mathbb{A}^3 - 0)/\mathbb{G}_m](\operatorname{Spec}\mathbb{Z})$, we obtain $\mathbb{P}\left(H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)(\mathbb{Z}) = (\mathbb{A}^3 - 0)(\operatorname{Spec}\mathbb{Z})/\mathbb{G}_m(\operatorname{Spec}\mathbb{Z})$. Elements of $(\mathbb{A}^3 - 0)(\operatorname{Spec}\mathbb{Z})$ precisely correspond to primitive binary quadratic forms. The action of $\mathrm{PGL}_2(\operatorname{Spec}\mathbb{Z})$ on $\mathbb{P}H^0\left(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))\right)(\operatorname{Spec}\mathbb{Z})$ lifts to the action of $\mathrm{GL}_2(\operatorname{Spec}\mathbb{Z})$ on $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(2))(\operatorname{Spec}\mathbb{Z})$ given in the definition of ν. Altogether, from $\mathscr{L}$, we have produced an equivalence class of primitive binary quadratic forms with $\mathbb{Z}$ coefficients, up to the simultaneous commuting actions of $\mathrm{GL}_1(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$.

Because $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ restricts to $\mathscr{L}$ along $X \to \mathbb{P}(g_*\mathscr{L}) \to \mathbb{P}^1_{\mathbb{Z}}$, the map we have constructed is inverse to ν. Since ν and its inverse preserve discriminant, we have obtained the desired bijection. □

Before proving Theorem 1.2.4, it will also be useful to have the following examples for $n = 1$ and $n = 2$. We note that the morally correct way to prove Theorem 1.2.4 in the cases $n = 1$ and 2 is to construct an analog of $\mathscr{S}^{(n)}$ for $n = 2$ and use $\mathscr{W} = \mathscr{S}^{(1)}$. But to shorten the proof, we will opt for a cheap workaround. One implication can be observed directly in the cases $n = 1$ and 2, as we now do.

Example 3.3.11. Let $g : X \to \operatorname{Spec} \mathbb{Z}$ be a degree 2 finite locally free cover. Suppose q is a quadratic form corresponding to the trivial element in $H^1(X, \mathbb{G}_m)$. We can choose a representative for q of the form $x^2 + axy + by^2$. We would like to show that for every n there is a degree n polynomial ξ with $\operatorname{Res}(q, \xi) = \pm 1$. Indeed, we may take $\xi = y^n$.

Example 3.3.12. We now consider the case $n = 2$. Let $g : X \to \operatorname{Spec} \mathbb{Z}$ be a degree 2 finite locally free cover. Suppose q is a quadratic form corresponds to an element of $H^1(X, \mathbb{G}_m)[2]$. In this case, q can either be taken to be $q = ax^2 + cy^2$ or $q = ax^2 + axy + cy^2$ [Cas78, Chapter 14, Lemma 4.1]. We would like to show that in both these cases, there is a degree 2 polynomial ξ with $\operatorname{Res}(q, \xi) = \pm 1$. In the first case, because q is primitive, we can find u, v with $uc - av = 1$. Then $\operatorname{Res}(ax^2 + cy^2, ux^2 + vy^2) = 1$. In the second case, again take u, v with $uc - av = 1$. Then we find $\operatorname{Res}(ax^2 + axy + cy^2, ux^2 + uxy + vy^2) = 1$.

Finally, we recall a standard fact we will need for the third proof of Theorem 1.2.4 and in subsequent sections.

Lemma 3.3.13. *For a finite locally free cover $g : X \to B$ of any degree, $H^1(B, g_*\mathbb{G}_m) \simeq H^1(X, \mathbb{G}_m)$.*

Proof. By the Leray spectral sequence,it suffices to show $R^1 g_* \mathbb{G}_m = 0$. Indeed, $R^1 g_* \mathbb{G}_m = 0$ holds because the stalks of $R^1 g_* \mathbb{G}_m$ are the Picard groups of semi-local rings, which we claim vanish.

The proof that Picard groups of semi-local rings vanish is standard, and we now recall it. We need to show any line bundle $\mathscr{L}$ over the spectrum of a semi-local ring B is trivial. Letting $Z \subset B$ denote the disjoint union of the closed points of B, we obtain $\mathscr{L}|_Z$ is trivial. We can lift the isomorphism $\phi : \mathscr{O}_Z \to \mathscr{L}|_Z$ to a map $\widetilde{\phi} : \mathscr{O}_B \to \mathscr{L}$ which reduces to ϕ on Z. By Nakayama's lemma, we obtain $\widetilde{\phi}$ is an isomorphism so $\mathscr{L}$ is trivial. □

3.3.14 Third proof of Theorem 1.2.4

To start, we assume $n \geq 3$. We will return to the cases $n = 1, 2$ at the end. Throughout this proof, we will conflate various groups and their quotients by the natural $\operatorname{Aut}_{X/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ action. This is justified because the statement of the theorem is also invariant under this $\mathbb{Z}/2\mathbb{Z}$ action.

Let $B = \operatorname{Spec} \mathbb{Z}$ and consider a finite locally free degree 2 cover $g : X \to B$. This includes spectra of orders in rings of integers, and technically also reducible covers of B.

We have maps

$$H^1(X,\mu_n) \simeq H^1(X,\mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \to H^1(B,g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m) \to H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m). \tag{3.3.3}$$

which induce a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathbb{G}_m) & \longrightarrow & H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ H^1(X,\mathbb{G}_m)[n] & \xrightarrow{\beta} & H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m)[n] \end{array} \tag{3.3.4}$$

which commutes because the corresponding diagram of sheaves on B commutes. Note also that this is compatible with the action of the nontrivial automorphism of $g : X \to B$ on the above diagram. The map γ is induced by (2.3.8) and α is induced by the Kummer sequence on X. The map β is induced by the composition $H^1(X,\mathbb{G}_m) \to H^1(B,g_*\mathbb{G}_m) \to H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m)$. Further, β is an isomorphism using Lemma 3.3.13 for the first map and the vanishing $H^1(B,\mathbb{G}_m) = H^2(B,\mathbb{G}_m) = 0$ for the second.

We first show that if there exists ξ with $(q,\xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}$ then q is n-torsion. By the proof of [Woo11a, Theorem 1.4] via the "Global Construction" of [Woo11a, p. 1763-1764], (or one can alternatively use Lemma 3.3.9,) since β is an isomorphism, we can view q as an element in $H^1(B,g_*\mathbb{G}_m)/\mathrm{Aut}_{X/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$ by taking the associated invertible sheaf $\mathscr{L}_q$ on X obtained as the restriction of $\mathscr{O}_{\mathbb{P}^1_B}(1)$ along the embedding $X \simeq V(q) \to \mathbb{P}^1_B$. Note that $\mathrm{Aut}_{(E_g,\sigma)/B}(B) \simeq \mathrm{Aut}_{X/\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z})$ are both $\mathbb{Z}/2\mathbb{Z}$, generated by the natural involution. Under the correspondence between elements of $H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)/\mathrm{Aut}_{(E_g,\sigma)/B}(B)$ and pairs $(q,\xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)$ with $V(q) \simeq X$ of Theorem 3.3.7, $\overline{\gamma}(q,\xi) = \overline{\beta}(\mathscr{L}_q)$, where $\overline{\gamma}$ and $\overline{\beta}$ mean the quotients of γ and β by the nontrivial automorphism of g. Therefore, any q for which there exists some ξ with $(q,\xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)$ (i.e., $\mathrm{Res}(q,\xi) \in \{\pm 1\}$) must lie in the image of $H^1(X,\mathbb{G}_m)[n]$.

Conversely, suppose q lies in the image of $H^1(X,\mathbb{G}_m)[n]$. The map α is surjective, and so we can lift $[q]$ to some element of $H^1(X,\mu_n)$, which then maps to an element of $H^1(B,g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$. Again by Theorem 3.3.7, this element yields a ξ so that $(q,\xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(B)$, meaning that $\mathrm{Res}(q,\xi) \in \{\pm 1\}$.

It remains to deal with the cases $n = 1$ and $n = 2$. Let $n \in \{1,2\}$. If there is some $\xi \in \mathbb{Z}[x,y]$ of degree n with $\mathrm{Res}(q,\xi) \in \{\pm 1\}$ we claim $\mathrm{Res}(q,\xi^k) = \pm 1$ for every $k \geq 1$. Indeed, if q and ξ have no roots in common modulo any prime, the same is true of q and ξ^k. Because we know the result holds for nk torsion when $k \geq 3$, we find that q is nk-torsion for every $k \geq 3$ and hence q is n-torsion. Conversely, q is n-torsion, there exists a polynomial

ξ of degree n with $\mathrm{Res}(q,\xi)=\pm 1$ by Example 3.3.11 when $n=1$ and by Example 3.3.12 when $n=2$. □

3.4 Cohomological Comparison

Throughout, we work in the flat topology so that all cohomology for flat sheaves is taken with respect to the flat topology. Let $g: X \to B$ be a finite locally free degree 2 morphism of schemes of relative degree 2. Perhaps Theorem 3.3.7 has convinced the reader that the cohomology group $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ may be computable. However, the natural next question is: why do we care? One strong motivation lies in its connection to $H^1(X,\mathbb{G}_m)[n]$. When $B = \operatorname{Spec}\mathbb{Z}$, this recovers the n-torsion in class groups of orders in quadratic fields. In the case $B=\operatorname{Spec}\mathbb{Z}$, we give a complete description of this relation in Lemma 3.4.1. For $g: X\to \operatorname{Spec}\mathbb{Z}$ a degree 2 finite locally free cover, the exact sequence (2.3.9) allows us to compare $H^1(X,\mathbb{G}_m)[n]$ to $H^1(\operatorname{Spec}\mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$.

Lemma 3.4.1. *Let $g: X \to \operatorname{Spec}\mathbb{Z}$ be a finite degree 2 locally free cover and let $n\geq 1$. There is natural surjection $H^1(\operatorname{Spec}\mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(X,\mathbb{G}_m)[n]$. The kernel of this map is isomorphic to*

$$\begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } \operatorname{disc}(g)\geq 0 \text{ and } X \text{ is irreducible} \\ \mathbb{Z}/\gcd(2,n)\mathbb{Z} & \text{if } X\simeq \operatorname{Spec}\mathbb{Z}[\sqrt{-1}] \text{ or } X\simeq \operatorname{Spec}(\mathbb{Z}\times\mathbb{Z}) \\ \mathbb{Z}/\gcd(3,n)\mathbb{Z} & \text{if } X\simeq \operatorname{Spec}\mathbb{Z}[\frac{\sqrt{-3}+1}{2}] \\ 1 & \text{otherwise} \end{cases} \tag{3.4.1}$$

Proof. Take $B=\operatorname{Spec}\mathbb{Z}$. By (2.3.8), we have a surjection $H^1(\operatorname{Spec}\mathbb{Z}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m)[n]$. Because $B=\operatorname{Spec}\mathbb{Z}$, $H^1(B,\mathbb{G}_m)=H^2(B,\mathbb{G}_m)=0$. Therefore, the natural map $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m)[n]\simeq H^1(B,g_*\mathbb{G}_m)[n]$ is an isomorphism. By Lemma 3.3.13, the latter is isomorphic to $H^1(X,\mathbb{G}_m)[n]$.

Therefore, it remains to analyze the kernel of this map, which by (2.3.5), is isomorphic to $\frac{H^0(B,g_*\mathbb{G}_m/\mathbb{G}_m)}{nH^0(B,g_*\mathbb{G}_m/\mathbb{G}_m)}$. Since $H^1(B,\mathbb{G}_m)=0$, this can be identified with

$$\mathbb{G}_m(X)/\mathbb{G}_m(\mathbb{Z})/n\left(\mathbb{G}_m(X)/\mathbb{G}_m(\mathbb{Z})\right) \simeq \mathbb{G}_m(X)/\mathbb{G}_m(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{G}_m(X)/\{\pm 1\}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}.$$

Therefore, the kernel is identified with the $\bmod n$ reduction of $\mathbb{G}_m(X)/\{\pm 1\}$, and it remains to analyze the unit group of X.

We now analyze $\mathbb{G}_m(X)$. The essential tool is Dirichlet's theorem. We first analyze the cases that X is integral. First, if X is the spectrum of an order in a real quadratic field, it has no roots of unity other than ± 1, and so its unit group is $\mathbb{Z} \times \{\pm 1\}$. On the other hand, if X is the spectrum of an order in an imaginary quadratic field, it will only have units other than ± 1 when $X = \operatorname{Spec} \mathbb{Z}[\sqrt{-1}]$, in which case the unit group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, or when $X = \operatorname{Spec} \mathbb{Z}[\frac{\sqrt{-3}+1}{2}]$, in which case the unit group is $\mathbb{Z}/6\mathbb{Z}$. This achieves all cases when X is integral, so it remains to address the cases that X is irreducible or not reduced. The only non-reduced case is $X = \operatorname{Spec} \mathbb{Z}[x]/(x^2)$, which has discriminant 0 and the unit group is $\mathbb{Z} \times \{\pm 1\}$, given by elements of the form $ax \pm 1$ for $a \in \mathbb{Z}$. If X is reducible, its normalization is $B \coprod B$, so $X = \operatorname{Spec} A$ for $A \subset \mathbb{Z} \times \mathbb{Z}$. The units in $\mathbb{Z} \times \mathbb{Z}$ are the four elements $(\pm 1, \pm 1)$. Since any subring of $\mathbb{Z} \times \mathbb{Z}$ contains $\pm(1, 1)$, there are no proper subrings of $\mathbb{Z} \times \mathbb{Z}$ containing all four units of $\mathbb{Z} \times \mathbb{Z}$, which completes our analysis. □

3.4.2 Examples

We next illustrate the usefulness of Lemma 3.4.1 and Theorem 3.3.7 with some examples. It is surprisingly elementary to verify the validity of Lemma 3.4.1 and Theorem 3.3.7 in specific cases, as we shall next see.

Remark 3.4.3. Fix $g : X \to \operatorname{Spec} \mathbb{Z}$ of discriminant d. For each primitive quadratic form q of discriminant d which lies in $H^1(X, \mathbb{G}_m)$, we can ask how many G_n orbits of degree n polynomials ξ there are with $\operatorname{Res}(q, \xi) = \pm 1$, up to the action of G_n. By combining Theorem 3.3.7 and Theorem 1.2.4 with Lemma 3.4.1 we can see there are no such values of ξ unless q lies in $H^1(X, \mathbb{G}_m)[n]$. In the latter case, the number of such orbits is determined by (3.4.1). We have verified this extensively using MAGMA in thousands of cases. Let us now see this carried out in some examples.

Example 3.4.4. Consider $K = \mathbb{Q}(\sqrt{-23})$ and $X = \operatorname{Spec} \mathcal{O}_K = \operatorname{Spec} \mathbb{Z}[\frac{\sqrt{-23}+1}{2}]$. We apply Remark 3.4.3 to this setting. We have $\operatorname{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$, with representatives for the three quadratic forms given by $q_1 := x^2 + xy + 6y^2, q_2 := 2x^2 + xy + 3y^2, q_3 := 2x^2 - xy + 3y^2$. Since these are all 3-torsion, we expect that for each quadratic form, there should exist some degree 3 polynomial ξ with $\operatorname{Res}(q_i, \xi) = \pm 1$. Indeed, we see $\operatorname{Res}(q_1, -y^3) = 1, \operatorname{Res}(q_2, -x^3 - xy^2 + y^3) = 1$, and $\operatorname{Res}(q_3, -x^3 - xy^2 - y^3) = 1$.

Example 3.4.5. Consider $K = \mathbb{Q}(\sqrt{-47})$ and $X = \operatorname{Spec} \mathcal{O}_K = \operatorname{Spec} \mathbb{Z}[\frac{\sqrt{-47}+1}{2}]$. This has $\operatorname{Cl}(K) \simeq \mathbb{Z}/5\mathbb{Z}$ with representatives given by the quadratic forms $q_1 = x^2 + xy + 12y^2, q_2 = 2x^2 + xy + 6y^2, q_3 = 2x^2 - xy + 6y^2, q_4 = 3x^2 + xy + 4y^2$, and $q_5 = 3x^2 - xy + 4y^2$. For each $1 \leq i \leq 5$, there is a degree 5 polynomial ξ_i so that $\operatorname{Res}(q_i, \xi_i) = 1$. Namely, $\xi_1 = -y^5, \xi_2 = -x^5 - 3x^3y^2 + x^2y^3 - xy^4 - y^5, \xi_3 = -x^5 - 3x^3y^2 - x^2y^3 - xy^4 + y^5, \xi_4 = -x^5 - x^4y - x^3y^2 + xy^4 + y^5, \xi_5 = -x^5 + x^4y - x^3y^2 + xy^4 - y^5$.

Example 3.4.6. In the case $K = \mathbb{Q}(\sqrt{-1})$ and $X = \operatorname{Spec} \mathscr{O}_K = \operatorname{Spec} \mathbb{Z}[\sqrt{-1}]$, the unique equivalence class of quadratic form has representative $x^2 + y^2$. Then, Lemma 3.4.1 predicts that for any positive integer n, there should be two orbits of pairs (q, ξ) for ξ of degree $2n$ with resultant 1. Indeed, the two orbits correspond to $\xi = x^{2n}$ and $\xi = (xy)^n$.

Example 3.4.7. In the case $K = \mathbb{Q}(\sqrt{-3})$ and $g : X = \operatorname{Spec} \mathscr{O}_K = \operatorname{Spec} \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \to \operatorname{Spec} \mathbb{Z}$, $\operatorname{Cl}(K)$ is the trivial group, but Lemma 3.4.1 predicts there should be two orbits of pairs (q, ξ) with resultant 1. Note that $\# H^1(X, g_* \mathbb{G}_m / \mathbb{G}_m \xrightarrow{\times 3} g_* \mathbb{G}_m / \mathbb{G}_m) = 3$, but the quotient of this group by the inversion action has size 2. We can take $q := x^2 + xy + y^2$ as a representative for the quadratic form. In this case, the two polynomials y^3 and $y^2 x$ have resultant 1 with q, and lie in distinct G_3 orbits. Note that $y^3 + y^2 x$ is another such polynomial which corresponds to the third element of $H^1(X, g_* \mathbb{G}_m / \mathbb{G}_m \xrightarrow{\times 3} g_* \mathbb{G}_m / \mathbb{G}_m)$ that is identified with $y^2 x$ under the automorphism of g.

Example 3.4.8. Consider the case $K = \mathbb{Q}(\sqrt{5})$ and $g : X = \operatorname{Spec} \mathscr{O}_K = \operatorname{Spec} \mathbb{Z}[\frac{\sqrt{5}+1}{2}] \to \operatorname{Spec} \mathbb{Z}$ which has discriminant 20. This has trivial $\operatorname{Cl}(K)$, and a representative for the quadratic form is given by $q = x^2 - 5y^2$. Then, Lemma 3.4.1 predicts there should be $2 = \#(\mathscr{O}_K^\times / 3\mathscr{O}_K^\times)/\operatorname{Aut}_{X/\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} \mathbb{Z})$ degree 3 polynomials ξ, up to the G_n action, having resultant ± 1 with q. Indeed, representatives for the two orbits are given by y^3 and $-4xy^2 - 9y^3$.

Chapter 4

Inflection Subschemes

Given a curve smooth relative curve C over a base B and a map $C \to \mathbb{P}^{n-1}$ our goal in §4.1 is to compute the relative inflection subscheme. On fibers, this corresponds to those points of C such that the nth order tangent vector is contained in a linear space of dimension 1 less than expected.

We will relate inflection subschemes to class groups of quadratic fields as follows. Take $C = \mathbb{P}^1$, and consider a quadratic form q as a subscheme of $\mathbb{P}^1$. When this inflection subscheme has a section not meeting the quadratic form q, the curve C corresponds to one of relatively few elements in the class group. This will be proven in §4.2, see Proposition 4.2.11.

4.1 The inflection subscheme

In this section, we introduce the inflection subscheme associated to a map from $\mathbb{P}^1 \to \mathbb{P}^n$. We start with defining inflection subschemes in §4.1.1, and recalling some of their basic properties. We then give a concrete formula to compute inflection subschemes in §4.1.8. We use this to bound the coefficients of the inflection subscheme. Further equivalent formulas are given later in Remark 6.2.6 and Lemma 6.2.5. Finally, in §4.1.12, we discuss how one can use inflection subschemes to identify which n coverings of the smooth locus of a singular genus 1 curve arise as torsors for the n torsion, in characteristic dividing n.

4.1.1 Generalities on inflection points

We will begin by reviewing generalities on inflection subschemes. Much of the following is standard. See for example, [PS17, Definition 2.2.2] and [PS17, Proposition 2.3.7] for generalizations of Definition 4.1.3 and Lemma 4.1.7 respectively. Technically [PS17] works under the hypothesis that $X \to B$ is a map of varieties of a field of characteristic 0, but those assumptions are not used in the above cited results.

Notation 4.1.2. Let B be a base scheme and let $f : C \to B$ be a smooth proper relative curve and $\pi : \mathbb{P}^{n-1}_B \to B$ the structure map. Suppose we are given a non-degenerate map $\iota : C \to \mathbb{P}^{n-1}_B$ specified by an invertible sheaf $\mathscr{L}$ on C together with n global sections $s^1, \ldots, s^n \in H^0(B, f_*\mathscr{L}) = H^0(C, \mathscr{L})$ which form an independent set in $f_*\mathscr{L}$. By independent set, we mean the map $\bigoplus_{i=1}^n \mathscr{O}_B \xrightarrow{(s^1,\ldots,s^n)} f_*\mathscr{L}$ is subbundle of rank n, i.e., the map is injective and the quotient is locally free.

Let

$$\begin{array}{ccc} & C \times_B C & \\ \swarrow^{\alpha} & & \searrow^{\beta} \\ C & & C \end{array} \tag{4.1.1}$$

denote the projections and let $\Delta \subset C \times_B C$ denote the diagonal subscheme, which is the image of C under the diagonal map to $C \times_B C$. For m a positive integer, let $m\Delta$ denote the mth order neighborhood of the diagonal. More precisely, if the diagonal scheme is $V(\mathscr{I}_\Delta)$ then $m\Delta$ is $V(\mathscr{I}_\Delta^{\otimes m})$. Define the mth order *jet scheme* associated to $\mathscr{L}$ to be $\mathscr{J}_{m,\mathscr{L}} := \alpha_*(\beta^*\mathscr{L})|_{m\Delta}$.

Definition 4.1.3. Retain notation as in Notation 4.1.2. Let $f^*\pi_*\mathscr{O}_{\mathbb{P}^{n-1}}(1) \to \mathscr{L}$ denote the composition of $f^*\pi_*\mathscr{O}_{\mathbb{P}^{n-1}}(1) \to f^*f_*\iota^*\mathscr{O}_{\mathbb{P}^{n-1}}(1) \simeq f^*f_*\mathscr{L} \to \mathscr{L}$. Let $\gamma_\iota : \alpha_*(\beta^*f^*\pi_*\mathscr{O}_{\mathbb{P}^{n-1}}(1))|_{n\Delta} \to \alpha_*(\beta^*\mathscr{L})|_{n\Delta}$ denote the induced map. We define the *inflection subscheme of C with respect to ι* as the closed subscheme $\operatorname{Inflection}(C, \iota) \subset C$ over which the rank of γ_ι is at most $n-1$. The scheme structure on $\operatorname{Inflection}(C, \iota) \subset C$ is obtained by locally taking the vanishing of the determinant of the matrix associated to γ_ι.

More generally, if we replace $\mathbb{P}^{n-1}_B$ by a rank $n-1$ projective bundle P over B, and consider a map $\iota : C \to P$, we define $\operatorname{Inflection}(C, \iota) \subset C$ by choosing a Zariski open cover $U \to B$ on which $P_U \simeq \mathbb{P}^{n-1}_U$ and letting $\operatorname{Inflection}(C, \iota)$ be the unique closed subscheme of C restricting to $\operatorname{Inflection}(C_U, \iota_U)$ over U.

Remark 4.1.4. It may be a bit opaque, but Definition 4.1.3 captures the usual notion of inflection point. Recall that the usual notion of inflection point says that when B is the spectrum of a field, a point $p \in C$ is an inflection point when the nth order neighborhood of C is contained in an $n-2$ plane. The fiber of $\alpha_*(\beta^*f^*\pi_*\mathscr{O}_{\mathbb{P}^{n-1}}(1))|_{n\Delta}$ over any point is identified with $H^0(C, \mathscr{L})$ while the fiber of $\alpha_*(\beta^*\mathscr{L})|_{n\Delta}$ over p is identified with $\mathscr{L}|_{\mathscr{O}_C/\mathscr{I}_{p/C}^{\otimes n}}$. The adjunction map is identified with the natural restriction map, restricting a section to its value on the nth order neighborhood of $p \in C$. Therefore, this map will drop rank precisely when the nth order neighborhood of C at p is contained in an $n-2$ plane.

We next give a way to explicitly compute the inflection subscheme in terms of equations. These specific equations could potentially be used to bound which regions to consider in a geometry of numbers argument.

Definition 4.1.5. For $C \to B$ a smooth proper relative curve, $r \in \mathbb{Z}_{\geq 1}$, and $\alpha \in K(C)$ a rational function, we let $\partial_x^{(r)}\alpha$ denote the rth divided power derivative of f. That is, when viewed as a formal expression, we have $r!\partial_x^{(r)}\alpha = \partial_x^r\alpha$, for $\partial_x^r\alpha$ the usual rth partial derivative of f.

Lemma 4.1.6. *Suppose we are given a point $\sigma : \operatorname{Spec} R \to C_R$ factoring through a given affine coordinate patch $\mathbb{A}^{n-1} \subset \mathbb{P}^{n-1}$ and that R has trivial Picard group. Let $U := \iota^{-1}(\mathbb{A}^n) \subset C$. Then a generator for the invertible sheaf $\mathscr{I}_{\sigma/C_R}/\mathscr{I}^2_{\sigma/C_R}$ induces an isomorphism of the total space of $\mathscr{O}_{C_R}/\mathscr{I}^{\otimes n}_{\sigma/C_R}$ with $R[x]/(x^n)$. Suppose we describe the map $U \to \mathbb{A}^{n-1}_R$ via the coordinates $(\sum_{j=0}^{n-1} s^{1,j}x^j, \ldots, \sum_{j=0}^{n-1} s^{n-1,j}x^j)$ for $s^{i,j} \in H^0(U, \mathscr{O}_U)$. Then, σ lies in* $\operatorname{Inflection}(C, \iota)$ *if and only if the matrix whose (i, j) entry is $\partial_x^{(j)} \sum_{k=0}^{n-1} s^{i,k}x^k$ has vanishing determinant.*

Proof. By definition, the $\operatorname{Inflection}(C, \iota)$ is where the matrix associated to

$$\gamma_\iota : \alpha_*(\beta^* f^* \pi_* \mathscr{O}_{\mathbb{P}^{n-1}}(1))|_{n\Delta} \to \alpha_*(\beta^* \mathscr{L})|_{n\Delta}$$

has vanishing determinant. We therefore wish to identify this matrix with the matrix whose (i, j) entry is $\partial_x^{(j)} \sum_{k=0}^{n-1} s^{i,k}x^k$

Note that $\mathscr{I}_{\sigma/C_R}/\mathscr{I}^2_{\sigma/C_R}$ is locally free because σ is a Cartier divisor by assumption that C is smooth over B. Therefore, it is free as it may be considered as a sheaf supported on $\operatorname{Spec} R$, which has trivial Picard group. Hence, $\mathscr{I}_{\sigma/C_R}/\mathscr{I}^2_{\sigma/C_R}$ is in fact free, and therefore has a global generator. It follows that the map

$$R[x]/(x^n) \to \operatorname{Spec} \oplus_{i=0}^{n-1} \operatorname{Sym}^i \mathscr{I}_{\sigma/C_R}/\mathscr{I}^2_{\sigma/C_R} \simeq \mathscr{O}_{C_R}/\mathscr{I}^{\otimes n}_{\sigma/C_R}$$

is an isomorphism.

Now, using this, we may describe the map $R[x]/(x^n)$ to $\mathbb{A}^{n-1}_R$ in terms of coordinates $(\sum_{j=0}^{n-1} s^{1,j}x^j, \ldots, \sum_{j=0}^{n-1} s^{n-1,j}x^j)$. By composing with a translation of $\mathbb{A}^{n-1}_R$, we may further assume $s^{i,0} = 0$ for all i. Let M denote the matrix whose (i, j) entry is $s^{i,j}$ for $1 \leq i \leq n-1, 1 \leq j \leq n-1$. Because we are assuming the nth section is invertible, we may as well set it to 1, and then we find that the locus where the matrix associated to γ_ι has vanishing determinant is also the locus where the determinant of M vanishes. The result then follows from the identity $\partial_x^{(j)} \sum_{k=0}^{n-1} s^{i,k}x^k = s^{i,j}$. □

We next compute the degree of the inflection subscheme in the case it has relative dimension 0.

Lemma 4.1.7. *Keeping notation from Notation 4.1.2, suppose $C \to B$ has genus g and $\mathscr{L}$ has degree d on C. If* $\mathrm{Inflection}(C, \iota)$ *does not contain the reduction of C, then* $\mathrm{Inflection}(C, \iota)$ *is generically finite over B of degree $nd + (-2+2g)\frac{n(n-1)}{2}$.*

Proof. It is enough to compute the degree in the case $B = \operatorname{Spec} k$. Begin with the exact sequence

$$0 \longrightarrow \mathscr{I}^{n-1}_{\Delta/C\times_B C}/\mathscr{I}^{n}_{\Delta/C\times_B C} \longrightarrow \mathscr{O}_{C\times_B C}/\mathscr{I}^{n}_{\Delta/C\times_B C} \longrightarrow \mathscr{O}_{C\times_B C}/\mathscr{I}^{n-1}_{\Delta/C\times_B C} \longrightarrow 0 \tag{4.1.2}$$

From the definition of the cotangent sheaf, we have identifications

$$\mathscr{I}^{n-1}_{\Delta/C\times_B C}/\mathscr{I}^{n}_{\Delta/C\times_B C} \simeq (N^{\vee}_{\Delta/C\times_B C})^{\otimes n-1} \simeq (\Omega^1_{C/B})^{\otimes n-1}.$$

Therefore, we obtain an exact sequence

$$0 \to \alpha^*\mathscr{L} \otimes (\Omega^1_{C/B})^{\otimes n-1} \to \alpha^*\mathscr{L} \otimes \mathscr{O}_{C\times_B C}/\mathscr{I}^{n}_{\Delta/C\times_B C} \to \alpha^*\mathscr{L} \otimes \mathscr{O}_{C\times_B C}/\mathscr{I}^{n-1}_{\Delta/C\times_B C} \to 0. \tag{4.1.3}$$

Pushing this forward along β yields the exact sequence

$$0 \longrightarrow \beta_*((\Omega^1_{C/B})^{\otimes n-1} \otimes \alpha^*\mathscr{L}) \longrightarrow \mathscr{J}_{n,\mathscr{L}} \longrightarrow \mathscr{J}_{n-1,\mathscr{L}} \longrightarrow 0. \tag{4.1.4}$$

which is right exact because Δ is finite over C. Note further that because $\Delta \to C$ is an isomorphism and $(\Omega^1_{C/B})^{\otimes n-1} \otimes \alpha^*\mathscr{L}$ is supported on C, $\deg\left((\Omega^1_{C/B})^{\otimes n-1} \otimes \alpha^*\mathscr{L}\right) = \deg\left(\beta_*\left((\Omega^1_{C/B})^{\otimes n-1} \otimes \alpha^*\mathscr{L}\right)\right)$.

From this, it follows that

$$\begin{aligned} \deg \mathscr{J}_{n,\mathscr{L}} &= \sum_{i=1}^{n} \deg(\Omega^1_{C/B})^{\otimes i-1} \otimes \alpha^*\mathscr{L} \\ &= \sum_{i=1}^{n}((-2+2g)\cdot(i-1)+d) \\ &= \frac{n(n-1)}{2}\cdot(-2+2g)+nd, \end{aligned}$$

as claimed. ◻

4.1.8 Computing the inflection subscheme for particular linear systems

The next step in our computations will be to compute the inflection subscheme explicitly for members of our particular linear system V_n. We give the explicit computation in Lemma 4.1.10 and use this to obtain bounds on the coefficients of the equation defining the inflection subscheme in Lemma 4.1.11. Inflection points enter the picture because curves with reducible inflection subschemes will correspond to points in the "cusps" of the region. Having the equation for the inflection subscheme will be useful because if the leading coefficient of the equation defining the inflection subscheme is 0, we will be able to deduce the inflection subscheme is reducible.

Definition 4.1.9. Retain notation from Notation 3.3.1. We define the *inflection subscheme associated to* (q, ξ) as $\mathrm{Inflection}(\iota, Z) \subset Z$.

We use $\mathrm{InflectionEquation}(q, \xi)$ to denote a choice of equation defining $\mathrm{Inflection}(\iota, Z) \subset Z \simeq \mathbb{P}^1$. This is not unique, but any two are related by the action of $\mathrm{GL}_2(\mathbb{Z})$ on $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1))$.

Lemma 4.1.10. *For $n \geq 3$ and R a ring, given $(q, \xi) \in V_n(R)$ there is a choice of* $\mathrm{InflectionEquation}(q, \xi)$ *as in Definition 4.1.9 given by* $q^n \cdot \partial_x^{(n-1)} \left(\frac{\xi}{q}\right)$.

Proof. By Lemma 3.3.2 we can choose a basis for the linear system $Z \to \mathbb{P}^{n-1}_{\mathbb{Z}}$ given by $(\xi, qx^{n-2}, qx^{n-3}y, \ldots, qy^{n-2})$. Therefore, by Lemma 4.1.6, $\mathrm{Inflection}(Z, \iota)$ is cut out by a power of q times the determinant of the matrix

$$\begin{pmatrix} \partial_x^{(1)} \frac{\xi}{q} & \partial_x^{(2)} \frac{\xi}{q} & \cdots & \partial_x^{(n-1)} \frac{\xi}{q} \\ \partial_x^{(1)} x^{n-2} & \partial_x^{(2)} x^{n-2} & \cdots & \partial_x^{(n-1)} x^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{(1)} x & \partial_x^{(2)} x & \cdots & \partial_x^{(n-1)} x \end{pmatrix}.$$

We see that all the terms below the anti-diagonal vanish, and therefore, the determinant of the above matrix is given as the product of the anti-diagonal entries. This product, up to a factor of ± 1 is

$$\partial_x^{(n-1)} \frac{\xi}{q} \cdot \partial_x^{(n-2)} x^{n-2} \cdots \partial_x^{(1)} x = \partial_x^{(n-1)} \frac{\xi}{q}.$$

We next need to determine what power of q to multiply this expression by. Using Lemma 4.1.7, taking $d = \deg(e + (n-2)f)|_Z = n$, we find that the degree of $\mathrm{Inflection}(Z, \iota)$ is $n^2 + \frac{n(n-1)}{2}(-2) = n$. Observe that the $\partial_x^{(n-1)} \frac{\xi}{q}$ has a factor of q^n in the denominator, and after multiplying by this factor, we obtain a polynomial of degree n. Therefore, $\mathrm{Inflection}(Z, \iota)$ is cut out of Z by the vanishing of $q^n \partial_x^{(n-1)} \left(\frac{\xi}{q}\right)$. □

The content of the following remark will not be needed in this paper, but could be useful to know in order to facilitate computations with $\mathrm{InflectionEquation}(q, \xi)$. Some further descriptions of the inflection equation, which may be quite helpful for computing it more quickly, are given in Remark 6.2.6. See also Lemma 6.2.5.

Lemma 4.1.11. *Suppose we are given $t, \lambda, \mu \in \mathbb{R}$ with $q = O_n(\lambda t^2)x^2 + O_n(\lambda)xy + O_n(\lambda t^{-2})y^2$ and $\xi = \sum_{i=0}^n O_n(\mu t^{n-2i})x^{n-i}y^i$. Then, for $0 \le i \le n$, the coefficient of $x^{n-i}y^i$ in $q^n \partial_y^{(n-1)} \left(\frac{\xi}{q}\right)$ is $O_n(\lambda^{n-1}\mu t^{n-2i})$.*

In particular, if (q, ξ) corresponds to a relative curve $Z \subset \mathbb{F}_{n-2}$ we can choose $\mathrm{InflectionEquation}(q, \xi)$ to be given by $\sum c_i x^{n-i} y^i$ with $c_i = O_n(\lambda^{n-1}\mu t^{n-2i})$.

Proof. To simplify matters, for the remainder of this proof, we dehomogenize all polynomials, and consider them only as polynomials in y. Observe that

$$q = \lambda t^2 \left(O_n(1) + O_n(t^{-2})y + O_n(t^{-4})y^2\right)$$

and so formally inverting it as a power series in y yields that $\frac{1}{q} = \frac{1}{\lambda t^2} \left(\sum_{i \ge 0} O_{n,i}(t^{-2i})\right) y^i$. Since the absolute value of the coefficient y^i in q^n is $O_n(\lambda^n t^{2n-2i})$, we find

$$\begin{aligned}
q^n \partial_y^{(n-1)} \left(\frac{\xi}{q}\right) &= \left(\sum_{i=0}^{2n} O_n(\lambda^n t^{2n-2i}) y^i\right) \partial_y^{(n-1)} \left(\left(\sum_{i=0}^{n} O_n(\mu t^{n-2i}) y^i\right) \frac{1}{t^2\lambda} \left(\sum_{i\ge 0} O_{n,i}\left(t^{-2i}\right) y^i\right)\right) \\
&= \lambda^{n-1}\mu t^{3n-2} \left(\sum_{i=0}^{2n} O_n(t^{-2i}) y^i\right) \partial_y^{(n-1)} \left(\left(\sum_{i=0}^{n} O_n(t^{-2i}) y^i\right) \left(\sum_{i\ge 0} O_{n,i}(t^{-2i}) y^i\right)\right) \\
&= \lambda^{n-1}\mu t^{3n-2} \left(\sum_{i=0}^{2n} O_n(t^{-2i}) y^i\right) \partial_y^{(n-1)} \left(\sum_{i\ge 0} O_{n,i}(t^{-2i}) y^i\right) \\
&= \lambda^{n-1}\mu t^{3n-2} \left(\sum_{i=0}^{2n} O_n(t^{-2i}) y^i\right) \left(\sum_{i\ge 0} O_{n,i}(t^{-2(i+n-1)}) y^i\right) \\
&= \lambda^{n-1}\mu t^{3n-2} \left(\sum_{i\ge 0} O_{n,i}(t^{-2n+2-2i}) y^i\right) \\
&= \lambda^{n-1}\mu t^{n} \left(\sum_{i\ge 0} O_{n,i}(t^{-2i}) y^i\right).
\end{aligned}$$

Therefore, the coefficient of y^i is $O_{n,i}(\lambda^{n-1}\mu t^{n-2i})$. Since we are only interested in computing these values for $0 \le i \le n$, we can absorb the dependence on i into the dependence on n, and obtain the coefficient of y^i is $O_n(\lambda^{n-1}\mu t^{n-2i})$.

The final statement regarding the inflection subscheme follows from Lemma 4.1.10 upon renaming the variables x and y. □

4.1.12 Identifying the inflection locus over additive curves in characteristic dividing n

We next aim to give a pleasant geometric description of how to interpret elements of the image $H^1(B, E^{\text{sm}}[n]) \hookrightarrow H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ in Lemma 4.1.17. The basic idea is that the issues completely stem from cuspidal genus 1 in characteristic dividing n. There are two geometric isomorphism classes of such curves: those where every point is an inflection point, and those with no smooth points being inflection points. An element will lie in the image of the cohomological map above so long as each fiber has *some* inflection point in the smooth locus. We now define the substack governing these curves where every point is an inflection point.

Definition 4.1.13. For $n \geq 3$, define $\mathscr{M}^{(n)}_{1,\text{inflection}} \subset \mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[1/n]$ as the substack parameterizing those tuples $(B, f : P \to B, \iota : C \to P) \in \mathscr{M}^{(n)}_{1,\text{cusp}}$ such that the restriction of Inflection(C, ι) to the smooth locus of $\iota(C)$ over B does not factor through a proper closed subscheme of B.

We next demonstrate that the above seemingly weaker definition implies that $C^{\text{sm}} \subset$ Inflection(C, ι).

Lemma 4.1.14. *Let $(B, f : P \to B, \iota : C \to P) \in \mathscr{M}^{(n)}_{1,\text{inflection}}$. For $\sigma : B \to C$ a section lying in the smooth locus, we have $\mathscr{O}_P(1)|_C \simeq \mathscr{O}_C(n\sigma)$ if and only if $C^{\text{sm}} \subset$* Inflection(C, ι).

Proof. Let C^{sm} denote the smooth locus of $C \to B$. We may identify $C^{\text{sm}} \simeq \mathbb{G}_a$ fppf locally on B in a way that σ maps to the identity. We wish to show the condition that the degree n invertible sheaf $\mathscr{O}_P(1)|_C$ has an nth root precisely when $C^{\text{sm}} \simeq$ Inflection(C, ι), and otherwise Inflection$(C, \iota) \cap C^{\text{sm}} = \emptyset$.

Every inflection point corresponds to an nth root of $\mathscr{O}_P(1)|_C$, so it is enough to show that when $\mathscr{O}_P(1)|_C \simeq \mathscr{O}_C(n\sigma)$, we have $C^{\text{sm}} \subset$ Inflection(C, ι).

Suppose therefore that $\mathscr{O}_C(\sigma)$ is an nth root of $\mathscr{O}_P(1)|_C$. Then, any point p in C^{sm} is n-torsion on $C^{\text{sm}} \simeq \mathbb{G}_a$ because we are working in characteristic dividing n. It remains to show that any n-torsion point lies in Inflection(C, ι). In terms of line bundles this means $\mathscr{O}_C(np - n\sigma) \simeq \mathscr{O}_C$, implying and so $\mathscr{O}_C(np) \simeq \mathscr{O}_C(n\sigma)$ and so the linear space at p spanned by the nth order neighborhood to C at p is contained in a hyperplane. This means p is an inflection point. □

Corollary 4.1.15. *Further, for any for any* $(B, f : P \to B, \iota : C \to P) \in \mathscr{M}^{(n)}_{1,\text{inflection}}$, $\text{Inflection}(C, B)$ *contains* C^{sm}, *for* C^{sm} *the smooth locus of* $C \to B$.

Proof. To verify the claim, we may work on an fppf cover on which $C \to B$ acquires a section lying in the smooth locus of C. It follows from Lemma 4.1.14 and the definition of $\mathscr{M}^{(n)}_{1,\text{inflection}}$ that there is a dense open over which $\mathscr{O}_P(1)|_C \simeq \mathscr{O}_C(n\sigma)$. The condition that two sheaves be isomorphic is a closed condition, and hence we obtain from another application of Lemma 4.1.14 that $\text{Inflection}(C, \iota) \supset C^{\text{sm}}$. □

Lemma 4.1.16. *The stack* $\mathscr{M}^{(n)}_{1,\text{inflection}}$ *is a closed substack of* $\mathscr{M}^{(n)}_{1,\text{cusp}}$.

Proof. It is enough to show $\mathscr{M}^{(n)}_{1,\text{inflection}}$ is a closed substack of $\mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}[1/n]$. This holds because for any map $B \to \mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}[1/n]$ given by $(B, f : P \to B, \iota : C \to P)$ the pullback to $\mathscr{M}^{(n)}_{1,\text{inflection}}$ is the closed subscheme of B which is the scheme theoretic image of $\text{Inflection}(C, \iota) \cap C^{\text{sm}}$ in B. □

We now give the geometric criterion for when an element of $H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ lies in the image of $H^1(B, E^{\text{sm}}[n])$. This is characterized by the unusual feature that the B point of $\mathscr{M}^{(n)}_1$ factors through a certain constructible subset of $\mathscr{M}^{(n)}_1$ which is not locally closed.

Lemma 4.1.17. *Suppose we are given a map* $B \to \mathscr{M}^{(n)}_1$ *corresponding to a genus* 1 *curve* $E \to B$ *and an* $\text{Aut}_{(E,e)/B}(B)$ *equivalence class of an n-covering in* $[T] \in H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ *(using Proposition 2.3.14). Then,* $[T]$ *lies in the image of* $H^1(B, E^{\text{sm}}[n]) \hookrightarrow H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ *if and only if the subscheme* $B \times_{\mathscr{M}^{(n)}_1} (\mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}/n\mathbb{Z})$ *factors through* $\mathscr{M}^{(n)}_{1,\text{inflection}} \subset \mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}/n\mathbb{Z}$.

Proof. Using the exact sequence (2.3.9), $H^1(B, E^{\text{sm}}[n]) \hookrightarrow H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ has image given as the kernel of the map $H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}}) \to H^0(B, E^{\text{sm}}/nE^{\text{sm}})$. Next, because the geometric fibers of E^{sm} are integral, the multiplication map only fails to be surjective on fibers where it is trivial. Further, the map $E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}}$ is only trivial on points where E is cuspidal and the characteristic of the residue field divides n, as can be checked on geometric fibers. Therefore, $E^{\text{sm}}/nE^{\text{sm}}$ is supported on the cuspidal locus in characteristic dividing n. Hence, to check if an element of $H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ maps to 0 in $H^0(B, E^{\text{sm}}/nE^{\text{sm}})$, we may freely replace B by $B \times_{\mathscr{M}^{(n)}_1} (\mathscr{M}^{(n)}_{1,\text{cusp}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}/n\mathbb{Z})$.

Now, we assume B factors through the cuspidal locus in characteristic dividing n. By Lemma 2.3.3 and Example 2.3.2 we may represent an element of $H^1(B, E^{\text{sm}} \xrightarrow{\times n} E^{\text{sm}})$ as a pair of a 1-cocycle and 0-cochain (s_{ij}, t_i) on a cover U_i of B such that $n \cdot s_{ij} = t_i - t_j$ on U_{ij}. However, since multiplication by n is the constant map to the identity, $n \cdot s_{ij} = t_i - t_j$

simplifies to the relation $t_i = t_j$. This implies that the above condition degenerates to the separate conditions that s_{ij} is a 1-cocycle and t_i is a 0-cocycle. Then, on the level of cocycles, the map $H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}}) \to H^0(B, E^{\mathrm{sm}}/nE^{\mathrm{sm}})$ simply sends (s_{ij}, t_i) to $t_i \in H^0(B, E^{\mathrm{sm}}/nE^{\mathrm{sm}}) = H^0(B, E^{\mathrm{sm}})$.

To conclude, it only remains to explain why the condition $t_i = 0$ corresponds to every point of E^{sm} being an inflection point. Our genus 1 curve lies in the image of $H^1(B, E^{\mathrm{sm}}[n]) \hookrightarrow H^1(B, E^{\mathrm{sm}} \xrightarrow{\times n} E^{\mathrm{sm}})$ (taken modulo $\mathrm{Aut}_{C/B}(B)$) if and only if its smooth locus is an $E^{\mathrm{sm}}[n]$ torsor. Because $t_i = 0$, we certainly have that t_i possesses an nth root, namely 0. Conversely, since $n \cdot t_i = 0$, $t_i = 0$ is the only section with an nth root. Therefore, when $t_i = 0$, we find $\mathscr{O}_P(1)|_C$ possesses an nth root, and so by Lemma 4.1.14, we have that $\mathrm{Inflection}(C, \iota) \supset C^{\mathrm{sm}}$. In particular, $\mathrm{Inflection}(C, \iota) \cap C^{\mathrm{sm}}$ does not factor through a proper closed subscheme of B, so B factors through $\mathscr{M}^{(n)}_{1,\mathrm{inflection}}$. Conversely, if B factors through $\mathscr{M}^{(n)}_{1,\mathrm{inflection}}$, we find $t_i = 0$ and the image of the class of E^{sm} in $H^0(B, E^{\mathrm{sm}})$ is 0. □

To conclude this section, we show how the dense open locus of the inflection subscheme contained in the smooth locus of $\iota(C)$ often is enough to recover C.

Lemma 4.1.18. *Let $n \geq 3$ and $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$. Let* $\mathrm{Inflection}(C, \iota)^\circ := \mathrm{Inflection}(C, \iota) \cap \iota(C)^{\mathrm{sm}}$ *for $\iota(C)^{\mathrm{sm}} \subset \iota(C)$ denote the smooth locus of $\iota(C) \to B$. Let $E^{\mathrm{sm}} := \mathrm{Pic}^0_{\iota(C)/B}$. Suppose $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$ lies in the image $H^1(B, E^{\mathrm{sm}}[n]) \to H^1(B, E^{\mathrm{sm}} \xrightarrow{\times[n]} E^{\mathrm{sm}})$ (taken modulo* $\mathrm{Aut}_{(E,e)/B}(B)$ *under the identification coming from Proposition 2.3.14). Then,* $\mathrm{Inflection}(C, \iota)^\circ$ *is the $E^{\mathrm{sm}}[n]$ torsor which gives rise to $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}(B)$ under the identification of Proposition 2.3.14.*

Proof. First let us check $\mathrm{Inflection}(C, \iota)^\circ$ is an $E^{\mathrm{sm}}[n]$ torsor when $(B, f : P \to B, \iota : C \to P, \tau : B \to C) \in \widetilde{\mathscr{M}}^{(n)}_{1,\mathrm{sing}}$ lies in the image $H^1(B, E^{\mathrm{sm}}[n]) \to H^1(B, E^{\mathrm{sm}} \xrightarrow{\times[n]} E^{\mathrm{sm}})$. As a first step, we construct an action of $E^{\mathrm{sm}}[n]$ on $\mathrm{Inflection}(C, \iota)^\circ$. Functorially, points of $\mathrm{Inflection}(C, \iota)^\circ$ can be described as sections $p : B \to \iota(C)^{\mathrm{sm}}$ such that $\mathscr{O}_{\iota(C)}(np) \simeq \mathscr{O}_P(1)|_{\iota(C)}$. Now, the translation action of $\mathrm{Pic}^0_{\iota(C)/B}$ on $\mathrm{Pic}^n_{\iota(C)/B}$ acts as the nth tensor power of the action on $\mathrm{Pic}^1_{\iota(C)/B}$. Therefore translation by any point of $\mathrm{Pic}^0_{\iota(C)/B}[n]$ acts trivially $\mathrm{Pic}^n_{\iota(C)/B}$. In particular, for $q \in E^{\mathrm{sm}}[n]$, letting $p + q \in \iota(C)^{\mathrm{sm}}$ denote the output of the action map $E^{\mathrm{sm}} \times \iota(C)^{\mathrm{sm}} \to \iota(C)^{\mathrm{sm}}$, we obtain $\mathscr{O}_{\iota(C)}(n(p + q)) \simeq \mathscr{O}_{\iota(C)}(np) \simeq \mathscr{O}_P(1)|_{\iota(C)}$. This implies that $p + q$ again lies in $\mathrm{Inflection}(C, \iota)^\circ$. This gives the desired action of $E^{\mathrm{sm}}[n]$ on $\mathrm{Inflection}(C, \iota)^\circ$.

We next wish to show $\mathrm{Inflection}(C, \iota)^\circ$ is an $E^{\mathrm{sm}}[n]$ torsor. Using Lemma 4.1.17, this is always the case in characteristic not dividing n, and additionally holds in characteristic dividing n when $\mathscr{O}_P(1)|_{\iota(C)}$ has an nth root fppf locally. Therefore, we may verify the claim after after base change along $\operatorname{Spec}\mathbb{Z}/n\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}$, and assume that $n = 0$ on the base scheme. In this case, our assumption that our given map $B \to \mathscr{M}^{(n)}_{1,\mathrm{cusp}}$ corresponds to an element in the image $H^1(B, E^{\mathrm{sm}}[n]) \to H^1(B, E^{\mathrm{sm}} \xrightarrow{\times[n]} E^{\mathrm{sm}})$ modulo $\operatorname{Aut}_{(E,e)/B}(B)$ implies $B \to \mathscr{M}^{(n)}_{1,\mathrm{cusp}}$ factors through $\mathscr{M}^{(n)}_{1,\mathrm{inflection}}$ by Lemma 4.1.17. By Corollary 4.1.15, we further find that $\operatorname{Inflection}(C,\iota)^\circ \simeq C^{\mathrm{sm}}$, which is indeed an $E^{\mathrm{sm}}[n] \simeq E^{\mathrm{sm}}$ torsor.

Finally, unwinding the proof of Proposition 2.3.14, (see also the proof of Lemma 2.3.10 and Lemma 2.3.12) shows that $\operatorname{Inflection}(C,\iota)^\circ$ does give rise to the the genus 1 curve in a Brauer-Severi scheme $C \to P$ corresponding to the map $B \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$. For the reader's convenience, we briefly recall the idea of this bijection. We are given E^{sm} and $\operatorname{Inflection}(C,\iota)^\circ$ so that $\operatorname{Inflection}(C,\iota)^\circ$ is a torsor for $E^{\mathrm{sm}}[n]$ embedded in P. We may embed $E^{\mathrm{sm}}[n]$ into an $n-1$ dimensional projective bundle by ne, for e the identity section. We then use $\operatorname{Inflection}(C,\iota)^\circ$ to dictate the transition functions over a cover which allows us to construct an E^{sm} torsor T embedded in P over B and containing $\operatorname{Inflection}(C,\iota)^\circ$. Then, the closure of T in P yields C. □

4.2 Bounding the number of inflection points with a section

In this section, we bound the number of n-coverings of a singular genus 1 associated to degree 2 covers of $\operatorname{Spec}\mathbb{Z}$ (though much can be generalized to any global field) such that their associated inflection subscheme has a section. Because the inflection subscheme is proper, having a section is equivalent to being trivial over $\operatorname{Spec}\mathbb{Q}$. The main results are Proposition 4.2.11, which handles the case that the degree 2 cover is normal and Lemma 4.2.14 which handles the non-normal case. We then combine these in Proposition 4.2.15 to bound the total size of these kernels summed over all degree 2 covers of bounded discriminant. The approach is quite loose. The bounds we obtain can likely be significantly improved.

The key to our whole approach is the next lemma which will be used to show that when the associated inflection subscheme has a section, it is pulled back from a μ_n torsor on the base, at least away from the branch locus of g.

Lemma 4.2.1. *Suppose B is an integral domain and $g : X \to B$ is a degree 2 locally free cover. The kernel of the composite map $H^1(X, \mu_n) \to H^1(B, g_*\mu_n) \to H^1(B, g_*\mu_n/\mu_n)$ is contained in the image of the pullback map $H^1(B, \mu_n) \to H^1(X, \mu_n)$.*

Proof. Let us begin with $[T] \in H^1(X, \mu_n)$ and trace where T goes under the above maps. We first send T to the Weil restriction as an abelian sheaf $g_*T \in H^1(B, g_*\mu_n)$. Next, g_*T is sent to its image $S \in H^1(B, g_*\mu_n/\mu_n)$ which we are assuming is the trivial torsor. The condition that S is trivial means that S has a section $B \to S$, and therefore the preimage of this section in $g_* T$ defines a μ_n torsor M over B with an inclusion $M \to g_* T$. By adjunction, this yields a map $g^* M \to T$ of μ_n torsors which must therefore be an isomorphism. Hence, T is the pullback of $[M] \in H^1(B, \mu_n)$. □

Later, in Proposition 4.2.11, we will need to identify the map appearing in Lemma 4.2.1 with the natural map. For non-étale $X \to B$, it is not the same, as can be deduced from Example 4.2.3 below. However, if we restrict to the étale locus of g, it will be the same. The key input for this is the following lemma.

Lemma 4.2.2. *For $g : X \to B$ a finite locally free cover of arbitrary degree, the map $H^1(X, \mu_n) \to H^1(B, g_* \mu_n)$ given by Weil restriction is a left inverse to the map $H^1(B, g_* \mu_n) \to H^1(X, g^* g_* \mu_n) \to H^1(X, \mu_n)$ given by pullback and the adjunction. That is, if we first pullback and apply adjunction, and then take the Weil restriction, we obtain the identity map on $H^1(B, g_* \mu_n)$.*

Proof. One way to see this is in terms of Čech cocycles. Begin with a torsor $T \in H^1(B, g_* \mu_n)$. To describe this torsor in terms of cocycles, choose a trivializing cover $U_i \to B$ of T. The torsor T can then be described as elements $t_{ij} \in g_* \mu_n(U_{ij})$. Under the identification $(g_* \mu_n)(U_{ij}) = \mu_n(U_{ij} \times_B X)$, we let $s_{ij} \in \mu_n(U_{ij} \times_B X)$ denote the cocycle data corresponding to $t_{ij} \in (g_* \mu_n)(U_{ij})$. When we apply the pullback map, we obtain a description of $g^* T$ as $g^*(t_{ij}) \in (g^* g_* \mu_n)(U_{ij}) = (g^* \mu_n)(U_{ij} \times_B X)$. By definition, the adjunction map then sends this torsor to the torsor S described in terms of cocycles again by $s_{ij} \in \mu_n(U_{ij} \times_B X)$, with s_{ij} as defined above. By definition, taking the Weil restriction then sends s_{ij} to t_{ij}. □

Example 4.2.3. Although Weil restriction is a left inverse to pullback and adjunction in Lemma 4.2.2, it is not in general a two-sided inverse. To see this, we can take $n = p$, $X = \operatorname{Spec} \mathbb{F}_p[\varepsilon]/(\varepsilon^2)$, and $B = \operatorname{Spec} \mathbb{F}_p$. We then have that $g_* \mu_p \simeq \mathbb{G}_a \times \mu_p$ and $H^1(B, g_* \mu_p) = H^1(B, \mathbb{G}_a) \times H^1(B, \mu_p) = 1$. The vanishing of $H^1(B, \mu_p)$ follows from the Kummer exact sequence. However, $H^1(X, \mu_p) \simeq \mathbb{F}_p \neq 1$, as can be deduced from the Kummer exact sequence on X, as it is given as the cokernel of the pth power map on $\mathbb{G}_m(X)$.

Our next step is to bound the size of $H^1(U, \mu_n)$, so that we can bound the size of those torsors whose associated inflection subscheme has a section using Lemma 4.2.1. We introduce some notation used for the next several results.

Notation 4.2.4. Let $n \geq 1$ be an integer. Let $B = \operatorname{Spec} \mathscr{O}_L$ for $\mathscr{O}_L$ the ring of integers of a global field L with characteristic not dividing n. Let $g : X \to B$ be a locally free finite degree 2 cover which is étale over a dense open subscheme $U \subset B$. Let η denote the generic point of B and let $Z := B - U$ denote the reduced closed complement and let r denote the number of points of Z.

The following general lemma on restricting torsors will be useful at several points in the argument.

Lemma 4.2.5. *Suppose X is a normal integral scheme and G is a finite flat group scheme over X. For $U \subset X$ a dense open subscheme, the restriction map $H^1(X, G) \to H^1(U, G|_U)$ is injective.*

Proof. It is enough to show that any two G torsors isomorphic on a dense open must be isomorphic. To prove this in turn, it suffices to show that a G torsor T with a section over U has a section over X. Because G is finite, the closure of the image of $U \to T$ defines a subscheme of T mapping finitely and birationally to X. Since X is normal, the map is an isomorphism and T is trivial. □

We now carry out a cohomological diagram chase in Lemma 4.2.6. This is used as input for our desired bound of n-coverings associated to normal X whose inflection subscheme has a section in Proposition 4.2.11.

Lemma 4.2.6. *With notation as in Notation 4.2.4, assume additionally that X is normal. The kernel of the composite map $H^1(X, \mu_n) \to H^1(g^{-1}(U), \mu_n) \to H^1(U, g_*\mu_n) \to H^1(U, (g_U)_*\mu_n/\mu_n)$ has order at most $n^{2r} \cdot \#H^1(B, \mu_n)$.*

Proof. By Lemma 4.2.5, the map $H^1(X, \mu_n) \to H^1(g^{-1}(U), \mu_n)$ is injective. Hence, it suffices to bound the size of the kernel of $H^1(g^{-1}(U), \mu_n) \to H^1(U, (g_U)_*\mu_n/\mu_n)$. By Lemma 4.2.1 applied to the map $g^{-1}(U) \to U$, the size of this kernel is bounded by $\#H^1(U, \mu_n)$. To complete the proof, we will show $\#H^1(U, \mu_n) \le n^{2r} \cdot \#H^1(B, \mu_n)$. From the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(B, \mathbb{G}_m)/nH^0(B, \mathbb{G}_m) & \longrightarrow & H^1(B, \mu_n) & \longrightarrow & H^1(B, \mathbb{G}_m)[n] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & H^0(U, \mathbb{G}_m)/nH^0(U, \mathbb{G}_m) & \longrightarrow & H^1(U, \mu_n) & \longrightarrow & H^1(U, \mathbb{G}_m)[n] & \longrightarrow & 0, \end{array} \tag{4.2.1}$$

the snake lemma shows we have an exact sequence

$$\operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0. \tag{4.2.2}$$

In order to show $\#H^1(U, \mu_n)$ is at most $H^1(B, \mu_n) \cdot n^{2r}$ is is enough to show $\# \operatorname{coker}\beta \le n^{2r}$. In turn, it is enough to bound both $\# \operatorname{coker}\alpha$ and $\# \operatorname{coker}\gamma$ by n^r.

To bound both $\# \operatorname{coker} \alpha$ and $\# \operatorname{coker} \gamma$, we will use the divisor exact sequence. Let $j : U \to B$ and $i : Z \to B$ denote the inclusions. Recall that B is normal and 1-dimensional, hence regular, hence locally factorial. We therefore obtain the sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow j_* \mathbb{G}_m \longrightarrow i_* \underline{\mathbb{Z}} \longrightarrow 0 \tag{4.2.3}$$

for $\underline{\mathbb{Z}}$ the constant sheaf associated to $\mathbb{Z}$ on Z. Observe that $H^1(Z, \underline{\mathbb{Z}}) = 0$ since there are no nontrivial homomorphisms from the profinite absolute Galois groups of points of Z to $\mathbb{Z}$. Further, $H^0(Z, \underline{\mathbb{Z}}) = \mathbb{Z}^r$, since Z has r connected components. From this, we obtain the exact sequence

$$H^0(B, \mathbb{G}_m) \longrightarrow H^0(U, \mathbb{G}_m) \longrightarrow \mathbb{Z}^r \longrightarrow H^1(B, \mathbb{G}_m) \longrightarrow H^1(U, \mathbb{G}_m) \longrightarrow 0. \tag{4.2.4}$$

We first use the above to bound $\# \operatorname{coker} \alpha$ by n^r. Let $Q := \operatorname{coker}\left(H^0(B, \mathbb{G}_m) \to H^0(U, \mathbb{G}_m)\right)$. By right exactness of the functor $\otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, we have $\operatorname{coker} \alpha = Q/nQ$. But Q injects into $\mathbb{Z}^r$ by (4.2.4), and hence has rank at most r, so $\# Q/nQ \leq n^r$.

To conclude, we bound $\# \operatorname{coker} \gamma$ by n^r. We have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(B, \mathbb{G}_m)[n] & \longrightarrow & H^1(B, \mathbb{G}_m) & \longrightarrow & \ker(H^1(B, \mathbb{G}_m) \to H^2(B, \mu_n)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & H^1(U, \mathbb{G}_m)[n] & \longrightarrow & H^1(U, \mathbb{G}_m) & \longrightarrow & \ker(H^1(U, \mathbb{G}_m) \to H^2(U, \mu_n)) & \longrightarrow & 0. \end{array} \tag{4.2.5}$$

Since $H^1(B, \mathbb{G}_m) \to H^1(U, \mathbb{G}_m)$ is a surjection via (4.2.4), the snake lemma applied to (4.2.5) yields a surjection $\ker(\delta) \to \operatorname{coker} \gamma$. By (4.2.4), $\ker(H^1(B, \mathbb{G}_m) \to H^1(U, \mathbb{G}_m))$ is generated by at most r elements, and therefore the same holds for the subgroup $\ker(\delta) \subset \ker(H^1(B, \mathbb{G}_m) \to H^1(U, \mathbb{G}_m))$. It follows that $\operatorname{coker} \gamma$ also has at most r generators. Since it is a $\mathbb{Z}/n\mathbb{Z}$ module, it has size at most n^r. □

Remark 4.2.7. When n is odd, it seems the kernel of the map in Lemma 4.2.6 is uniformly bounded. This uniform bound for odd n can be deduced from a version of Abhyankar's lemma. Suppose K/L is a degree 2 field extension and v is a place of K lying over a place w of L, neither of which have residue characteristic 2. The above mentioned version of Abhyankar's lemma implies that if a μ_n torsor T over K is the pullback of a μ_n torsor S over L, and T is unramified at a place v, then S is unramified at w.

This observation can be used to somewhat improve the bounds somewhat in the case n is odd. However, the bounds we establish below are sufficient for our purposes, and so we do not take up this optimization.

Thus far we have investigated elements of $H^1(X, \mu_n)$ whose associated inflection subscheme has a section. The following lemma allows us to relate this to elements of the hypercohomology group $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ whose associated inflection subscheme has a section.

Lemma 4.2.8. *Let $B = \operatorname{Spec} \mathbb{Z}$ and let $g : X \to B$ be a connected finite locally free cover of arbitrary degree and n a positive integer. If n is even and g does not factor through $\operatorname{Spec} \mathbb{Z}[i]$, the natural composite map*

$$H^1(X, \mu_n) \simeq H^1(X, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \to H^1(B, g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m) \to H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m). \tag{4.2.6}$$

is a surjection with kernel $\mathbb{Z}/2\mathbb{Z}$. Otherwise, (4.2.6) *is an isomorphism.*

Proof. The distinguished triangle associated to the three complexes

$$\begin{array}{ccccc} \mathbb{G}_m & \longrightarrow & g_*\mathbb{G}_m & \longrightarrow & g_*\mathbb{G}_m/\mathbb{G}_m \\ \downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} & & \downarrow \\ \mathbb{G}_m & \longrightarrow & g_*\mathbb{G}_m & \longrightarrow & g_*\mathbb{G}_m/\mathbb{G}_m \end{array} \tag{4.2.7}$$

on B gives rise to an exact sequence on hypercohomology

$$H^1(B, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \xrightarrow{\quad \alpha \quad} H^1(B, g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m) \longrightarrow$$

$$\longrightarrow H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \longrightarrow H^2(B, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m). \tag{4.2.8}$$

By computing the first and last terms using the exact sequence associated to the distinguished triangle (2.3.5), we find $H^1(B, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \simeq \mathbb{Z}/\gcd(2, n)\mathbb{Z}$ and $H^2(B, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \simeq 0$.

We also find

$$H^1(X, \mu_n) \simeq H^1(X, \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \simeq H^1(B, g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m), \tag{4.2.9}$$

as we now explicate. The first isomorphism holds due to the quasi-isomorphism of complexes $\mu_n \simeq \left(\mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m\right)$. The second isomorphism can be deduced, for example, by applying (2.3.5) to both $\mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m$ on X and $g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m$ on B and then using Lemma 3.3.13. Therefore, in the case n is odd, we obtain our desired isomorphism $H^1(X, \mu_n) \simeq H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$.

It only remains to analyze the case that n is even. We have already seen the map

$H^1(X, \mu_n) \simeq H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ is always surjective, so we now aim to identify the kernel. Under the identification (4.2.9), the kernel can be identified with the image of $\mathbb{Z}/2\mathbb{Z} \simeq H^1(B, \mu_n) \to H^1(X, \mu_n)$. We can interpret this map geometrically as pulling back a μ_n torsor over B to the corresponding μ_n torsor over X. Since n is even, there is precisely one nontrivial μ_n torsor over B given by $\operatorname{Spec} \mathbb{Z}[i]$. We wish to identify when $\operatorname{Spec} \mathbb{Z}[i]$ pulls back to the trivial μ_n torsor on X. The pullback is given by $\operatorname{Spec} \mathbb{Z}[i] \times_B X$. Therefore, the torsor is trivial precisely when $\operatorname{Spec} \mathbb{Z}[i] \times_B X \to X$ has a section, or equivalently when X has a map to $\operatorname{Spec} \mathbb{Z}[i]$. Hence, this pullback map is 0 precisely when g factors through $\operatorname{Spec} \mathbb{Z}[i]$. □

We next use the above to explain the relation between our hypercohomology group and the usual n-Selmer group of a number field. Both Remark 4.2.9 and Lemma 4.2.10 will not be used in what follows, but may be helpful to those who have previously encountered the n-Selmer group of a number field.

Remark 4.2.9 (Relation to n-Selmer group of a number field)**.** Let $B = \operatorname{Spec} \mathbb{Z}$, let K be a number field, let $\mathscr{O}_K$ be the associated ring of integers, and let $g : \operatorname{Spec} \mathscr{O}_K =: X \to B$ denote the structure map. Then, the hypercohomology group $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ is closely related by Lemma 4.2.8 to $H^1(X, \mu_n)$, the n-Selmer group of K.

We now explain the relation between these two groups in more detail. Classically, the n-Selmer group of a number field is defined as

$$\operatorname{Sel}_n(K) := \left\{\alpha \in K^\times : \text{ there exists a fractional ideal } \mathfrak{a} \text{ of } K \text{ with } (\alpha) = \mathfrak{a}^n\right\} / \left(K^\times\right)^n.$$

From a cohomological perspective, this n-Selmer group is equivalently described as the flat cohomology group $H^1(X, \mu_n)$, as we show below in Lemma 4.2.10.

Sticking to the cohomological perspective, for general bases B and finite locally free covers $g : X \to B$, we have a natural map $H^1(X, \mu_n) \to H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ given in (4.2.6). This map has previously made an appearance in (3.3.3). For general bases B, one can gain a fairly concrete understanding of the kernel and cokernel of this map by relating it to invariants such the units in the base B, the class group or Picard group of B, the Brauer group of B, etc.

In the case of interest where $B = \operatorname{Spec} \mathbb{Z}$, the comparison between these two groups is fairly clean. The map from the n-Selmer group of the number field to $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ is surjective with kernel of order at most 2 by Lemma 4.2.8.

Lemma 4.2.10. *For K a number field and $X = \operatorname{Spec} \mathscr{O}_K$, the natural map $H^1(X, \mu_n) \to \operatorname{Sel}_n(K)$ is an isomorphism.*

Proof. Geometrically, elements of $H^1(X, \mu_n)$ correspond to a n-torsion invertible sheaves $\mathscr{L}$ on X together with a choice of isomorphism $\iota : \mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$, modulo automorphisms of $\mathscr{L}$. To see the claimed equivalence, we can view $\mathrm{Sel}_n(K) \subset H^1(K, \mu_n) \simeq K^\times/(K^\times)^n$. There is a restriction map $H^1(X, \mu_n) \to H^1(K, \mu_n)$. We claim the image of this map is contained in $\mathrm{Sel}_n(K)$. Indeed, viewing an element of $H^1(X, \mu_n)$ as an invertible sheaf $\mathscr{L}$ with a specified isomorphism $\mathscr{L}^{\otimes n} \simeq \mathscr{O}_X$, the ideal corresponding to $\mathscr{L}^{\otimes n}$ under this isomorphism lies in $\mathrm{Sel}_n(K)$ because it has valuation which is a multiple of n at every closed point of X. The resulting map $H^1(X, \mu_n) \to \mathrm{Sel}_n(K)$ is an injection by Lemma 4.2.5. It is an isomorphism because both $H^1(X, \mu_n)$ and $\mathrm{Sel}_n(K)$ extensions of $\mathrm{Cl}(K)[n]$ by $\mathscr{O}_K^\times/\left(\mathscr{O}_K^\times\right)^n$, and therefore have the same size. □

Next, we use the above results to bound the number of elements of $H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ with a section. This will enable us to achieve our desired bound on the number of n-coverings whose associated inflection subscheme has a section when X is normal.

Proposition 4.2.11. *With notation as in Notation 4.2.4, assume also that X is normal. Then, for ρ the natural map appearing in Lemma 4.2.8,*

$$\#\ker\left(H^1(X, \mu_n) \xrightarrow{\rho} H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right) = O_{n,B}(n^{2r}). \tag{4.2.10}$$

If additionally $B = \operatorname{Spec}\mathbb{Z}$, we also have

$$\#\ker\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right) = O_n(n^{2r}). \tag{4.2.11}$$

Proof. Using Lemma 4.2.8, (4.2.11) follows from (4.2.10), so it remains to prove (4.2.10).

Consider the composition

$$H^1(X, \mu_n) \xrightarrow{\alpha} H^1(U, g_*\mu_n/\mu_n) \xrightarrow{\beta} H^1(\eta, (g_\eta)_*\mu_n/\mu_n) \xrightarrow{\gamma} H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m),$$

where the map α is given as the composite of restriction to U with the map of Lemma 4.2.1. Note that g is étale over U so as to ensure that the above composition is the natural restriction map ρ. This follows from Lemma 4.2.2 once we verify that the natural pullback and adjunction map $H^1(U, (g_U)_*\mu_n) \to H^1(g^{-1}(U), \mu_n)$ is an isomorphism. For this, from the Leray spectral sequence, it is enough to know $R^1(g_U)_*\mu_n = 0$. Indeed, $R^1(g_U)_*\mu_n = 0$ from the Kummer exact sequence because $R^1(g_U)_*\mathbb{G}_m = 0$ and the multiplication by n map on $(g_U)_*\mathbb{G}_m$ is surjective. The surjectivity follows from the fact that g_U is étale, and so $(g_U)_*\mathbb{G}_m$ is a torus. (If the map were instead ramified at a point whose residue characteristic divides n, the multiplication by n map on $g_*\mathbb{G}_m$ would fail to be surjective.)

Remark 4.2.12. The issue for checking $\gamma \circ \beta \circ \alpha$ agrees ρ boils down to verifying commutativity of the square

$$\begin{array}{ccc} H^1(g^{-1}(U), \mu_n) & \longrightarrow & H^1(g^{-1}(U), \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^1(U, g_*\mu_n) & \longrightarrow & H^1(U, g_*\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m). \end{array} \tag{4.2.12}$$

Since g is étale over U, all the maps above will be isomorphisms, and the square will commute. However, when g fails to be étale, and moreover $R^1g_*\mathbb{G}_m \neq 0$, the above square may fail to commute. One can use Example 4.2.3 to produce an example where this fails to commute (when g is not étale over U).

Having identified the map in (4.2.10), We wish to show $\# \ker \gamma \circ \beta \circ \alpha = n^{2r} \cdot O_{n,B}(1)$. For this, is suffices to separately show $\# \ker(\alpha) = n^{2r} \cdot O_{n,B}(1)$, β is injective, and γ is an isomorphism.

We have $\# \ker(\alpha) = n^{2r} \cdot O_{n,B}(1)$ by Lemma 4.2.6. Next, β is injective by Lemma 4.2.5 and spreading out. This uses that g is étale over U so that $g_*\mu_n$ is finite locally free by [BLR90, §7.6, Proposition 5]. Finally, γ is an isomorphism by the exact sequence (2.3.9) because g is generically étale so that $g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m$ is a multiplication by n map between tori over η, hence surjective. □

The remainder of the section is essentially concerned with bootstrapping the above result to non-normal schemes X. The issue is that Lemma 4.2.5 no longer applies, and so we now need to bound the size of the kernel of the restriction $H^1(X, \mu_n) \to H^1(g^{-1}(U), \mu_n)$. We will do so by understanding the kernel of the restriction from X to its normalization, and then composing with the further restriction to U.

Lemma 4.2.13. *Keeping notation as in Notation 4.2.4, let $B = \operatorname{Spec} \mathbb{Z}$ and let $\widetilde{g} : \widetilde{X} \to X$ denote the normalization of X. Then,*

$$\# \ker \left(H^1(X, \mu_n) \to H^1(\widetilde{X}, \mu_n) \right) \leq n^{3+r}. \tag{4.2.13}$$

Proof. By the Kummer exact sequence,

$$\begin{aligned}&\#\ker\left(H^1(X,\mu_n)\to H^1(\widetilde{X},\mu_n)\right)\\&\le\#\ker\left(\mathbb{G}_m(X)/n\mathbb{G}_m(X)\to\mathbb{G}_m(\widetilde{X})/n\mathbb{G}_m(\widetilde{X})\right)\cdot\#\ker\left(H^1(X,\mathbb{G}_m)[n]\to H^1(\widetilde{X},\mathbb{G}_m)[n]\right)\end{aligned}$$

To finish, we will bound $\#\ker\left(H^1(X,\mathbb{G}_m)[n]\to H^1(\widetilde{X},\mathbb{G}_m)[n]\right)$ by n^{r+1} and we will bound $\#\ker\left(\mathbb{G}_m(X)/n\mathbb{G}_m(X)\to\mathbb{G}_m(\widetilde{X})/n\mathbb{G}_m(\widetilde{X})\right)$ by n^2.

The easier part is the n^2 bound. Observe, $\#\frac{\mathbb{G}_m(X)}{n\mathbb{G}_m(X)}\le n^2$ by Dirichlet's unit theorem because $\mathbb{G}_m(X)$ injects into $\mathbb{G}_m(\widetilde{X})$.

We finish the proof by bounding $\#\ker\left(H^1(X,\mathbb{G}_m)[n]\to H^1(\widetilde{X},\mathbb{G}_m)[n]\right)$. To this end, we recall a general fact about degree 2 locally free covers of $\operatorname{Spec}\mathbb{Z}$. Write $X=\operatorname{Spec}R$ for R an order in the regular degree 2 algebra $\mathscr{O}_K$ over $\mathbb{Z}$, with $\widetilde{g}:\widetilde{X}=\operatorname{Spec}\mathscr{O}_K\to\operatorname{Spec}\mathbb{Z}$ the structure map. (Here, $\mathscr{O}_K$ will either be a ring of integers or $\mathbb{Z}\times\mathbb{Z}$). Let $\mathfrak{f}$ denote the conductor of R in $\mathscr{O}_K$. We claim that in fact $\mathfrak{f}=(f)$ for some $f\in\mathbb{Z}$. One may deduce this from the following observation: if ω is a generator of $\mathscr{O}_K$ over $\mathbb{Z}$, then we can take f so that $f\omega$ is a generator for R over $\mathbb{Z}$. In this case (f) is the conductor.

Let f denote the conductor associated to $\widetilde{X}\to X$ as in the previous paragraph. From the exact sequence appearing in the proof of [Neu99, Chapter 1, Theorem 12.12] (coming from [Neu99, Chapter 1, Theorem 12.9] and [Neu99, Chapter 1, Theorem 12.11]) we have

$$(\mathbb{Z}/f\mathbb{Z})^\times\longrightarrow H^1(X,\mathbb{G}_m)\longrightarrow H^1(\widetilde{X},\mathbb{G}_m)\longrightarrow 0. \tag{4.2.14}$$

Let A denote the image of the map $(\mathbb{Z}/f\mathbb{Z})^\times\to H^1(X,\mathbb{G}_m)$. Taking n-torsion of the sequence

$$0\longrightarrow A\longrightarrow H^1(X,\mathbb{G}_m)\longrightarrow H^1(\widetilde{X},\mathbb{G}_m)\longrightarrow 0 \tag{4.2.15}$$

yields an exact sequence

$$0\longrightarrow A[n]\longrightarrow H^1(X,\mathbb{G}_m)[n]\longrightarrow H^1(\widetilde{X},\mathbb{G}_m)[n] \tag{4.2.16}$$

and hence an isomorphism $A[n]\simeq\ker\left(H^1(X,\mathbb{G}_m)[n]\to H^1(\widetilde{X},\mathbb{G}_m)[n]\right)$. For each odd prime $p\mid f$, the unit group of the p-Sylow subgroup of $(\mathbb{Z}/f\mathbb{Z})^\times$ is cyclic, while for $p=2$ there may be 2 generators. Therefore, $(\mathbb{Z}/f\mathbb{Z})^\times$ has at most $r+1$ generators. Hence, the same holds for A and $A[n]$. In particular,

$$\#\ker\left(H^1(X, \mathbb{G}_m)[n] \to H^1(\widetilde{X}, \mathbb{G}_m)[n]\right) = \#A[n] \le n^{r+1}.$$

□

Combining the previous results yields the following bound on the number of n-coverings which become trivial on the generic fiber in the non-normal case.

Lemma 4.2.14. *Let $B = \operatorname{Spec}\mathbb{Z}$ and keep notation as in Notation 4.2.4. Then,*

$$\begin{aligned}&\#\ker\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right)\\ &\le n^{3r} \cdot O_n(1).\end{aligned} \tag{4.2.17}$$

Proof. Using Lemma 4.2.8, we find

$$\begin{aligned}&\#\ker\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right)\\ &\le \#\ker\left(H^1(X, \mu_n) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right).\end{aligned}$$

We will bound the latter kernel. Let $\widetilde{g} : \widetilde{X} \to B$ be the normalization of X. Because $\eta \to X$ factors through $\widetilde{X}$,

$$\#\ker\left(H^1(X, \mu_n) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right)$$

is bounded by

$$\#\ker\left(H^1(X, \mu_n) \to H^1(\widetilde{X}, \mu_n)\right) \cdot \#\ker\left(H^1(\widetilde{X}, \mu_n) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right).$$

From Proposition 4.2.11, the second term is bounded by $O_n(n^{2r})$. From Lemma 4.2.13, the first term is bounded by $n^{r+3} = O_n(n^r)$. □

Using the above bound, we are prepared to bound the total number of n-coverings over all quadratic algebras of bounded discriminant which are trivial on the generic fiber.

Proposition 4.2.15. *Let $B = \operatorname{Spec}\mathbb{Z}$. For $g : X \to B$ ranging over all finite degree* 2 *locally* *free covers,*

$$\sum_{g:X\to B,\mathrm{disc}(g)\le Y} \#\ker\left(H^1(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m) \to H^1(\eta, (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} (g_\eta)_*\mathbb{G}_m/\mathbb{G}_m)\right) \tag{4.2.18}$$

is $O_n(Y(\log Y)^{n^3-1})$.

Proof. Keeping notation as in Notation 4.2.4, we can write $n^{3r} = \prod_{p|\mathrm{disc}(g)} n^3$. By Lemma 4.2.14, up to a constant depending only on n, we can therefore bound (4.2.18) by

$$\sum_{g:X\to B,\mathrm{disc}(g)\le Y} \prod_{p|\mathrm{disc}(g)} n^3. \tag{4.2.19}$$

Recall next that given any integer d, there is a most one degree 2 cover of $\operatorname{Spec}\mathbb{Z}$ of discriminant d. One way to prove this is that any degree 2 cover of $\operatorname{Spec}\mathbb{Z}$ can be expressed as $X = \operatorname{Spec}\mathbb{Z}[x]/(x^2 + ax + b)$ for a unique $a \in \{0, 1\}$ and $b \in \mathbb{Z}$. A geometric way to see this is to use the embedding $X \to \mathbb{P}(g_*\mathscr{O}_X)$ over $\operatorname{Spec}\mathbb{Z}$.

Therefore, since each discriminant appears at most once in (4.2.19), it is enough to bound

$$\sum_{d\le Y}\left(\prod_{p|d} n^3\right).$$

The remainder of this proof is a standard application of a Tauberian theorem. Taking $a_d := \prod_{p|d} n^3$, we are then trying to sum the first Y coefficients of the Dirichlet series $\sum_{d=1}^\infty a_d d^{-s}$. We can write this in the product expansion

$$\sum_d a_d d^{-s} = \prod_{\text{p prime}}\left(1 + n^3 \cdot \left(\frac{1}{p^{-s}} + \frac{1}{p^{-2s}} + \frac{1}{p^{-3s}} + \frac{1}{p^{-4s}} + \cdots\right)\right).$$

Let $\Re(s)$ denote the real part of s. We claim that on $\Re(s) > 1$, $\sum_d a_d d^{-s}$ is bounded by a constant times $\zeta(s)^{n^3}$, for $\zeta(s)$ the Riemann Zeta function. This bound holds because on $\Re(s) > 1$, we can bound the part of the product expansion corresponding to p by

$$1 + n^3 \cdot \left(\frac{1}{p^{-s}} + \frac{1}{p^{-2s}} + \frac{1}{p^{-3s}} + \frac{1}{p^{-4s}} + \cdots\right) \le 1 + n^3 p^{-s} + O_n(p^{-2s}).$$

Using the above, we find $\sum_d a_d d^{-s}$ converges whenever $\Re(s) > 1$. Further, since $\zeta(s)$ has a pole of order 1 at 1 and no other poles with real part $s = 1$, we find $\sum_d a_d d^{-s}$ has a pole of order n^3 at 1 and no other poles in the region $\Re(s) \ge 1$. It follows from a Tauberian theorem as in [Nar83, Corollary p. 121-122] that $\sum_{d \leq Y} a_d = O(Y\left(\log Y\right)^{n^3-1})$. □

Chapter 5

Geometry of numbers

In this section, the group scheme G_n and the affine space V_n will play a central role, so it may be useful to review their definitions given in Definition 2.1.1 and Definition 2.1.2. Recall that we use the notation a_i, b_j associated to a pair (q, ξ) as $(q, \xi) := (\sum_{i=0}^{2} a_i y^i x^{2-j}, \sum_{j=0}^{n} b_j y^j x^{n-j})$ for $a_i \in \mathbb{Z}, b_j \in \mathbb{Z}$ with $0 \leq i \leq 2, 0 \leq j \leq n$.

In this section, we will will investigate some of the geometry of numbers related to counting points on (q, ξ) such that $\operatorname{Res}(q, \xi) = \pm 1$ which lie in a fundamental domain for the $G_n(\mathbb{Z})$ action on $G_n(\mathbb{R})$. We are not actually able to count points on this hypersurface, so in the end we are not able to get any statistical results relating class groups of quadratic fields. First, we enumerate the real orbits and determine their stabilizers in § 5.1. Following this, we introduce the fundamental domain for the G_n action in § 5.2, and describe related notation in § 5.3, § 5.4, and § 5.5. Then, in § 5.6, we show how to count a certain "trivial" subset of class group elements, which we view as living in a cusp. Finally, in § 5.7 we record an averaging lemma applicable for a non-unimodular groups. Although this is specifically written in our setting, it seems to apply quite a bit more generally.

5.1 The real orbits

We begin with a computation of the orbits of $G_n(\mathbb{R})$ on $V_n^{\operatorname{Res} \in \mathbb{G}_m}(\mathbb{R})$. This is fairly simple when n is odd, though somewhat involved when n is even. The main computation is completed in Proposition 5.1.7.

We begin with some examples of real orbits along with their stabilizers.

Remark 5.1.1. To compute stabilizer groups in the following string of examples, we use that the stabilizer of a point $\operatorname{Spec} \mathbb{R} \to [V_n^{\operatorname{Res} \in \mathbb{G}_m}/G_n]$ can be identified with the automorphism group of the corresponding genus 1 curve via the map $\operatorname{Spec} \mathbb{R} \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ by

Theorem 3.3.7. The composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R} \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$ corresponds to a nodal genus 1 degree n curve C in $\mathbb{P}^n_{\mathbb{C}}$ which has automorphism group $\mu_n \rtimes \mathbb{Z}/2\mathbb{Z}$ (the dihedral group of order $2n$). Here, μ_n is identified with translation by n-torsion elements of $\mathrm{Pic}^0_{C/\mathbb{C}}$ and $\mathbb{Z}/2\mathbb{Z}$ is generated by any automorphism switching the preimages of the nodes on the normalization of C and fixing one of the n inflection points in the smooth locus of C. So, given a point (q, ξ), we will be able to compute the stabilizer group in $G_n(\mathbb{R})$ by identifying the $2n$ elements of $G_n(\mathbb{C})$ and then observing which of them lie in $G_n(\mathbb{R})$.

Example 5.1.2. When n is odd, there is a real orbit of positive discriminant given by $(q, \xi) = (xy, x^n + y^n)$. To compute the stabilizer group, we will use the strategy outlined in Remark 5.1.1. The stabilizer group over $\mathbb{C}$ is generated by those elements switching x and y and sending $x \mapsto \zeta x, y \mapsto \zeta^{-1} y$ for ζ an nth root of unity. Of these, the only real elements are the identity and the one switching x and y, so the stabilizer is $\mathbb{Z}/2\mathbb{Z}$.

Example 5.1.3. Let n be even. There are two real orbits of positive discriminant: $(q, \xi) = (xy, x^n + y^n)$, which has positive resultant, and $(xy, x^n - y^n)$ which has negative resultant. Note that when n is even, the sign of the resultant is preserved under the $G_n(\mathbb{R})$ action, and therefore these are distinct orbits.

In the first case, the computation analogous to Example 5.1.2: the real stabilizer group of is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by the element switching x and y, and the element with $\zeta = -1$. In the second case, let η be a primitive $2n$th root of unity. We still have automorphisms over $\mathbb{C}$ given by powers of $x \mapsto \eta^2 x, y \mapsto \eta^{-2} y$. We also have the order two automorphism $x \mapsto \eta y, y \mapsto \eta^{-1} x$. Here, we see the only automorphisms defined over $\mathbb{R}$ is the identity automorphism and $x \mapsto -x, y \mapsto -y$. Hence, the real stabilizer group is $\mathbb{Z}/2\mathbb{Z}$.

Example 5.1.4. When n is odd, there is a real orbit of negative discriminant given by $q = x^2 + y^2, \xi = (x + iy)^n + (x - iy)^n$. By changing variables from Example 5.1.3, we see the automorphism group over $\mathbb{C}$ is given by switching $x + iy$ and $x - iy$, and sending $x + iy \mapsto \zeta(x + iy), x - iy \mapsto \zeta^{-1}(x - iy)$ for ζ an nth root of unity. The automorphism switching $x + iy$ and $x - iy$ is given by $x \mapsto x, y \mapsto -y$ and so is defined over $\mathbb{R}$. The only real automorphism of the second sort occurs for $\zeta = 1$. Therefore, the real stabilizer group over $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z}$.

Example 5.1.5. When n is even, there is an orbit of negative discriminant given by $q = x^2 + y^2, \xi = (x + iy)^n + (x - iy)^n$. The automorphisms over $\mathbb{C}$ are given by switching $x + iy$ and $x - iy$, and sending $x + iy \mapsto \zeta(x + iy), x - iy \mapsto \zeta^{-1}(x - iy)$ for ζ an nth root of unity. This time, the automorphism switching $x + iy$ and $x - iy$ is defined over $\mathbb{R}$. The only real automorphism of the second sort occurs when $\zeta = \pm 1$. Therefore, the real stabilizer group over $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The following lemma gives a sort of "non-example" of a real orbit.

Lemma 5.1.6. *For n even, there is a map Spec* $\mathbb{R} \to [V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n]$ *which does not lift to a map Spec* $\mathbb{R} \to V_n^{\mathrm{Res}\in\mathbb{G}_m}$.

Proof. Using the equivalence $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n] \simeq [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$ of Lemma 3.3.6, we will equivalently describe a map $\operatorname{Spec}\mathbb{R} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$ which does not lie in the image of $V_n^{\mathrm{Res}\in\mathbb{G}_m}$. Via Lemma 3.3.2, this corresponds to not being given as the map from $\mathbb{P}^1 \to \mathbb{P}^n$ induced by the sections $(\xi, qx^{n-2}, \ldots, qy^{n-2})$ for $x, y \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$ a basis and $(q, \xi) \in H^0(\mathbb{P}^1, \mathscr{O}(2) \oplus \mathscr{O}(n))$.

We will construct the map $\operatorname{Spec}\mathbb{R} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$ by descending it from a map $\operatorname{Spec}\mathbb{C} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$. So, begin with the unique map $\operatorname{Spec}\mathbb{C} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$, which corresponds to a nodal genus 1 curve C in $\mathbb{P}^{n-1}_{\mathbb{C}}$. Choose a complex point $e \in C$ so that C is embedded by $\mathscr{O}_C(ne)$. Note that this makes the smooth locus of C isomorphic to $\mathbb{G}_m$, and endows it with the structure of a group scheme. To descend this to a map $\operatorname{Spec}\mathbb{R} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$, we will use the description of descent in [Poo17, Proposition 4.4.2(i)]. It is enough to specify an map $\widetilde{\sigma} : C \to C$ over the $\operatorname{Spec}\mathbb{R}$ map $\sigma : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$ given by complex conjugation which sends $\mathscr{O}_C(ne)$ to its pullback under complex conjugation, for which $\widetilde{\sigma} \circ^{\sigma} \widetilde{\sigma}$ is the identity. Note that the automorphisms of C induced by inversion on $\mathbb{G}_m$ and translation by the nontrivial 2-torsion point both preserve $\mathscr{O}_C(ne)$; the translation action preserves $\mathscr{O}_C(ne)$ by the assumption that n is even. Letting $\mathbb{P}^1 \to C$ denote the normalization, consider the map $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ over σ which on $\mathbb{G}_m \subset \mathbb{P}^1$ is given by $x \mapsto -\frac{1}{\sigma(x)}$. Concretely, this sends $a + bi \in \mathbb{C}^\times$ to $-\frac{1}{a-bi}$. This map α induces our desired map $\widetilde{\sigma} : C \to C$ over σ which yields the desired descent data, and produces a nodal genus 1 curve X over $\operatorname{Spec}\mathbb{R}$ corresponding to a map $\operatorname{Spec}\mathbb{R} \to [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n]$.

To conclude, we prove that X does not lie in the image of a $\operatorname{Spec}\mathbb{R}$ point of $V_n^{\mathrm{Res}\in\mathbb{G}_m}$. The key to this is to observe that X has no $\mathbb{R}$ points. Indeed, any $\mathbb{R}$-point of X would correspond to a $\mathbb{C}$ point of $\mathbb{P}^1$ which is invariant under the map α. However, if $a + bi \in \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$ were invariant under α then we would have $a + bi = -\frac{1}{a-bi}$, which implies $-a^2 - b^2 = 1$. This is impossible because $a, b \in \mathbb{R}$. Therefore, X has no $\mathbb{R}$ points. If X were the image of (q, ξ) then it would have many infinitely many real points given by evaluating the tuple $(\xi, qx^{n-2}, \ldots, qy^{n-2})$ at real values of x and y. Therefore, X does not lie in the image of any $(q, \xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}(\mathbb{R})$. □

Putting the above together with some cohomological calculations yields the following description of all the real orbits with their stabilizers.

Proposition 5.1.7. *When $n \geq 3$ is odd, the action of $G_n(\mathbb{R})$ on $V_n^{\mathrm{Res}\in\mathbb{G}_m}(\mathbb{R})$ has two orbits corresponding to generically étale degree 2 covers. The orbits correspond to $\operatorname{disc}(q) > 0$ and $\operatorname{disc}(q) < 0$. Both of these orbits have stabilizer in $G_n(\mathbb{R})$ equal to $\mathbb{Z}/2\mathbb{Z}$.*

When $n \geq 3$ is even, the action of $G_n(\mathbb{R})$ on $V_n^{\mathrm{Res}\in\mathbb{G}_m}(\mathbb{R})$ has three orbits corresponding corresponding to generically étale degree 2 covers. The three orbits correspond to $\operatorname{disc}(q) < 0$ which has stabilizer $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\operatorname{disc}(q) > 0$ and $\operatorname{Res}(q, \xi) < 0$, which has stabilizer $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\operatorname{disc}(q) > 0$ and $\operatorname{Res}(q, \xi) > 0$, which has stabilizer $\mathbb{Z}/2\mathbb{Z}$.

Proof. Using Theorem 3.3.7, we wish to identify all degree 2 generically étale covers $g : X \to \operatorname{Spec} \mathbb{R}$ and corresponding elements of $H^1(\operatorname{Spec} \mathbb{R}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ lying in the image of $V_n^{\mathrm{Res}\in\mathbb{G}_m}(\mathbb{R})$. We also wish to identify the sizes of the corresponding automorphism groups for each such element.

To simplify matters, because $\operatorname{Spec} \mathbb{R}$ has characteristic 0, we have

$$H^1(\operatorname{Spec} \mathbb{R}, g_*\mu_n/\mu_n) \simeq H^1(\operatorname{Spec} \mathbb{R}, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$$

Now, $\operatorname{Spec} \mathbb{R}$ has two degree 2 étale covers, $g' : \operatorname{Spec} \mathbb{R} \coprod \operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{R}$ and $g'' : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$. The first corresponds to q of positive discriminant and the second corresponds to q of negative discriminant.

We first examine the case of the cover g', corresponding to the discriminant being positive. By chasing the exact sequence on cohomology we find

$$\begin{aligned} H^1(\operatorname{Spec} \mathbb{R}, g'_*\mu_n/\mu_n) &\simeq H^1(\operatorname{Spec} \mathbb{R} \coprod \operatorname{Spec} \mathbb{R}, \mu_n)/H^1(\operatorname{Spec} \mathbb{R}, \mu_n) \\ &\simeq H^1(\operatorname{Spec} \mathbb{R}, \mu_n) \simeq \left(\mathbb{R}^\times\right)/\left(\mathbb{R}^\times\right)^n. \end{aligned}$$

This is $\mathbb{Z}/2\mathbb{Z}$ when n is even and trivial when n is odd. Therefore, the examples enumerated in Example 5.1.2 and Example 5.1.3 are exhaustive. Note that the automorphism of g is given by inversion and therefore acts trivially on $\mathbb{Z}/2\mathbb{Z}$.

Next, we analyze the case of negative discriminant, corresponding to the cover g''. Here, we have $H^1(\operatorname{Spec} \mathbb{R}, g''_*\mu_n/\mu_n) \simeq H^2(\operatorname{Spec} \mathbb{R}, \mu_n)$ because

$$H^1(\operatorname{Spec} \mathbb{R}, g''_*\mu_n) \simeq H^1(\operatorname{Spec} \mathbb{C}, \mu_n) = \mathbb{C}^\times/\left(\mathbb{C}^\times\right)^n = 1.$$

Because $H^1(\operatorname{Spec} \mathbb{R}, \mathbb{G}_m) = 1$ and $H^2(\operatorname{Spec} \mathbb{R}, \mathbb{G}_m) \simeq \mathbb{Z}/2\mathbb{Z}$, we find $H^1(\operatorname{Spec} \mathbb{R}, g''_*\mu_n/\mu_n)$ is the n-torsion in $\mathbb{Z}/2\mathbb{Z}$, and hence either $\mathbb{Z}/2\mathbb{Z}$ when n is even, or trivial when n is odd.

When n is odd, it follows that Example 5.1.4 is exhaustive. It remains to deal with the case that q has negative discriminant and n is even. We have one orbit described in Example 5.1.5. As we saw above, there is only 1 other $\operatorname{Spec} \mathbb{R}$ points of $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n]$ over

g''. However, because the point produced in Lemma 5.1.6 does not lie over g', as we have already enumerated all such points, it must lie over g''. Therefore, there is only a single orbit over g'' when n is even and the discriminant is negative, given in Example 5.1.5.

The corresponding automorphism groups were calculated in Examples 5.1.2-5.1.5. □

We conclude our discussion of the real orbits with a speculative remark that will not be needed in what follows. In order to state this remark, we will need the following useful alternate description of the sheaf $g_* \mathbb{G}_m / \mathbb{G}_m[n]$. The uninterested reader may skip ahead to §5.2.

Lemma 5.1.8. *Let $g : X \to B$ be a degree 2 finite locally free morphism. We have an isomorphism of sheaves on B in the flat topology $g_* \mu_n / \mu_n \simeq (g_* \mathbb{G}_m / \mathbb{G}_m)[n]$.*

Proof. We have a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{\times n} & \mathbb{G}_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_*\mu_n & \longrightarrow & g_*\mathbb{G}_m & \xrightarrow{\times n} & g_*\mathbb{G}_m & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (g_*\mathbb{G}_m/\mathbb{G}_m)[n] & \longrightarrow & g_*\mathbb{G}_m/\mathbb{G}_m & \xrightarrow{\times n} & g_*\mathbb{G}_m/\mathbb{G}_m & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.1.1}$$

Since the second and third nonzero columns are exact, and the first row is exact, the snake lemma shows the first nonzero column is exact. Hence, $g_* \mu_n / \mu_n \simeq (g_* \mathbb{G}_m / \mathbb{G}_m)[n]$. □

Remark 5.1.9. Let $E_n \subset V_n^{\mathrm{Res} \in \mathbb{G}_m}$ denote the open locus corresponding to pairs (q, ξ) where $V(q)$ is étale over B. In this remark, for even n, we construct four maps $H^0(B, [E_n/G_n]) \to H^2(B, \mu_2)$. We conjecture the these four maps agree, though we have not verified this. Knowing these maps agree would give an alternate method of verifying Proposition 5.1.7 and similar results. The second map relates this construction to the canonical quadratic form on $\mathscr{S}^{(n)} \simeq [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$.

Our first map is given as the composition $H^0(B, [E_n/G_n]) \to H^1(B, G_n) \to H^1(B, \mathrm{PGL}_2) \to H^2(B, \mu_2)$ where the first map above is the boundary map, the second comes from the quotient $G_n \to \mathrm{PGL}_2$ of Lemma 2.1.4, and the last map is the boundary map associated to $\mu_2 \to \mathrm{SL}_2 \to \mathrm{PGL}_2$.

For the second map, we have maps

$$[E_n/G_n] \to \mathscr{V}^{\mathrm{smile},(n)} \simeq [\widetilde{\mathscr{H}}^{(n)}_{\mathrm{sing}}/\mathrm{PGL}_n] \to [\mathscr{H}^{(n)}/\mathrm{PGL}_n]$$

which our second map

$$H^0(B,[E_n/G_n]) \to H^0(B,[\mathscr{H}^{(n)}/\mathrm{PGL}_n]) \to H^1(B,\mathrm{PGL}_n) \to H^2(B,\mu_n) \to H^2(B,\mu_2).$$

The map $H^1(B,\mathrm{PGL}_n) \to H^2(B,\mu_n)$ is induced by the boundary map coming from the identification $\mathrm{PGL}_n \simeq \mathrm{PSL}_n$ and the exact sequence (2.3.13) and the map $\mu_n \to \mu_2$ is raising to the $n/2$th power.

Our third map is described as follows. We will first construct maps $\mathrm{Res} : [E_n/G_n] \to B(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{disc} : [E_n/G_n] \to B\mu_2$ which can be understood as the "resultant map" and the "discriminant map." Namely, the discriminant map is induced from the map $E_n \to B(\mathbb{Z}/2\mathbb{Z})$ sending (q,ξ) to the degree 2 étale cover $V(q)$ of B. The resultant maps is induced by the resultant map $E_n \to \mathbb{G}_m$ and the map $G_n \to G_n/U_n \to \mathbb{G}_m$ obtained by first quotienting by the unipotent radical and then by the copy of $\mathrm{GL}_2 \subset G_n$. Because n is even, for any choice of orbit $G_n \to E_n$, we obtain a commuting diagram

$$\begin{array}{ccccc} G_n & \longrightarrow & E_n & \longrightarrow & [E_n/G_n] \\ \downarrow & & \downarrow & & \\ \mathbb{G}_m & \xrightarrow{\times 2} & \mathbb{G}_m & \longrightarrow & B\mu_2 \end{array} \tag{5.1.2}$$

which induces the map $[E_n/G_n] \to B\mu_2$. Therefore, combining the resultant and discriminant maps, we obtain a map $H^0(B,[E_n/G_n]) \to H^0(B, B\mu_2 \times B(\mathbb{Z}/2\mathbb{Z})) \simeq H^1(B,\mu_2) \times H^1(B,\mathbb{Z}/2\mathbb{Z})$. Combining this map with the cup product map $H^1(B,\mu_2) \times H^1(B,\mathbb{Z}/2\mathbb{Z}) \to H^2(B,\mu_2)$ gives our third map $H^0(B,[E_n/G_n]) \to H^2(B,\mu_2)$.

For our fourth and final map, we start by observing that associated to any element $s \in H^0(B,[E_n/G_n])$, we have a corresponding degree 2 étale cover $g : X \to B$. We can then lift s to an element of $H^0(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$. Because g is generically étale, the natural inclusion $H^1(B, g_*\mu_n/\mu_n) \to H^0(B, g_*\mathbb{G}_m/\mathbb{G}_m \xrightarrow{\times n} g_*\mathbb{G}_m/\mathbb{G}_m)$ is an isomorphism. Upon identifying $g_*\mu_n/\mu_n \simeq g_*\mathbb{G}_m/\mathbb{G}_m[n]$ via Lemma 5.1.8, raising to the $n/2$th power induces a map $g_*\mathbb{G}_m/\mathbb{G}_m[n] \to g_*\mathbb{G}_m/\mathbb{G}_m[2]$, and hence a map $g_*\mu_n/\mu_n \to g_*\mu_2/\mu_2$. This yields the map on cohomology $H^1(B, g_*\mu_n/\mu_n) \to H^1(B, g_*\mu_2/\mu_2)$. Finally, composing this with the boundary map $H^1(B, g_*\mu_2/\mu_2) \to H^2(B,\mu_2)$ yields our fourth map $H^0(B,[E_n/G_n]) \to H^2(B,\mu_2)$.

5.2 Defining the fundamental domain

We now set up a counting problem which is equivalent to determining average sizes of n-torsion in class groups of quadratic fields. We note that we are not sure how to approach this question, and so we only set it up here. We begin by introducing a some notation. This sort of notation is fairly standard in the type of counting arguments pioneered by Bhargava as in [Bha05].

Using the description of the structure of G_n from Lemma 2.1.4, it will be convenient to have a name for its maximal reductive quotient, which can also be realized as a Levi subgroup of G_n using Lemma 2.1.4.

Definition 5.2.1. Let $G_n^{\text{reductive}}$ denote $G_n/U_n \simeq \mathbb{G}_m^2 \times \mathrm{GL}_2/\mu_{n-2}$, the maximal reductive quotient of G_n. We also view $G_n^{\text{reductive}} \subset G_n$ as a Levi subgroup generated by the subgroups of G_n determined by (2.1.1) and (2.1.3).

5.3 Notation for coordinates on G_n

We now set notation for the NAK decomposition of $\mathrm{SL}_2(\mathbb{R})$. Define $N, A, K \subset \mathrm{SL}_2(\mathbb{R})$ by

$$N := \left\{ \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} : m \in \mathbb{R} \right\} \tag{5.3.1}$$

$$A := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\} \tag{5.3.2}$$

$$K := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} \simeq \mathrm{SO}_2(\mathbb{R}) \tag{5.3.3}$$

Then, every element of $\mathrm{SL}_2(\mathbb{R})$ can be written uniquely as a product nak for $m \in N, a \in A, k \in K$, essentially by Gram Schmidt decomposition. Choose a fixed left Haar measure dg on G_n. (This choice will not matter much for us, as we will only be concerned with everything up to constants anyway, but it is important to fix one left Haar measure to work with.)

Let $N(t)$ be the subset of $\mathbb{R}$ (depending on t) lying in the usual fundamental domain for the action of $N \times A$ on $\mathbb{H}$, see [Ser73, VII,§1, Theorem 1]. Concretely, $N(t)$ is the locus of points n so that $t^2 i + n$ has real part of absolute value at most $1/2$ and complex norm at least 1. In particular, when $t > 1$ this is simply $N(t) = [-1/2, 1/2)$ and otherwise for $t > 0$ it is subset of $[-1/2, 1/2)$ such that $t^2 i + n$ lies above the unit circle, or on the unit circle and with non-positive real coordinate.

We will next introduce coordinates for the maximal central torus $\mathbb{G}_m^2(\mathbb{R}) \subset G_n^{\text{reductive}}(\mathbb{R})$. Define

$$\Lambda := \left\{ \begin{pmatrix} \lambda \cdot \mathrm{id}_{H^0(\mathbb{P}^1, \mathscr{O}(2))} & 0 \\ 0 & \mathrm{id}_{H^0(\mathbb{P}^1, \mathscr{O}(n))} \end{pmatrix} \in \mathrm{Aut}(H^0(\mathbb{P}^1, \mathscr{O}(2)) \oplus H^0(\mathbb{P}^1, \mathscr{O}(n))) : \lambda \in \mathbb{R}_{>0} \right\} \tag{5.3.4}$$

$$M := \left\{ \begin{pmatrix} \mathrm{id}_{H^0(\mathbb{P}^1, \mathscr{O}(2))} & 0 \\ 0 & \mu \cdot \mathrm{id}_{H^0(\mathbb{P}^1, \mathscr{O}(n))} \end{pmatrix} \in \mathrm{Aut}(H^0(\mathbb{P}^1, \mathscr{O}(2)) \oplus H^0(\mathbb{P}^1, \mathscr{O}(n))) : \mu \in \mathbb{R}_{>0} \right\}. \tag{5.3.5}$$

That is, we use λ for the coordinate scaling $H^0(\mathbb{P}^1, \mathscr{O}(2))$ and μ for the coordinate scaling $H^0(\mathbb{P}^1, \mathscr{O}(n))$.

Finally, we let $(\alpha_0, \ldots, \alpha_{n-2}) \in \mathbb{R}^{n-1}$ denote the coordinates for $U_n(\mathbb{R}) \simeq \mathbb{G}_a^{n-1}(\mathbb{R}) \simeq \mathbb{R}^{n-1}$. We will often use α to denote such a tuple $(\alpha_0, \ldots, \alpha_{n-2})$.

5.4 Notation for the fundamental domain

We now construct a fundamental domain $\mathscr{F}$ for the left action of $G_n(\mathbb{Z})$ on $G_n(\mathbb{R})$:

By Lemma 2.1.4, $G_n = U_n \rtimes G_n^{\text{reductive}}$. Hence, we can write every $g \in G_n$ uniquely as αh for $\alpha \in U_n, h \in G_n^{\text{reductive}}$. Therefore, it suffices to construct $\mathscr{F}_{G_n^{\text{reductive}}}$, a fundamental domain for the action of $G_n^{\text{reductive}}(\mathbb{Z})$ on $G_n^{\text{reductive}}(\mathbb{R})$, and $\mathscr{F}_{U_n}$, a fundamental domain for the action of $U_n(\mathbb{Z})$ on $U_n(\mathbb{R})$. We will then define

$$\mathscr{F} := \mathscr{F}_{U_n(\mathbb{R})} \cdot \mathscr{F}_{G_n^{\text{reductive}}(\mathbb{R})} \subset G_n.$$

We now specify choices of $\mathscr{F}_{G_n^{\text{reductive}}}$ and $\mathscr{F}_{U_n}$. Identifying $U_n \simeq \mathbb{G}_a^{n-1}$, we will take

$$\mathscr{F}_{U_n} = \{\alpha_0, \ldots, \alpha_{n-2} \in \mathbb{G}_a^{n-1}(\mathbb{R}) : 0 \le \alpha_i < 1 \text{ for } 0 \le i \le n-2\}.$$

For $\mathscr{F}_{G_n^{\text{reductive}}}$, we will take the following fundamental domain, essentially coming from Gauss' classical fundamental domain for SL_2. Using notation for N, A, K, Λ and M as in §5.3, we define

$$\mathscr{F}_{G_n^{\text{reductive}}} := \left\{ \lambda\mu m t k : \lambda \in \Lambda, \mu \in M, t \ge \frac{3^{1/4}}{\sqrt{2}}, m \in N(t), k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta \in [0, \pi) \right\}. \tag{5.4.1}$$

Lemma 5.4.1. *The set $\mathscr{F}$ is a fundamental domain for the left action of $G_n(\mathbb{Z})$ on $G_n(\mathbb{R})$*

Proof. Since $\mathscr{F}_{G_n^{\text{reductive}}}$ is a fundamental domain for the left action of $U_n(\mathbb{Z})$ on $U_n(\mathbb{R})$ and $\mathscr{F} = \mathscr{F}_{U_n} \cdot \mathscr{F}_{G_n^{\text{reductive}}}$, this follows from Lemma 5.4.2 and Lemma 5.4.3 below. □

Lemma 5.4.2. *The set $\mathscr{F}_{G_n^{\text{reductive}}}$ is a fundamental domain for the left action of $G_n^{\text{reductive}}(\mathbb{Z})$ on $G_n^{\text{reductive}}(\mathbb{R})$.*

Proof. Using Lemma 2.1.4, we can identify $G_n^{\text{reductive}}$ as a central extension of PGL_2 by $\mathbb{G}_m \times \mathbb{G}_m$. As $\mathrm{PGL}_2 = \mathrm{SL}_2/\mu_2$, we claim the exact sequence $\mu_2 \to \mathrm{SL}_2 \to \mathrm{PGL}_2$ identifies $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$. Indeed this claim may be deduced from a straightforward diagram chase using that $H^i(\operatorname{Spec}\mathbb{Z}, \mu_2) \to H^i(\operatorname{Spec}\mathbb{R}, \mu_2)$ are isomorphisms for $i = 0, 1$, and $H^1(\operatorname{Spec}\mathbb{Z}, \mathrm{SL}_2) = H^1(\operatorname{Spec}\mathbb{R}, \mathrm{SL}_2) = 0$. Because the pullback of $G_n^{\text{reductive}} \to \mathrm{PGL}_2$ along $\mathrm{SL}_2 \to \mathrm{PGL}_2$ is $\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SL}_2$, we find that a fundamental domain for the action of $(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SL}_2)(\mathbb{Z})$ on $(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SL}_2)(\mathbb{R})$ maps isomorphically to a fundamental domain for the action of $G_n^{\text{reductive}}(\mathbb{Z})$ on $G_n^{\text{reductive}}(\mathbb{R})$. The coordinates M and Λ encapsulate the fundamental domain for the first two factors of $\mathbb{G}_m$ while the coordinates N, A, and K encapsulate the fundamental domain for SL_2 by [Ser73, Chapter VII, Theorem 1]. □

Lemma 5.4.3. *Suppose G_1 and G_2 are algebraic groups over Spec $\mathbb{Z}$ and $G = G_1 \rtimes G_2$. For $i \in \{1, 2\}$, let $\mathscr{F}_i$ be a fundamental domain for the left action of $G_i(\mathbb{Z})$ on $G_i(\mathbb{R})$. Then $\mathscr{F} := \mathscr{F}_1 \cdot \mathscr{F}_2$ is a fundamental domain for the left action of $G(\mathbb{Z})$ on $G(\mathbb{R})$.*

Proof. Choose an arbitrary $g \in G(\mathbb{R})$. We may write g uniquely as αh for $\alpha \in G_1(\mathbb{R})$ and $h \in G_2(\mathbb{R})$. We want to show there is a unique $(\alpha', h') \in G_1(\mathbb{Z}) \times G_2(\mathbb{Z})$ so that $\alpha' h' \alpha h \in \mathscr{F}$. We can write $\alpha' h' \alpha h = \alpha'(h'\alpha(h')^{-1})h'h$. For this product to be in the fundamental domain $\mathscr{F}$, we need both $h'h \in \mathscr{F}_2$ and $\alpha'(h'\alpha(h')^{-1}) \in \mathscr{F}_1$. Because $\mathscr{F}_2$ is a fundamental domain for the left action of $G_2(\mathbb{Z})$ on $G_2(\mathbb{R})$, there is a unique $h' \in G_2(\mathbb{Z})$ so that $h'h \in \mathscr{F}_2$. Fixing this h', there is a unique α' so that $\alpha'(h'\alpha(h')^{-1}) \in \mathscr{F}_1$. □

5.5 Notation for the orbits

As shown in Proposition 5.1.7, there are either 2 or 3 orbits of $G_n(\mathbb{R})$, on $V_n(\mathbb{R})$ of nonzero discriminant, depending on whether n is even or odd. Hence, similarly, there are either 2 or 3 orbits of $G_n(\mathbb{R})$ on $V_n(\mathbb{R})$ of nonzero discriminant.

Let $\mathscr{O}$ be one of these three orbits. Concretely, when n is odd, we can think of $\mathscr{O} \in \{+, -\}$ corresponding to whether the discriminant is positive or negative. When n is even, we have

$\mathscr{O} \in \{-, +-, ++\}$ where the first sign corresponds to the sign of the discriminant and the second (if it exists) corresponds to the sign of the resultant. Let $v^{\mathscr{O}} \in V_n(\mathbb{Z})$ denote a fixed vector in the orbit corresponding to $\mathscr{O}$. As a concrete choice, in the n odd case we can take $v^+ = (xy, x^n + y^n), v^- = (x^2 + y^2, (x + iy)^2 + (x - iy)^n)$ while in the n even case we can take $v^- = (x^2 + y^2, (x + iy)^2 + (x - iy)^n), v^{++} = (xy, x^n + y^n), v^{+-} = (xy, x^n - y^n)$ as in Examples 5.1.2-5.1.5.

For $S \subset V_n(\mathbb{R})$, define the weighted sets, i.e., multisets,

$$S_{[Y,Y'],Z} := \{(q, \xi) \in S : Y \leq |\operatorname{disc}(q)| \leq Y', 0 < |\operatorname{Res}(q, \xi)| \leq Z\},$$
$$S_{[Y,Y'],=Z} := \{(q, \xi) \in S : Y \leq |\operatorname{disc}(q)| \leq Y', |\operatorname{Res}(q, \xi)| = Z\},$$

where we give $v \in S$ weight $\frac{1}{\operatorname{Stab}_{G_n(\mathbb{Z})}(v)}$. Also, when $S \subset G_n(\mathbb{R})v^{\mathscr{O}}$, define

$$N(S)_{[Y,Y'],Z} := \#\{(q, \xi) \in S_{[Y,Y'],Z} \cap \mathscr{F} v^{\mathscr{O}}\},$$
$$N(S)_{[Y,Y'],=Z} := \#\{(q, \xi) \in S_{[Y,Y'],=Z} \cap \mathscr{F} v^{\mathscr{O}}\},$$

where we count each element of the weighted set with its weighting specified above. For general $S \subset V_n(\mathbb{R})$ not necessarily in a single orbit, let

$$N(S)_{[Y,Y'],Z} := \sum_{\text{orbit representatives } v^{\mathscr{O}}} N(S \cap G_n(\mathbb{R})v^{\mathscr{O}})_{[Y,Y'],Z},$$
$$N(S)_{[Y,Y'],=Z} := \sum_{\text{orbit representatives } v^{\mathscr{O}}} N(S \cap G_n(\mathbb{R})v^{\mathscr{O}})_{[Y,Y'],=Z}.$$

Let $n^{\mathscr{O}} := \operatorname{Stab}_{G_n(\mathbb{R})}(v^{\mathscr{O}})$. Concretely, by Proposition 5.1.7, when n is odd we have $n^+ = n^- = 2$, while when n is even we have $n^- = n^{+-} = 4$ and $n^{++} = 2$.

5.6 The main and cusp regions

We would like to count points in $V_n(\mathbb{Z})$. This corresponds to a certain hypersurface in affine space, and we are unsure how to do this. We can at least divide this up into a "main" region and a "cusp" region, and count the points in the cusp region. Unfortunately, we do not have a good understanding of how to count points in the main region.

Definition 5.6.1. For $S \subset V_n(\mathbb{Z})$ let S^{main} denote the set of $(q, \xi) \in S$ such that the associated inflection subscheme as in Definition 4.1.9 does not restrict to the trivial torsor over $\operatorname{Spec} \mathbb{Q}$

via the identification of Lemma 4.1.18. Because the inflection subscheme is proper, this is equivalent to requiring that it does not have a section over $\operatorname{Spec}\mathbb{Z}$. Let $S^{\mathrm{cusp}} := S - S^{\mathrm{main}}$ denote the set of $(q,\xi) \in S$ such that the associated inflection subscheme is generically trivial.

We now record a bound on the number of elements of V_n^{cusp}, which is essentially a translation of Proposition 4.2.15 and Lemma 4.1.18. Again, we are unsure how to deal with counting points in the main region S^{main}.

Lemma 5.6.2. *We have*

$$N(V_n(\mathbb{Z})^{\mathrm{cusp}} \cap G_n(\mathbb{R})v^{\mathscr{O}})_{Y,=1} = O_n(Y(\log Y)^{n^3-1}).$$

Proof. Given a degree 2 cover $g : X \to \operatorname{Spec}\mathbb{Z}$ of discriminant d, let E_g^{sm} be the group scheme associated to the cover $X \to \operatorname{Spec}\mathbb{Z}$ of discriminant $d := \operatorname{disc}(q)$ as in Notation 3.2.1. Then, Lemma 4.1.18 shows an element $(q,\xi) \in V_n^{\mathrm{Res}\in\mathbb{G}_m}$ with $\operatorname{disc} q = d$ whose inflection subscheme has as section over $\operatorname{Spec}\mathbb{Z}$, and hence over $\operatorname{Spec}\mathbb{Q}$, corresponds to the trivial cohomology class in the associated group $H^1(\operatorname{Spec}\mathbb{Q}, E_g^{\mathrm{sm}}[n]) \simeq H^1(\operatorname{Spec}\mathbb{Q}, E_g^{\mathrm{sm}} \xrightarrow{\times n} E_g^{\mathrm{sm}})$, taken modulo automorphisms of g. Note here that $E_g^{\mathrm{sm}} \simeq g_*\mathbb{G}_m/\mathbb{G}_m$ by Lemma 3.2.7 and there are two automorphisms of g identified with the two automorphisms of the elliptic curve (E_g,σ). It follows that elements of $V_n(\mathbb{R})v^{\mathscr{O}} \cap (V_n(\mathbb{Z})_{[Y/2,Y],=1})^{\mathrm{cusp}}$ corresponding to a pair (q,ξ) with discriminant d lie in the kernel $H^1(\operatorname{Spec}\mathbb{Z}, E_g^{\mathrm{sm}} \xrightarrow{\times n} E_g^{\mathrm{sm}}) \to H^1(\operatorname{Spec}\mathbb{Q}, E_g^{\mathrm{sm}} \xrightarrow{\times n} E_g^{\mathrm{sm}})$. Then, Proposition 4.2.15 yields yields the bound $N(V_n(\mathbb{Z})^{\mathrm{cusp}} \cap G_n(\mathbb{R})v^{\mathscr{O}})_{Y,=1} = O_n(Y(\log Y)^{n^3-1})$. □

5.7 An averaging lemma for non-unimodular groups

Although we will not use it again, we would now like to give a version of an standard averaging lemma which is applicable to non-unimodular groups. Since left and right Haar measures on G_n do not agree, the proof of the averaging method as in [BS15, Theorem 2.5] does not apply.

To state this, we need the following notation.

Notation 5.7.1. Let B_0 denote a fixed nonempty open bounded subset of $G_n(\mathbb{R})$ with B_0^{-1} also bounded and let dg denote a left Haar measure on G_n. For $B \subset G$, we use $\operatorname{Vol}(B) := \int_B dg$. Fix $v^{\mathscr{O}} \in V_n(\mathbb{Z})$ and let $S \subset G_n(\mathbb{R})v^{\mathscr{O}}$. For $H \subset G_n(\mathbb{R})$ a subset, use $\#\{S_{[Y,Y'],Z} \cap Hv^{\mathscr{O}}\}$ to denote the multiset where any given v in this subset is given weight equal to the number of $h \in H$ with $hv^{\mathscr{O}} = v$. In the case H is a fundamental domain, this implies that each $v \in G_n(\mathbb{R})v^{\mathscr{O}}$ is counted with multiplicity $\frac{\mathrm{Stab}_{G_n(\mathbb{R})}(v)}{\mathrm{Stab}_{G_n(\mathbb{Z})}(v)}$.

Remark 5.7.2. The following lemma gives a version of the usual averaging trick, as appears in [BS15, Theorem 2.5] which applies to non-unimodular groups. The reason we need to be careful is that left and right Haar measures may no longer agree, and so the usual proof of [BS15, Theorem 2.5] does not apply.

Lemma 5.7.3. *With notation as in Notation 5.7.1 and §5.5,*

$$n^{\mathscr{O}} N(S)_{[Y,Y'],Z} = \frac{\int_{g \in B_0^{-1}} \#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} g^{-1} v^{\mathscr{O}}\right\} dg}{\mathrm{Vol}(B_0^{-1})} = \frac{\int_{g \in \mathscr{F}} \#\left\{S_{[Y,Y'],Z} \cap g B_0 v^{\mathscr{O}}\right\} dg}{\mathrm{Vol}(B_0^{-1})}. \tag{5.7.1}$$

Proof. To start, we justify the first equality. The key to this is the observation that $n^{\mathscr{O}} N(S)_{[Y,Y'],Z} = \#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} v^{\mathscr{O}}\right\}$. Indeed, both sides count elements of $V_n(\mathbb{Z})$ with discriminant in the range $[Y, Y']$ and resultant at most Z, so it suffices to check each such element appears with the same multiplicity on both sides. On the right hand side, an element v appears with multiplicity $\frac{\mathrm{Stab}_{G_n(\mathbb{R})}(v)}{\mathrm{Stab}_{G_n(\mathbb{Z})}(v)}$ while in $N(S)_{[Y,Y'],Z}$ it appears with multiplicity $\frac{1}{\mathrm{Stab}_{G_n(\mathbb{Z})}(v)}$. Because $n^{\mathscr{O}} = \mathrm{Stab}_{G_n(\mathbb{R})}(v) = \mathrm{Stab}_{G_n(\mathbb{R})}(v^{\mathscr{O}})$ as $v \in G_n(\mathbb{R}) \cdot v^{\mathscr{O}}$, it follows that v appears with the same multiplicity on both sides.

Given this, we next note that $\#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} v^{\mathscr{O}}\right\} = \#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} g^{-1} v^{\mathscr{O}}\right\}$ because if $\mathscr{F}$ is a fundamental domain for the left $G_n(\mathbb{Z})$ action on $G_n(\mathbb{R})$ then, for any $h \in G_n(\mathbb{R})$, $\mathscr{F} \cdot h$ is also a fundamental domain for the same left action. It follows that

$$n^{\mathscr{O}} N(S)_{[Y,Y'],Z} = \frac{\int_{g \in B_0^{-1}} \#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} v^{\mathscr{O}}\right\} dg}{\mathrm{Vol}(B_0^{-1})} = \frac{\int_{g \in B_0^{-1}} \#\left\{S_{[Y,Y'],Z} \cap \mathscr{F} g^{-1} v^{\mathscr{O}}\right\} dg}{\mathrm{Vol}(B_0^{-1})}.$$

To conclude, we only need to verify the numerators in the latter equality of (5.7.1) agree.

Indeed,

$$\begin{aligned}
\int_{g\in B_0^{-1}} \#\left\{S_{[Y,Y'],Z}\cap \mathscr{F}g^{-1}\cdot v^{\mathscr{O}}\right\}\,dg &= \int_{g\in B_0^{-1}} \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in \mathscr{F}g^{-1}\\ hv^{\mathscr{O}}=v}} 1\,dg\\
&= \int_{g\in B_0^{-1}} \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} 1_{h\in\mathscr{F}g^{-1}}\,dg\\
&= \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} \int_{g\in B_0^{-1}} 1_{g\in h^{-1}\mathscr{F}}\,dg\\
&= \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} \mathrm{Vol}(B_0^{-1}\cap h^{-1}\mathscr{F})\,dg\\
&= \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} \mathrm{Vol}(hB_0^{-1}\cap \mathscr{F})\,dg\\
&= \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} \int_{g\in\mathscr{F}} 1_{g\in hB_0^{-1}}\,dg\\
&= \int_{g\in\mathscr{F}} \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in G_n(\mathbb{R})\\ hv^{\mathscr{O}}=v}} 1_{h\in gB_0}\,dg\\
&= \int_{g\in\mathscr{F}} \sum_{\substack{v\in S_{[Y,Y'],Z}\\ h\in gB_0\\ hv^{\mathscr{O}}=v}} 1\,dg\\
&= \int_{g\in\mathscr{F}} \#\left\{S_{[Y,Y'],Z}\cap gB_0v^{\mathscr{O}}\right\}\,dg. \qquad \square
\end{aligned}$$

Chapter 6

The secant variety to the rational normal curve

In this chapter, we relate the Cohen-Lenstra heuristics to several other problems regarding counting points on varieties. The relation is not very precise, and, for the most part, we only discuss it in these introductory remarks. The subsequent parts of the section merely describe the connection between various geometric spaces, leaving the reader to infer precise statements about the Cohen-Lenstra heuristics if they so desire.

The first map we describe goes from the space of pairs of polynomials or unit resultant to the secant variety to the rational normal curve. Namely, starting with the space of pairs (q, ξ) having unit resultant, where ξ has degree n, in § 6.1, we describe a way to obtain a point on the secant variety to the rational normal curve in $\mathbb{P}^n$. This sends integral points in the unit resultant space to integral points missing the rational normal curve (in all fibers over $\operatorname{Spec} \mathbb{Z}$).

In this way, we relate the problem of counting points on the complement of a hypersurface in affine space to counting points on the complement of a codimension 2 subset of a projective variety. Our hope was that this would make it easier to count the resulting points. It appears this is the case when $n \leq 4$. However, when $n \geq 5$, it seems that most integral points on this secant variety *do* meet the codimension 2 subvariety given by the rational normal curve, and so it is not enough to count points on the projective secant variety. This is discussed further in Remark 6.5.6 and Remark 6.5.7. These also suggest that one should be able to relate the Cohen-Lenstra heuristics to counting secant lines to the rational normal curve which contain a point of small height (as opposed to counting all points on these secant lines). But again this unfortunately seems like a difficult problem.

There is also a map from this secant variety to the space of inflection subschemes. The composition of these two maps was previously discussed in § 4.1. This second map essentially inserts certain signed binomial coefficients to the coordinates, and we relate it to the classical notion of apolarity in § 6.2. This gives a way to rephrase the Cohen-Lenstra heuristics in terms of counting these inflection subschemes, but again we do not make this precise.

The final parts of the section are concerned with understanding what (a version of the) Batyrev Manin conjecture would predict for counting points on the secant variety to the rational normal curve. To this end, we work out the intersection theory of the secant variety in § 6.3. As a side comment, we describe a fun relation between the secant lines to the rational normal curve and a Veronese embedding of the $(\mathbb{P}^2)^\vee$ of lines on $\mathbb{P}^2$ in § 6.4. Finally, in § 6.5, we describe the predictions of the Batyrev Manin conjecture.

6.1 The map from the unit resultant space to the complement of the rational normal curve in the secant variety to the rational normal curve

Previously, we had been investigating the open in affine space $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ of pairs (q, ξ) of unit resultant. There is a map from $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ to the space of binary n-ic forms, given by taking the inflection subscheme. This factors through a map to the secant variety to the rational normal curve, which we next describe. This map quotients $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ by the action of $\mathbb{G}_a^{n-1} \rtimes \mathbb{G}_m$ where $\mathbb{G}_m$ action takes a pair (q, ξ) and sends it to $(\lambda q, \lambda^{-n+1}\xi)$ and $\mathbb{G}_a^{n-1}$ adds multiples of q to ξ. Our goal is to obtain a better understanding of this secant variety.

Let $\mathrm{AffineSec}^{(n)} \subset \mathbb{A}^{n+1}$ be the affine cone over the secant variety to the rational normal curve in $\mathbb{P}^n$. Let $\mathrm{AffineSec}^{(n),\circ}$ be the open subscheme of $\mathrm{AffineSec}^{(n)}$ which is the complement of the preimage of the rational normal curve. This has an action of GL_2 coming from permuting the coordinates on the vector bundle whose projectivization is $\mathbb{P}^1$, which is identified with the rational normal curve.

Proposition 6.1.1. *There is an isomorphism of stacks* $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/G_n] \to [\mathrm{AffineSec}^{(n),\circ}/\mathrm{GL}_2]$. *In fact, this comes from a map* $T_n : V_n^{\mathrm{Res}\in\mathbb{G}_m} \to \mathrm{AffineSec}^{(n),\circ} \subset \mathbb{A}^{n+1}$ *inducing an isomorphism* $[V_n^{\mathrm{Res}\in\mathbb{G}_m}/\mathbb{G}_a^{n-1} \rtimes \mathbb{G}_m] \simeq \mathrm{AffineSec}^{(n),\circ}$.

Proof. We produce a map $T_n : V_n^{\mathrm{Res}\in\mathbb{G}_m} \to \mathrm{AffineSec}^{(n),\circ}$ realizing the former as a $\mathbb{G}_a^{n-1} \rtimes \mathbb{G}_m$ torsor over the latter.

To define T_n, we will describe where we send a pair (q, ξ) of unit resultant. Define $f_1, \ldots, f_n$ by $(x^{n-2}q, x^{n-3}yq, \ldots, y^{n-2}q, \xi) =: (f_1, \ldots, f_n) \in H^0(\mathbb{P}^1, \mathscr{O}(n))^n$. Write $f_i = \sum_j a_i^j x^j y^{n-j}$. Let T_n denote the map $V_n^{\mathrm{Res}\in\mathbb{G}_m} \to \mathbb{A}^{n+1}$ which sends $(q, \xi) \mapsto (t_k)_{1\le k\le n+1}$

where t_k is the $n \times n$ minors of the $n \times (n+1)$ matrix $A := (a_i^j)_{1\le i\le n, 0\le j\le n}$ with the kth column omitted.

This map T_n is invariant under the action of $\mathbb{G}_m$ scaling q by $\lambda \in \mathbb{G}_m$ and ξ by λ^{-n+1}, as the resulting action scales each of the minors by $\lambda^{n+1} \cdot \lambda^{-n+1} = 1$. It is also invariant under the action of $\mathbb{G}_a^{n-1}$, since adding rows of a matrix to other rows does not change the values of its minors. Further, these actions are both free on $V_n^{\mathrm{Res}\in\mathbb{G}_m}$, and so $V_n^{\mathrm{Res}\in\mathbb{G}_m}$ is a $\mathbb{G}_a^{n-1} \rtimes \mathbb{G}_m$ torsor over its image. It remains to show the image is the affine cone over the secant variety to the rational normal curve, minus the cone over rational normal curve.

The coordinates of the resulting point $T_n(q, \xi)$ in $\mathbb{A}^{n+1}$ can be dually thought in terms of the n-dimensional space of hyperplanes containing the point $T_n(q, \xi)$. In these terms, we are looking for the n-dimensional space of hyperplanes spanned by ξ and multiples of q of homogeneous degree n. This corresponds to the intersection point of the span of q on the rational normal curve with ξ (since hyperplanes containing this intersection point are the same as hyperplanes containing the span of the two points defining q and ξ). Now, the map T_n constructed above sends (q, ξ) a point of $\mathbb{A}^{n+1} - 0$ whose image in $\mathbb{P}^n$ is the point of intersection of the degree n forms in the span of q and ξ. Indeed, this holds because the minors constructed above express the Plücker coordinates on n-dimensional spaces of hyperplanes in $\mathbb{A}^{n+1}$.

This implies that, upon quotienting by the scaling action of $\mathbb{G}_m$, geometrically, the map we have algebraically constructed above sends (q, ξ) to the point of intersection of the line spanned by q and ξ. (Note again that we are composing with the map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ here.)

We now want to understand the image of the map T_n, and show that it lies on the secant variety to the rational normal curve, but does not meet the rational normal curve. Begin with a degree 2 cover $g : X \to B$. Given a point (q, ξ), from q one obtains a degree 2 subscheme of $\mathbb{P}^1$, which can be identified with a degree 2 subscheme of the rational normal curve. The line spanned by this degree 2 subscheme defines a secant line to the rational normal curve. The intersection of ξ, viewed as a hyperplane, with this line defines a point on this line. The condition that the resultant is a unit forces this intersection point not to meet q. That is, it does not meet the rational normal curve. □

So, the prediction for average size of n-torsion in class groups of quadratic fields coming from the Cohen lenstra heuristics can be rephrased in terms of counting points on the secant variety to the rational normal curve in $\mathbb{P}^n$, missing the rational normal curve, modulo a certain GL_2 action. However, one must be careful about what the GL_2 action is, as we now remark.

Remark 6.1.2. Using Proposition 6.1.1 it is tempting to think we may have found a direct way to show the bijection between binary cubic forms up to the PGL_2 action and 3-torsion subgroups of the class group, which is usually derived via class field theory (under the bijection between order 3 quotients of the class group and degree 3 unramified extensions).

Namely, using Proposition 6.1.1 we have given a way to directly produce order 3-subgroups of the class group from binary cubic forms, missing the rational normal cubic in $\mathbb{P}^3$, up to a PGL_2 action. That is, beginning with a pair (q, ξ) of unit resultant, which we we can produce an explicit cubic form as the minors of an associated matrix, as in the proof of Proposition 6.1.1. and therefore obtain a binary cubic form, up to the PGL_2 action. If X is the double cover of $\operatorname{Spec}\mathbb{Z}$ isomorphic to $V(q)$, we have shown such orbits correspond to elements of $H^1(X, \mu_3)/\pm 1$. This is very nearly the same as an order 3 subgroup of the class group, and from the map $H^1(X, \mu_3) \to \mathrm{Cl}(X)[3]$, given an element of $H^1(X, \mu_3)/\pm 1$, we can obtain a subgroup of the class group of order dividing 3.

So, may seem at first we have somehow bypassed class field theory to obtain a description of PGL_2 orbits of binary cubic forms as subgroups of the class group (as opposed to quotients).

However, this is in fact a very different parameterization that the usual one, appearing, for example, in [BST13, §2]. The reason for this difference is that the discriminant of the binary cubic form we obtain by sticking the viewing a point $(a, b, c, d) \in \mathbb{A}^4$ as a form $ax^3 + bx^2y + cxy^2 + dy^3$ does not agree with the discriminant of the binary quadratic form. Really, the space $\mathbb{A}^4$ parameterizes the dual space to binary cubic forms, and so it is not natural to think of it as a binary cubic form in the above way.

To see this in a concrete example, begin with the quadratic form $q = x^2 + xy + y^2$, which is the unique quadratic form of discriminant -3. Since the associated ring of integers has cube roots of unity, $H^1(X, \mu_3)/\pm 1$ has order 2, with the two representative pairs (q, ξ) given by (q, y^3) and (q, y^2x). The associated cubic forms we obtain are obtained by taking the minors of the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which gives the cubic forms $x^2y + y^2x$ and $-x^3 - x^2y + y^3$. These cubic forms have discriminants 1 and 23. However, when one puts in factors of $(-1)^i\binom{n}{i}$ in front of the coefficient of x^iy^{n-i}, one obtains binary cubics forms which have discriminant 81 (which is -27 times the intended discriminant -3 of the quadratic form; this -27 is a universal constant that would appear for any initial choice of quadratic form). So when one puts in these factors, one does respect the discriminant, and this recovers the parameterization given in [Bha04, Theorem 13] also used in [BV16].

We note also that the map of Proposition 6.1.1 is not GL_2 equivariant for the x and y variables we have been using in this remark. As an example of this non-equivariance, when one transforms by the linear operator sending $x \mapsto x + y, y \mapsto y$, one fixes y^3 and sends q as above to $x^2 + 3xy + y^2$, and the resulting cubic form is $6y^3 + 3y^2x + yx^2$. On the other hand, if one first constructs the cubic form, one gets $x^2y + y^2x$ and if one then applies the linear operator, one gets $x^2y + 3xy^2 + 2y^3$.

6.2 Apolarity

We now want to understand the relation between the secant variety to the rational normal curve and inflection subschemes. This is best understood through the notion of apolarity, rooted in classical algebraic geometry. See [Dol12, Chapter 1], especially [Dol12, (1.65)] for some description, although it is not completely obvious why our formulation is related to the formulation there.

Consider the rational normal curve $\mathbb{P}^1 \to \mathbb{P}^n$. There is a corresponding *apolarity map* $\mathbb{P}^1 \to (\mathbb{P}^n)^\vee$ given by

$$\begin{aligned} \phi_n : \mathbb{P}^1 &\to (\mathbb{P}^n)^\vee \\ [\beta, \alpha] &\mapsto [\beta^n, -\binom{n}{1}\beta^{n-1}\alpha, \binom{n}{2}\beta^{n-2}\alpha^2, \dots, (-1)^n\alpha^n]. \end{aligned}$$

In characteristic 0 or characteristic more than n, this is isomorphic to a rational normal curve, but it may fail to be so in characteristic at most n.

We start by giving a geometric description of the apolarity map.

Lemma 6.2.1. *The image of the apolarity map parameterizes hyperplanes whose intersection with the rational normal curve consists of a single geometric point, i.e., have order n tangency to the rational normal curve at their unique intersection point.*

Proof. Beginning with the rational normal curve $\rho : \mathbb{P}^1 \to \mathbb{P}^n$ and a point $\rho([\alpha, \beta])$ the hyperplane $\phi_n([\beta, \alpha])$ vanishes to order n at $\rho([\alpha, \beta])$ because the hyperplane $\phi_n([\beta, \alpha])$, viewed as a hyperplane in $\mathbb{P}^n$ is given as $(\beta x - \alpha y)^n$ if we take $x^n, x^{n-1}y, \dots, y^n$ as the coordinates on $\mathbb{P}^n$. This restricts to the degree n subscheme supported at the point $[\alpha, \beta]$ in $\mathbb{P}^1$. □

We want to relate the apolarity map to points on the secant variety to the rational normal curve and inflection subschemes. Given a point in projective space, the next lemma tells us when a point is an inflection point for the corresponding linear series on the rational normal curve.

Lemma 6.2.2. *Let $p \in \mathbb{P}^n = \mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}(n))$ be a point determined by a sub linear system $W \subset H^0(\mathbb{P}^1, \mathscr{O}(n))$ of codimension 1. Then, W corresponds to a hyperplane $[W] \in (\mathbb{P}^n)^\vee$. The inflection points of the resulting map $\mathbb{P}^1 \to \mathbb{P}W$ are identified with the intersection of the hyperplane $[W]$ with $\phi_n(\mathbb{P}^1)$.*

Proof. A point t is an inflection point of the map $\mathbb{P}^1 \to \mathbb{P}W$ if and only if the curvilinear order n subscheme of $\mathbb{P}^1$ supported at t is contained in a hyperplane H. Because $\mathbb{P}^1 \to \mathbb{P}W$ is the projection of the map $\mathbb{P}^1 \to \mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}(n))$ from p, the condition that the order n subscheme is contained in H is equivalent to the order n subscheme associated to the map $\mathbb{P}^1 \to \mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}(n))$ being contained in the cone at p over H. In other words, it is equivalent to the order n subscheme on the rational normal curve being contained in a hyperplane that passes through p. The hyperplane $[W] \in (\mathbb{P}^n)^\vee$ precisely corresponds to hyperplanes in $\mathbb{P}^n$ passing through p. Therefore, we are looking to describe hyperplanes passing through p which meet the rational normal curve to order n. By Lemma 6.2.1, these are parameterized by $[W] \cap \phi_n(\mathbb{P}^1)$. □

Using the previous lemma, we can easily deduce a description of the inflection subscheme in terms of sections of a linear series.

Proposition 6.2.3. *Suppose the codimension 1 subspace W of Lemma 6.2.2 is spanned by the elements $f_1, \ldots, f_n \in H^0(\mathbb{P}^1, \mathscr{O}(n))$, with $f_i = \sum_j a_i^j x^j y^{n-j}$. Then the corresponding inflection subscheme of $\mathbb{P}^1$ associated to $\mathbb{P}^1 \to \mathbb{P}W$ is given by $\sum_{k=0}^n (-1)^k \binom{n}{k} t_k x^k y^{n-k}$ where the t_k is the $n \times n$ minors of the $n \times (n+1)$ matrix $A := (a_i^j)_{1 \le i \le n, 0 \le j \le n}$ with the kth column omitted.*

Proof. Using Lemma 6.2.2, we are trying to compute the intersection of W with the image of the apolarity map. First, let us compute the coordinates of the hyperplane W. If $e_0, \ldots, e_n$ are the coordinates on $\mathbb{P}^n$ then the dual has Plucker coordinates given by $e_0 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots e_n$. In these terms, the kth Plucker coordinate associated to the subspace spanned by $f_1, \ldots, f_n$ has coefficients given as the minor of the matrix A where the kth column is omitted. Now, restricting this along the apolarity map, which we recall is given by the linear system $[\beta^n, -\binom{n}{1}\beta^{n-1}\alpha, \binom{n}{2}\beta^{n-2}\alpha^2, \ldots, (-1)^n\alpha^n]$ which multiplies the ith coordinate by $(-1)^i\binom{n}{i}$, we find that the inflection subscheme is given by $\sum_{k=0}^n (-1)^k \binom{n}{k} t_k x^k y^{n-k}$, as desired. □

Lemma 6.2.4. *Keep notation as in Proposition 6.2.3. Let $r_n := \prod_{i=0}^{n-1}((n-i)! \cdot i!)$. Consider the matrix $B := \left(\partial_x^i \partial_y^{n-1-i} f_j\right)_{0 \le i \le n-1, 1 \le j \le n}$. Then $\det B$ is divisible by r_n. Further, $\det B / r_n \in H^0(\mathbb{P}^1, \mathscr{O}(n))$ gives an alternate expression (at least up to sign) for the inflection subscheme.*

Sketch. I do not know a particularly nice way to see this, other than by directly cranking out the algebra and computing that the coefficient of $x^k y^{n-k}$ in $\det B/r_n$ agrees with the corresponding coefficient $(-1)^k \binom{n}{k} t_k$ determined in Proposition 6.2.3. For example, it is fairly direct to see this for the coefficients of x^n and y^n. For the other coefficients, there is a quite a bit of cancellation upon expanding $\det B$, and one finds that the remaining terms give the desired expression. □

Combining the above, we describe the map sending $(q, \xi) \in V_n^{\mathrm{Res} \in \mathbb{G}_m})$ to its corresponding inflection subscheme as a composition of the map to the secant variety to the rational normal curve and the apolarity map.

Lemma 6.2.5. *There is a natural map* $V_n^{\mathrm{Res} \in \mathbb{G}_m} \to \mathbb{A}^{n+1}$ *which is invariant for the* $\mathbb{G}_m \times \mathbb{G}_a^{n-1}$ *action. This is induced as the composition of the map* $V_n^{\mathrm{Res} \in \mathbb{G}_m} \to \mathbb{A}^{n+1}$ *of Proposition 6.1.1 with the map*

$$\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$$

$$(x_0, \ldots, x_{n+1}) \mapsto ((-1)^i \binom{n}{n-i} x_{n-i})_{0 \le i \le n}$$

inducing the apolarity map.

Proof. Via Proposition 6.2.3, one can compute the $(n-k)$th coefficient as $(-1)^k \binom{n}{k} t_k$ for t_k the $n \times n$ minor omitting column k of the matrix whose rows are $x^{n-2}q, x^{n-1}q, \ldots, y^{n-2}q, \xi$. Using this description, we see that the map is the composition of the map specified in Proposition 6.1.1 with the above map $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ as in the statement, inducing the apolarity map when restricted to the rational normal curve. □

Remark 6.2.6. The map of Lemma 6.2.5 can explicitly be given in several other ways as well. A concrete formula, as shown in Lemma 4.1.10, gives this map as sending q, ξ to $q^n \cdot \partial_x^{(n-1)} \left(\frac{\xi}{q}\right)$.

One can give another explicit description of InflectionEquation(q, ξ). With notation for r_n as in Lemma 6.2.4, at least up to sign, one can show that $r_n \cdot$ InflectionEquation(q, ξ) is the determinant of the matrix

$$\begin{pmatrix} \partial_x^{n-1}(x^{n-2}q) & \partial_x^{n-2}\partial_y(x^{n-2}q) & \cdots & \partial_y^{n-1}(x^{n-2}q) \\ \partial_x^{n-1}(x^{n-3}yq) & \partial_x^{n-2}\partial_y(x^{n-3}yq) & \cdots & \partial_y^{n-1}(x^{n-3}yq) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{n-1}(y^{n-2}q) & \partial_x^{n-2}\partial_y(y^{n-2}q) & \cdots & \partial_y^{n-1}(y^{n-2}q) \\ \partial_x^{n-1}(\xi) & \partial_x^{n-2}(\xi) & \cdots & \partial_y^{n-1}(\xi) \end{pmatrix}.$$

6.2.7 Recovering the quadratic form

So, given (q, ξ) we have demonstrated several ways to obtain the inflection subscheme InflectionEquation(q, ξ) associated q, ξ. Given the InflectionEquation(q, ξ), we cannot in general recover ξ, but we can recover q.

Lemma 6.2.8. *Suppose* $\mathrm{Res}(q, \xi)$ *is a unit and* $f := \mathrm{InflectionEquation}(q, \xi) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} s_i x^i y^{n-i}$. *Then, up to a sign, we can recover* q *as* $(-s_{j+1}s_{j+3} + s_{j+2}^2)x^2 + (s_{j+1}s_{j+2} - s_j s_{j+3})xy + (-s_j s_{j+2} + s_{j+1}^2)y^2$ *for* $0 \leq j \leq n-3$. *Further, it can also be recovered, up to a universal constant as the determinant*

$$\begin{pmatrix} \partial_x^j \partial_y^{n-1-j} f & \partial_x^{j+1} \partial_y^{n-1-(j+1)} f \\ \partial_x^{j+1} \partial_y^{n-1-(j+1)} f & \partial_x^{j+2} \partial_y^{n-1-(j+2)} f \end{pmatrix}.$$

Proof. Given InflectionEquation$(q, \xi) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} s_i x^i y^{n-i}$, unwinding the proof of Proposition 6.2.3 shows that the point $[s_0, \ldots, s_n] \in \mathbb{P}^n$ is the corresponding point on the secant variety to the rational normal curve from which we are projecting and taking the inflection subscheme; therefore, the linear system of hyperplanes vanishing on $[s_0, \ldots, s_n]$ is spanned by $x^{n-2}q, x^{n-3}yq, \ldots, y^{n-2}q, \xi$. Given this data, we now wish to recover q. We know that q, when viewed as a linear form on $\mathbb{P}^2$ (instead of a quadratic form on $\mathbb{P}^1$) must vanish at the points $[s_j, s_{j+1}, s_{j+2}]$ and $[s_{j+1}, s_{j+2}, s_{j+3}]$ with the first corresponding to $x^j y^{n-2-j} q$ vanishing on $[s_0, \ldots, s_n]$ and the second condition corresponding to $x^{j+1} y^{n-2-(j+1)} q$ vanishing on $[s_0, \ldots, s_n]$. So, we are looking for a quadratic form vanishing on these two points in $\mathbb{P}^2$. The quadratic form then has plucker coordinates given by the minors of

$$\begin{pmatrix} s_j & s_{j+1} & s_{j+2} \\ s_{j+1} & s_{j+2} & s_{j+3} \end{pmatrix},$$

as was claimed in the statement.

The final claim regarding the matrix of partials is an elementary algebra exercise. □

6.3 Intersection theory on the abstract secant variety

Now, let's work over $\mathbb{P}^2_{\mathbb{Z}} \simeq \mathrm{Hilb}^2_{\mathbb{P}^1/\mathbb{Z}}$, which we think of as the space parameterizing quadratic forms $ax^2 + bxy + cy^2$ and try to gain a better understanding of this secant variety to the rational normal curve.

We now construct the bundles whose projectivizations yields the secant varieties.

Definition 6.3.1. Let H denote the Hilbert scheme of degree 2 subschemes of $\mathbb{P}^1$, which is abstractly isomorphic to $\mathbb{P}^2 = \mathrm{Hilb}^2_{\mathbb{P}^1} = \mathrm{Sym}^2_{\mathbb{P}^1}$. We have the universal degree 2 subscheme $U \subset \mathbb{P}^1 \times H$, which carries an invertible sheaf $\mathscr{O}_U(1)$, which we define as the pull back from $\mathscr{O}_{\mathbb{P}^1_H}(1)$. Let $g : U \to H$ denote the projection. We define $W^{(n)} := g_*\mathscr{O}_U(n)$, a rank 2 vector bundle on $H \simeq \mathbb{P}^2$. We define $P^{(n)} := \mathbb{P}g_*\mathscr{O}_U(n)$.

We'd next like to understand some basic invariants of $P^{(n)}$, like its degree. First, we want to show it is the abstract secant variety to the rational normal curve in $\mathbb{P}^n$.

Notation 6.3.2. Define the maps

$$\begin{array}{ccccc} U & \xrightarrow{\iota} & \mathbb{P}^1_H & \xrightarrow{s} & \mathbb{P}^1_{\mathbb{Z}} \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle t} \\ & & H & \xrightarrow{h} & \operatorname{Spec}\mathbb{Z} \end{array} \tag{6.3.1}$$

Also, let $\pi : \mathrm{AffineSec}^{(n)} \to H$ denote the projection map.

Lemma 6.3.3. *For $n \geq 1$, the variety $P^{(n)}$ has a map to $\mathbb{P}^n$, identifying it as the abstract secant variety to the rational normal curve in $\mathbb{P}^n$.*

Proof. With U and H as in Notation 6.3.2 Consider the diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathbb{P}^1_H & \longrightarrow & \mathbb{P}^n_H \\ & \searrow & & \swarrow{\scriptstyle r} & \\ & & H & & \end{array} \tag{6.3.2}$$

The abstract secant variety to the rational normal curve in $\mathbb{P}^n$ is by definition the secant variety to U in $\mathbb{P}^n_H$, and the secant variety is the image of this under the map $\mathbb{P}^n_H \to \mathbb{P}^n$. Consider the surjection of sheaves $r_*\mathscr{O}_{\mathbb{P}^n}(1) \to r_*(\mathscr{O}_{\mathbb{P}^n}(1)|_U)$. The latter sheaf is locally free of rank 2, being the pushforward of a locally free sheaf from a degree 2 cover. Further, as long as $n \geq 1$, the above restriction map is a surjection of sheaves, as may be verified on fibers, where we see there is an $n-1$-dimensional subspace of sections of $H^0(\mathbb{P}^n, \mathscr{O}(1))$ vanishing our the degree 2 subscheme corresponding to the given point of H. Note that the source is actually a globally free sheaf on H, and hence the above restriction map corresponds to a map from H to the Grassmannian of lines in $\mathbb{P}^n$. In other words, it yields a projective bundle whose fiber over a point of H is the line vanishing on the corresponding degree 2 subscheme in $\mathbb{P}^n$. This is precisely the abstract secant variety. Finally, we can identify $\mathscr{O}_{\mathbb{P}^n}(1)|_U \simeq \mathscr{O}_{\mathbb{P}^1}(n)|_U$, which is what we were calling $\mathscr{O}_U(n)$. Additionally, the composition $U \to \mathbb{P}^1_H \to \mathbb{P}^n_H \to H$ identifies $g_*\mathscr{O}_U(n)$ with $r_*(\mathscr{O}_{\mathbb{P}^n}(1)|_U)$. □

Definition 6.3.4. For $\phi : P^{(n)} \to \mathbb{P}^n$ the map of Lemma 6.3.3, we define the (embedded) secant variety, $\mathrm{Sec}^{(n)} \subset \mathbb{P}^n$, to be the image of ϕ.

Our next goal is to compute identify $W^{(n)}$ for low n. The following lemma will be useful for this.

Lemma 6.3.5. *There is a natural map $f^*\mathscr{O}_H(-1) \to \mathscr{O}_{\mathbb{P}^1_H}(2)$ induced by $U \subset \mathbb{P}^1_H$. Above, $\mathscr{O}_H(-1)$ denotes the line bundle corresponding to $\mathscr{O}_{\mathbb{P}^2}(-1)$ under the isomorphism $H \simeq \mathbb{P}^2$.*

Proof. It is equivalent to give a map $f^*\mathscr{O}_H(-1) \otimes \mathscr{O}_{\mathbb{P}^1_H}(-2) \to \mathscr{O}_{\mathbb{P}^1_H}$. This is given as the kernel of the map $\mathscr{O}_{\mathbb{P}^1_H} \to \mathscr{O}_U$. Indeed, we can think of $U \subset H \times \mathbb{P}^1 \simeq \mathbb{P}^2 \times \mathbb{P}^1$ as a $(2, 1)$ divisor, given by the equation $ax^2 + bxy + cy^2$, for a, b, c the coordinates of $\mathbb{P}^2$ and x, y the coordinates on $\mathbb{P}^1$. □

Lemma 6.3.6. *We have*

$$W^{(0)} \simeq \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$$
$$W^{(1)} \simeq \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}$$
$$W^{(2)} \simeq T_{\mathbb{P}^2}(-1)$$

We note that $W^{(0)}$ is not especially meaningful, and will not be used in what follows, because the map from $\mathbb{P}^1$ to $\mathbb{P}^0$ is certainly not an embedding, so $W^{(0)}$ is not really connected to a secant variety.

Proof. For the first, statement, the cover $U \to \mathbb{P}^2$ is branched over the degree 2 discriminant divisor, and we get an exact sequence $\mathscr{O}_{\mathbb{P}^2} \to g_*\mathscr{O}_U \to \mathscr{O}_{\mathbb{P}^2}(-1)$. Since $\mathbb{P}^2$ has vanishing first cohomology, there are no extensions of line bundles, so the above sequence splits. For the next two isomorphisms, we use the natural surjection $\mathrm{Sym}^n(g_*\mathscr{O}_U(1)) \to W^{(n)}$. When $n = 1$ this is a map of rank 2 bundles and hence an isomorphism. The source is the trivial rank 2 bundle because it receives a map from $\mathscr{O}_{\mathbb{P}^2} \otimes H^0(\mathbb{P}^1, \mathscr{O}(1)) \to g_*\mathscr{O}_U(1)$ coming from the adjunction, which is an isomorphism, as can be verified on fibers over points of $\mathbb{P}^2$, where the two sections of x and y map injectively to "functions on the degree two scheme given by U restricted to that fiber." When $n = 2$, we get an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathrm{Sym}^2(g_*\mathscr{O}_U(1)) \longrightarrow W^{(2)} \longrightarrow 0 \qquad (6.3.3)$$

with the first map given by the coordinates on $\mathbb{P}^2 \simeq H$, i.e., from Lemma 6.3.5. We can recognize this a twist of the Euler exact sequence by $\mathscr{O}_{\mathbb{P}^2}(-1)$ giving an isomorphism $T_{\mathbb{P}^2}(-1) \simeq W^{(2)}$. □

We next wish to compute the degree of $W^{(n)}$. The following exact sequence will be the key to computing this degree.

Lemma 6.3.7. *For $n \geq 2$, there is an exact sequence*

$$0 \longrightarrow \mathrm{Sym}^{n-2} W^{(1)} \otimes \mathscr{O}_H(-1) \longrightarrow \mathrm{Sym}^n W^{(1)} \longrightarrow W^{(n)} \longrightarrow 0 \tag{6.3.4}$$

where above $\mathscr{O}_H(1)$ denotes the line bundle corresponding to $\mathscr{O}_{\mathbb{P}^2}(1)$ under the isomorphism $H \simeq \mathbb{P}^2$.

Proof. Consider the exact sequence

$$0 \longrightarrow f^*\mathscr{O}_H(-1) \otimes \mathscr{O}_{\mathbb{P}^1_H}(n-2) \longrightarrow \mathscr{O}_{\mathbb{P}^1_H}(n) \longrightarrow \iota_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(n) \longrightarrow 0 \tag{6.3.5}$$

given by the ideal sheaf of U twisted by $\mathscr{O}_{\mathbb{P}^1_H}(n)$. Pushing this forward by f gives the exact sequence above upon noting that $R^1 f_*\mathscr{O}_{\mathbb{P}^1_H}(n-2) = 0$ by cohomology and base change and using $f_*\mathscr{O}_{\mathbb{P}^1_H}(1) \simeq g_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(1) = W^{(1)}$. □

Remark 6.3.8. The exact sequence of Equation 6.3.4 should be thought of a way of realizing the unit resultant condition. Namely, if we think of a quadratic form as having coefficients a, b, c, then the first map can be thought of as the matrix of linear forms

$$\begin{pmatrix} a & 0 & 0 & \cdots \\ b & a & 0 & \cdots \\ c & b & a & \cdots \\ 0 & c & b & \cdots \\ 0 & 0 & c & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We are now ready to compute the degree of $W^{(n)}$, and more generally its Chern class.

Corollary 6.3.9. *The Chern class of $W^{(n)}$ on $H \simeq \mathbb{P}^2$ is given by $1 + (n-1)x + \binom{n}{2}x^2$, where x is the class of the hyperplane on $\mathbb{P}^2$. In particular, $W^{(n)}$ has degree $n-1$ but is not a sum of line bundles of $n \geq 2$.*

Proof. Using the exact sequence (6.3.4), since $W^{(1)}$ is the trivial rank 2 bundle by Lemma 6.3.6, we find that the Chern class of $W^{(n)}$ is the inverse of the Chern class of $\mathrm{Sym}^{n-2} W^{(1)} \otimes \mathscr{O}_H(-1) \simeq \mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus(n-1)}$. This has Chern class $(1-x)^{n-1} = 1 - (n-1)x + \binom{n-1}{2}x^2$ and then we indeed find that $1 + (n-1)x + \binom{n}{2}x^2$ is inverse to $1 - (n-1)x + \binom{n-1}{2}x^2$, as their product is 1.

The degree claim follows from the Chern class calculation, as it is the first Chern class. The fact that these bundles are not sums of line bundles follows from the computation of the second Chern class, and the constraints that the line bundles would have to have positive degrees summing to $n-1$, using the surjection of Lemma 6.3.7. Since the product of their degrees would have to be $\binom{n}{2}$, this is impossible once $n \geq 2$. □

Having computed the Chern class of $W^{(n)}$, we easily deduce its Chow ring.

Lemma 6.3.10. *The chow ring of $P^{(n)}$ is isomorphic to*

$$\mathbb{Z}[x,z]/(x^3, z^2 - (n-1)xz + \binom{n}{2}x^2),$$

where x is the class of the hyperplane on $\mathbb{P}^2$ and z is the class of $\mathscr{O}_{P^{(n)}}(1)$

Proof. This follows from the formula for Chow rings of projective bundles and the above computation of the Chern class of $W^{(n)}$ as in Corollary 6.3.9 using [EH16, Theorem 9.6]. Though, as it may cause confusion, be aware that in [EH16, Theorem 9.6] there they use $\mathbb{P}\mathscr{E}^\vee$ for what we are calling $\mathbb{P}\mathscr{E}$. □

Lemma 6.3.11. *The map $P^{(n)} \to \mathbb{P}\operatorname{Sym}^n t_*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1) \simeq \mathbb{P}^n_{\mathbb{Z}}$ is induced by a surjection $g^*h^*t_*\mathscr{O}(n) \to \iota^*\mathscr{O}_{\mathbb{P}^1_H}(1)$. In particular, this is induced by the line bundle $\mathscr{O}_{P^{(n)}}(1)$ on $P^{(n)}$.*

Proof. The map $P^{(n)} \simeq \mathbb{P}(h_*g_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(n)) \to \mathbb{P}(t_*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n))$ is induced by the natural restriction map

$$t_*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \to h_*f_*s^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \to h_*f_*\iota_*\iota^*s^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) = h_*g_*\iota^*s^*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) = h_*g_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(n)$$

coming from the base change map and adjunction. In particular, because $g_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(n) \simeq \pi_*\mathscr{O}_{P^{(n)}}(1)$ (by definition of $W^{(n)}$) the above map corresponds by adjunction to a surjection $h^*t_*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \to g_*\iota^*\mathscr{O}_{\mathbb{P}^1_H}(n)$ and hence to a surjection $\pi^*h^*t_*\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(n) \to \mathscr{O}_{P^{(n)}}(1)$. Therefore, the resulting map $P^{(n)} \to \mathbb{P}^n$ is induced by the complete linear system associated to the invertible sheaf $\mathscr{O}_{P^{(n)}}(1)$. Note that this indeed has rank $n+1$ as can be seen by pushing forward (6.3.4) along h. □

6.4 The Veronese embedding

We next observe a pleasant geometric fact that describes the $\mathbb{P}^2$ parameterizing lines on the secant variety under the Plucker embedding into the grassmannian of lines on $\mathbb{P}^n$. Recall we are still using the notation of Notation 6.3.2.

Proposition 6.4.1. *The secant lines to the rational normal curve in $\mathbb{P}^n$ sweep out a family of lines over $\mathbb{P}^2 \simeq \mathrm{Hilb}^2_{\mathbb{P}^1}$ which defines a map from $\mathbb{P}^2$ to the grassmannian of lines in $\mathbb{P}^n$. Upon composing with the Plucker embedding, this can be identified with the $(n-1)$-Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^{\binom{n+2}{2}-1} \simeq \mathbb{P}(\wedge^2 h_*(\mathrm{Sym}^n W^{(1)}))$.*

Proof. The map from $\mathrm{Hilb}^2_{\mathbb{P}^1}$ to the grassmannian of lines is induced by the natural surjection $\mathrm{Sym}^n W^{(1)} \to W^{(n)}$ of (6.3.4), which is identified with a rank 2 quotient of the trivial rank n bundle by Lemma 6.3.6. Indeed, under this map, the fiber over a degree 2 subscheme of $\mathbb{P}^1$ corresponds to the line joining the corresponding degree 2 subscheme of the rational normal curve. This defines a map from H to the grassmannian of lines in the projective space $\mathbb{P}(h_*(\mathrm{Sym}^n W^{(1)})))$. The composition with the Plucker embedding to the projectivization of $\wedge^2 h_*(\mathrm{Sym}^n W^{(1)})$ is induced by the surjection $\wedge^2(\mathrm{Sym}^n W^{(1)}) \to \wedge^2 W^{(n)}$. Since $W^{(n)}$ is a rank 2 bundle of degree $(n-1)$, using Corollary 6.3.9, we find that $\wedge^2 W^{(n)}$ is a degree $n-1$ line bundle on $H \simeq \mathbb{P}^2$, and hence isomorphic to $\mathscr{O}_{\mathbb{P}^2}(n-1)$.

It remains to show that the corresponding map $\mathbb{P}^2 \to \mathbb{P}(\wedge^2 h_*(\mathrm{Sym}^n W^{(1)}))$ is induced by the complete linear system. To see this is the case, we wish to verify that the map of bundles $\alpha : h_*(\wedge^2(\mathrm{Sym}^n W^{(1)})) \to h_*(\wedge^2 W^{(n)})$ is an isomorphism. This is shown in Lemma 6.4.2. □

Lemma 6.4.2. *The natural map $h_*(\wedge^2(\mathrm{Sym}^n W^{(1)})) \to h_*(\wedge^2 W^{(n)})$ is an isomorphism.*

Proof. For $\mathscr{F}$ a sheaf, we use $\mathrm{Sym}_2 \mathscr{F}$ to denote the kernel of the surjection $\mathscr{F} \otimes \mathscr{F} \to \wedge^2 \mathscr{F}$. Whenever we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{6.4.1}$$

of sheaves on a scheme, there is a corresponding exact sequence

$$0 \longrightarrow \mathrm{Sym}_2 A \longrightarrow A \otimes B \longrightarrow \wedge^2 B \longrightarrow \wedge^2 C \longrightarrow 0. \tag{6.4.2}$$

We take A, B, C to be the three terms of (6.3.4), and our goal is to show the induced map on global sections from $h_* \wedge^2 B \to h_* \wedge^2 C$ is an isomorphism. Equivalently, if we let $K := \ker\left(\wedge^2 B \to \wedge^2 C\right)$, it is enough to show $h_* K = R^1 h_* K = 0$.

With this reduction in hand, we now want to check $h_* K = 0$. Observe that $A \otimes B \simeq \mathscr{O}(-1)^{\oplus n-1} \otimes \mathscr{O}^{\oplus n+1}$ while $\mathrm{Sym}_2 A \simeq \mathscr{O}(-2)^{\oplus \binom{n}{2}}$. It follows that $R^1 h_* \mathrm{Sym}_2 A = 0$ and $h_*(A \otimes B) = 0$, implying $h_* K = 0$. Similarly, we find $R^1 h_* K = 0$. □

Remark 6.4.3. In fact, Proposition 6.4.1 can also be understood geometrically in terms of the representation theory of SL_2 and "geometric plethysm" as in [FH91, §11.3]. In those terms, if V denotes the standard 2-dimensional representation of SL_2, then the rational normal curve in $\mathbb{P}^n$ corresponds to $\mathrm{Sym}^n V$. The map from the $\mathbb{P}^2$ of lines joining points on the rational normal curve to the grassmannian of lines composed with the Plucker embedding corresponds to $\wedge^2(\mathrm{Sym}^n V)$. The $(n-1)$-Veronese of $\mathbb{P}^2$ can be identified with $\mathrm{Sym}^{n-1}(\mathrm{Sym}^2 V)$. In these terms, Proposition 6.4.1 is closely related to the isomorphism SL_2 representations $\wedge^2(\mathrm{Sym}^n V) \simeq \mathrm{Sym}^{n-1}(\mathrm{Sym}^2 V) \simeq \mathrm{Sym}^{n-1}(\mathrm{Sym}^2 V)$ coming from [FH91, 11.34 and 11.35]. These are both isomorphic to $\mathrm{Sym}^{2n-2} V \oplus \mathrm{Sym}^{2n-6} V \oplus \mathrm{Sym}^{2n-10} V \oplus \cdots \oplus \mathrm{Sym}^{2n-2-4i} V \oplus \cdots$. Geometrically, these representations in $\mathrm{Sym}^{n-1}(\mathrm{Sym}^2 V)$ can be identified with the product of i copies of the conic in $\mathbb{P}^2$ (thought of as a rational normal curve in $\mathbb{P}^2$) times the span of those degree $(n-1)-2i$ powers of linear forms which are tangent to the conic. See [FH91, §11.3], especially [FH91, p. 159-160] for more on this and related statements. We have not carefully worked out what all the subrepresentations correspond to geometrically in $\wedge^2(\mathrm{Sym}^n V)$, but, at least the top dimensional one $\mathrm{Sym}^{2n-2} V$ corresponds to the variety of tangent lines to the rational normal curve in $\mathbb{P}^n$.

6.5 Batyrev Manin for the secant variety

We'd next like to verify the Batyrev Manin Conjecture for $P^{(n)}$, or at least understand what it predicts. Recall this says that if a is the minimum rational power of $\mathscr{O}(1)$ for which $\mathscr{O}(1)^{\otimes a} + K_{P^{(n)}}$ is effective as a rational divisor class, then $\mathscr{O}(1)$ should have asymptotically about B^a points in a box of side length B. We next compute the canonical sheaf and the effective cone. We now identify the hilbert scheme H with $\mathbb{P}^2$.

Lemma 6.5.1. *The canonical sheaf $P^{(n)}$ is $\mathscr{O}_{P^{(n)}}(-2) \otimes \pi^* \mathscr{O}_{\mathbb{P}^2}(n-4)$.*

Proof. We use two exact sequences: the relative Euler exact sequence

$$0 \longrightarrow \Omega_{P^{(n)}/\mathbb{P}^2} \longrightarrow \mathscr{O}_{P^{(n)}}(-1) \otimes \pi^* W^{(n)} \longrightarrow \mathscr{O}_{P^{(n)}} \longrightarrow 0 \tag{6.5.1}$$

and the relative tangent exact sequence

$$0 \longrightarrow \pi^* \Omega_{\mathbb{P}^2} \longrightarrow \Omega_{P^{(n)}} \longrightarrow \Omega_{P^{(n)}/\mathbb{P}^2} \longrightarrow 0 \tag{6.5.2}$$

The first exact sequence shows $\omega_{P^{(n)}/\mathbb{P}^2} \simeq \mathscr{O}(-2) \otimes \pi^* \det W^{(n)}$. Note that $\det W^{(n)} \simeq \mathscr{O}_{\mathbb{P}^2}(n-1)$ by Corollary 6.3.9. Plugging this the second exact sequence, and using $\omega_{\mathbb{P}^2} \simeq \mathscr{O}_{\mathbb{P}^2}(-3)$, we get $\omega_{P^{(n)}} \simeq \mathscr{O}_{W^{(n)}}(-2) \otimes \mathscr{O}_{\mathbb{P}^2}(n-4)$. □

Lemma 6.5.2. *The effective cone of $P^{(n)}$ is spanned by $[U]$ and $\pi^* \mathscr{O}_{\mathbb{P}^2}(1)$.*

Proof. Certainly the two classes above are effective. Let H denote the class of $\pi^*\mathscr{O}_{\mathbb{P}^2}(1)$. It is therefore enough to show that $A_\varepsilon := [U] - \varepsilon H$ is not effective for any $\varepsilon > 0$. If it were effective, we would have $A_\varepsilon \cdot \mathscr{O}(1) \cdot \mathscr{O}(1) \geq 0$, since $\mathscr{O}(1)$ defines an embedding to projective space away from U and so there are sections of $\mathscr{O}(1)$ not vanishing on any given proper closed subscheme. However, $\mathscr{O}(1)$ contracts U to a curve, and so $U \cdot \mathscr{O}(1) \cdot \mathscr{O}(1) = 0$, and therefore $A_\varepsilon \cdot \mathscr{O}(1) \cdot \mathscr{O}(1) < 0$, implying such a divisor is not in the effective cone. □

Using the above, we can explicitly describe the effective cone in terms of $\mathscr{O}(1)$ and $\pi^*\mathscr{O}_{\mathbb{P}^2}(1)$.

Lemma 6.5.3. *The class of U in $P^{(n)}$ is $2z + (2-n)x$, so $\mathscr{O}(U) \simeq \mathscr{O}(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(2-n)$. In particular, the effective cone of $P^{(n)}$ is generated by $\mathscr{O}(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(2-n)$ and $\pi^*\mathscr{O}_{\mathbb{P}^2}(1)$.*

Proof. We can use intersection theory, using the method of undetermined coefficients. Since U is a divisor, working in the Chow ring of $P^{(n)}$ via the description of Lemma 6.3.10, we can write $[U] = ax + bz$. Since $U \to H = \mathbb{P}^2$ is a degree 2 map, we find $(ax + bz) \cdot x^2 = 2x^2 z$. Also, since U is contracted under the map induced by $\mathscr{O}(1)$ (the line bundle whose class is z), we find $(ax + bz) \cdot z^2 = 0$, because the intersection of a hyperplane with itself will miss the rational normal curve in the secant variety.

Now, it is a calculation. The first relation shows $b = 2$. We want to find a using $(ax + 2z) \cdot z^2 = 0$. Expanding this, we find $0 = (ax + 2z)((n-1)xz - \binom{n}{2}x^2) = (a(n-1) + 2(n-1)^2 - 2\binom{n}{2})x^2 z$. Solving for a gives $a = 2 - n$. Therefore, the class of U is $(2-n)x + 2z$.

The final statement follows from Lemma 6.5.2. □

We also give another way to directly determine the effective cone specific in the case $n = 3$. This is more or less superseded by the previous results, and we encourage the reader to skip the following proof.

Lemma 6.5.4. *The effective cone of $P^{(3)}$ is spanned by $\pi^*\mathscr{O}_{\mathbb{P}^2}(1)$ and $\mathscr{O}(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(-1)$.*

Proof. The key observation is to show that $\pi^*\mathscr{O}_{\mathbb{P}^2}(d)$ sweeps out a degree $2d$ surface in $\mathbb{P}^3$ under the map induced by $\mathscr{O}(1)$. Once we show this, it will follow that $\mathscr{O}(2d) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(-d)$ is effective, but $\mathscr{O}(2d) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(-d-1)$ and $\mathscr{O}(2d+1) \otimes \pi^*\mathscr{O}_{\mathbb{P}^2}(-d-1)$ are not effective. This characterizes the effective cone.

The key observation is that any line in $\mathbb{P}^2$ has preimage mapping to a quadric surface. Once we know this, it follows that a degree d curve has preimage mapping to a degree $2d$ surface. Because the class in the chow ring is well defined up to rational equivalence, and flat pullback and proper pushforward induce homomorphisms of Chow rings, it is enough to check this in the case of a particular line in $\mathbb{P}^2$. Let's just concentrate on a line tangent to the discriminant locus at a point p. The discriminant curve maps to the rational normal curve. The tangent line maps to the cone over p joining p to other points on the rational normal curve. The projection of this cone from p gives a plane conic, which implies that this cone is the cone over a plane conic, and hence is a quadric cone. Therefore, it has degree 2 as desired. □

Proposition 6.5.5. *Batyrev Manin holds for $P^{(3)}$.*

Proof. Using Lemma 6.5.1, the canonical divisor is $\mathcal{O}(-2) \otimes \pi^* \mathcal{O}(-1)$. By Lemma 6.5.4, we find that the minimum tensor power a of $\mathcal{O}(1)$ so that $\mathcal{O}(a) \otimes K = \mathcal{O}(a-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(-1)$ is effective is $a = 4$. Hence, we would like to know that there are B^4 points in a ball of height B for $P^{(3)}$. But $\mathcal{O}(1)$ maps $P^{(3)}$ surjectively to $\mathrm{Sec}^{(3)} \simeq \mathbb{P}^3$, and so we find this indeed has B^4 points in a ball of height B. □

Let's now summarize what Batyrev Manin would predict.

Remark 6.5.6. According to Batyrev-Manin, the asymptotic number of points on our variety in a ball of size B should be approximately B^a (times a power of $\log B$) where a is the minimum rational number for which $\mathcal{O}(a) + K$ which is in the effective cone. We have seen $K = \mathcal{O}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(n-4)$, and so the desired minimum value of a would be when $\mathcal{O}(a-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(n-4)$ is effective.

Using the computation of the effective cone from Lemma 6.5.2, we saw the effective cone was spanned by $\pi^* \mathcal{O}_H(1)$ and $\mathcal{O}(2) \otimes \pi^* \mathcal{O}_H(2-n)$. Therefore, once $n \geq 4$, we will only have $\mathcal{O}(a-2) \otimes \pi^* \mathcal{O}_H(n-4)$ effective when $a \geq 2$.

Working this out explicitly for some low values of n, we find the following. When $n = 3$, we want $\mathcal{O}(a-2) \otimes \pi^* \mathcal{O}_H(-1)$ to be in the effective cone spanned by $\pi^* \mathcal{O}_H(1)$ and $\mathcal{O}(2) \otimes \pi^* \mathcal{O}_H(-1)$. This happens when $a-2$ is at least 2 so a is at least 4, and Batyrev-Manin predicts about B^4 points. Note that this is consistent with the fact that when $n = 3$, the embedded secant variety to the rational normal curve is all of $\mathbb{P}^3$, and so there are indeed asymptotically B^4 (up to a constant) points in $\mathbb{P}^3$ in a ball of radius B. When $n = 4$, we want $\mathcal{O}(a-2)$ to be effective, when the effective cone is spanned by $\pi^* \mathcal{O}_H(1)$ and $\mathcal{O}(2) \otimes \pi^* \mathcal{O}_H(-2)$. This happens when $a \geq 2$. When $n = 5$, we want $\mathcal{O}(a-2) \otimes \pi^* \mathcal{O}_H(1)$ to be in the effective cone spanned by $\pi^* \mathcal{O}_H(1)$ and $\mathcal{O}(2) \otimes \pi^* \mathcal{O}_H(-3)$. Again, this happens when $a \geq 2$.

Remark 6.5.7. Let us now discuss the number of points on the secant variety to the rational normal curve when $n \geq 4$. One can see that if one fixes a single line on the secant variety of low height, (say the line joining $[1, 0, \ldots, 0]$ and $[0, \ldots, 0, 1]$,) then this line already has B^2 points. On the other hand, it seems the Cohen-Lenstra heuristics (non-obviously) would likely imply that once $n \geq 5$, there should be fewer than B^2 n-torsion quadratic forms in a ball of size B.

What is going on here is that we are counting all points on the secant variety, when we should really only be counting integral points not meeting the rational normal curve. When $n = 4$, it seems a positive proportion of points do not meet the rational normal curve, but once $n \geq 5$, it seems that asymptotically 100% of points meet the rational normal curve.

Chapter 7

The cohomology of the resultant 1 hypersurface over finite fields

7.1 The main result on cohomology of the resultant 1 hypersurface, remarks, and questions

We would like to compute the étale cohomology of the Resultant 1 hypersurface over $\mathbb{F}_q$. By this we mean those pairs of (not necessarily monic) polynomials (q, ξ) of degrees 2 and n with resultant 1.

Remark 7.1.1. Our motivation for calculating this cohomology was for its potential use to calculating the major arcs if one wanted to apply the circle method to try to count points on the resultant 1 hypersurface. This is closely related to the Cohen-Lenstra heuristics via Theorem 1.2.4. Of course, it would likely be much more difficult to bound the contribution from the minor arcs. It also seemed like a fun exercise in using weights to calculate etale cohomology groups.

The idea is to follow the strategy of [FW17] with suitable modifications.

Definition 7.1.2. We use $\mathrm{Poly}^{(a,b)}$ to denote the open subscheme of $\mathbb{A}^{a+b}$ parameterizing pairs of homogeneous (not necessarily monic) polynomials of degrees a and b in the x and y variables with no common root. We also use $R_k^{(a,b)}$ to denote pairs of (not necessarily monic) homogeneous polynomials of degree a and b with some common factor of degree k. We will use $\mathrm{Poly}_1^{(a,b)}$ to denote those pairs of homogeneous polynomials of degrees a and b with exactly 1 common root.

For this section, we work over a finite field $\mathbb{F}_q$ and choose a prime ℓ relatively prime to

q. The main result we would like to show is the following. Let $\mathrm{Res} : \mathrm{Spec}\,\mathrm{Sym}^\bullet H^0(\mathscr{O}(a) \oplus \mathscr{O}(b)) \to \mathbb{A}^1_t$ denote the map sending a pair of polynomials to their resultant.

Theorem 7.1.3. *Fix a prime power q and a prime ℓ which is relatively prime to q. For $n \geq 2$, the open subscheme $\mathrm{Poly}^{(2,n)} = \mathrm{Res}^{-1}(\mathbb{G}_m)$ has cohomology*

$$H^i_{c,\text{ét}}(\mathrm{Poly}^{(2,n)}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+4) & \text{if } i = 2n+8 \\ \mathbb{Q}_\ell(n+3) & \text{if } i = 2n+7 \\ \mathbb{Q}_\ell(n+2) & \text{if } i = 2n+5 \\ \mathbb{Q}_\ell(n+1) & \text{if } i = 2n+4 \\ 0 & \text{else.} \end{cases}$$

Remark 7.1.4. In accordance with this, the point count over $\mathbb{F}_q$ is $q^{n+4} - q^{n+3} - q^{n+2} + q^{n+1}$. This follows from the Grothendieck Lefschetz trace formula. The weights indicate the trace of Frobenius on the nonzero cohomology groups, and hence their contribution to the point counts. This can be deduced by working in the Grothendieck ring and using $\mathbb{G}_m$ fibrations to allow the leading coefficient to not be monic. essentially following the same method as [HP19, Proposition 18]. They state it in characteristic 0, but the only issue with working in characteristic p is that the potentially may have to invert radicial surjective maps in the Grothendieck ring. That it has a "bijective homeomorphism" (i.e., radicial surjective map) to a scheme with the claimed class is explained in [FW19]. Alternatively, see [VW20].

Corollary 7.1.5. *Let n be odd. Then, the resultant 1 hypersurface $\mathrm{Res}^{-1}(t = 1) \subset \mathrm{Spec}\,\mathrm{Sym}^\bullet H^0(\mathscr{O}(2) \oplus \mathscr{O}(n))$ has $\#(\mathrm{Res}^{-1}(t = 1))(\mathbb{F}_q) = q^{n+3} + q^{n+1}$. Furthermore,*

$$H^i_{c,\text{ét}}(\mathrm{Res}^{-1}(t = 1), \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+3) & \text{if } i = 2n+7 \\ \mathbb{Q}_\ell(n+1) & \text{if } i = 2n+4 \\ 0 & \text{else.} \end{cases}$$

Proof. The first statement follows from the second and the Grothendieck Lefschetz trace formula. Because n is odd, the unit resultant open subscheme has cohomology isomorphic to the tensor product of the cohomology of $\mathbb{G}_m$ with the resultant 1 hypersurface. The action of $\mathbb{G}_m$ is given by $\lambda \cdot (q, \xi) = (\lambda^{-1} q, \lambda^{(n+1)/2} f)$. This uses that n is odd. Now, if we take the product of $\mathbb{G}_m$ with the resultant 1 hypersurface, we get an isomorphism to the unit resultant open subscheme $\mathrm{Poly}^{(2,n)}$ and so the claim follows from the Kunneth theorem. □

The above theorem suggests some interesting questions in topology.

Question 7.1.6. 1. Can one compute the cohomology of pairs of polynomials of arbitrary degrees (not just those where the first has degree 2) which no common root?

2. Can one compute the cohomology of the space of triples of polynomials where no three share a common root?

3. What about the space of polynomials where no two of the three share a common root?

4. Similarly, what about the analogous questions for more than 3 polynomials?

Remark 7.1.7. These questions are closely related to computing the cohomology of configuration spaces of tuples multi-colored points, on $\mathbb{P}^1$. It is quite possible that the following method of proof, or something similar, could be used to answer these. One paper discussing some closely related topics to the following question is [FWW19].

7.2 Computing the cohomology

In this section, we carry out the proof of Theorem 7.1.3. In what follows, we will need the Leray-Hirsch theorem. This is well known in topology, and presumably standard in étale cohomology. In any case, here is a simple proof of the version we will need.

Lemma 7.2.1 (Leray-Hirsch)**.** *Suppose $X \to Y$ is a $\mathbb{G}_m$ torsor and we have a map $X \to \mathbb{G}_m$ so that the induced map $f : X \to Y \times \mathbb{G}_m$ is finite flat. Then, for ℓ prime to q, the induced map $\phi : H^\bullet_{c,\text{ét}}(Y, \mathbb{Q}_\ell) \otimes H^\bullet_{c,\text{ét}}(\mathbb{G}_m, \mathbb{G}_\ell) \to H^\bullet_{c,\text{ét}}(X, \mathbb{Q}_\ell)$ is an isomorphism.*

Proof. The "transfer map" or norm map given by pulling back cohomology from $Y \times \mathbb{G}_m$ to X and then pushing forward down to $Y \times \mathbb{G}_m$ is multiplication by $\deg f$ on the cohomology of $Y \times \mathbb{G}_m$. Therefore, we obtain that ϕ is an injection. The Leray spectral sequence for the cohomology of X as a $\mathbb{G}_m$ torsor over Y gives an upper bound on the dimensions of the cohomology of X by that of $Y \times \mathbb{G}_m$. Because the cohomology of $Y \times \mathbb{G}_m$ injects into that of X, as demonstrated above, the injection must be an isomorphism. □

We now want to compute the cohomology of the unit resultant hypersurface to prove Theorem 7.1.3.

We have an exact sequence on compactly supported cohomology induced by the open and closed inclusions

$$R_1^{(2,n)} \longrightarrow R_0^{(2,n)} \longleftarrow \text{Poly}^{(2,n)}. \tag{7.2.1}$$

Observe $R_0^{(2,n)}$ is simply an affine space of dimension $n+4$, so for our goal, it is equivalent to compute the cohomology of $R_1^{(2,n)}$ as

$$H^i_{c,\text{ét}}(R_1^{(2,n)}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+3) & \text{if } i = 2n+6 \\ \mathbb{Q}_\ell(n+2) & \text{if } i = 2n+4 \\ \mathbb{Q}_\ell(n+1) & \text{if } i = 2n+3 \\ 0 & \text{else} \end{cases}$$

For this, we will use the cofiber sequence

$$R_2^{(2,n)} \longrightarrow R_1^{(2,n)} \longleftarrow \text{Poly}_1^{(2,n)}. \tag{7.2.2}$$

The idea will now be to compute the cohomology of the left and right terms, and then analyze the long exact sequence via weights.

Lemma 7.2.2. *We have*

$$H^i_{c,\text{ét}}(R_2^{(2,n)}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+2) & \textit{if } i = 2n+4 \\ \mathbb{Q}_\ell(n+1) & \textit{if } i = 2n+2 \\ \mathbb{Q}_\ell(n-1) & \textit{if } i = 2n-1 \\ 0 & \textit{else.} \end{cases}$$

Proof. We can write $R_2^{(2,n)}$ as a disjoint union of two schemes: the first is $\mathbb{A}^{n+1}$ corresponding to those pairs $(0, \xi)$; the second is $(\mathbb{A}^3 - 1) \times \mathbb{A}^{n-1}$ consisting of those pairs (q, ξ') with $q \neq 0$. The map $(\mathbb{A}^3 - 1) \times \mathbb{A}^{n-1} \to R_2^{(2,n)}$ is given by $(q, \xi') \mapsto (q, q\xi') \in R_2^{(2,n)}$. The long exact sequence on compactly supported cohomology for these two pieces then gives the claim, essentially because the cohomology of these two pieces do not interfere with each other. □

The next lemma is the heart of our computation, and is somewhat involved.

Lemma 7.2.3. *We have*

$$H^i_{c,\text{ét}}(\text{Poly}_1^{(2,n)}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+3) & \textit{if } i = 2n+6 \\ \mathbb{Q}_\ell(n+1)^{\oplus 2} & \textit{if } i = 2n+3 \\ \mathbb{Q}_\ell(n-1) & \textit{if } i = 2n \\ 0 & \textit{else.} \end{cases}$$

Proof. The idea here to construct a certain $\mathbb{G}_m$ bundle over $\mathrm{Poly}_1^{(2,n)}$ which is a product, making its cohomology easy to compute. And we then use Leray Hirsch to relate the cohomology of this $\mathbb{G}_m$ bundle to the cohomology of $\mathrm{Poly}_1^{(2,n)}$.

Consider the subset $U_n = (\mathbb{A}^2 - 0) \times \mathrm{Poly}^{(1,n-1)} \subset \mathrm{Spec}\,\mathrm{Sym}^\bullet H^0(\mathscr{O}(1)) \times H^0(\mathscr{O}(1) \oplus \mathscr{O}(n-1))$ where the first factor $\mathbb{A}^2 - 0$ indicates the degree 1 polynomial is nonzero. There is a map

$$\begin{aligned} U_n &\to \mathrm{Poly}_1^{(2,n)} \\ (l,(m,g)) &\mapsto (lm, lg) \end{aligned}$$

making U_n a $\mathbb{G}_m$ bundle over $\mathrm{Poly}_1^{(2,n)}$.

The cohomology of U_n is easy to compute from the following lemma.

Lemma 7.2.4. *We have*

$$H^i_{c,\text{ét}}(\mathrm{Poly}^{(1,n-1)}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+2) & \text{if } i = 2n+4 \\ \mathbb{Q}_\ell(n+1) & \text{if } i = 2n+3 \\ \mathbb{Q}_\ell(n) & \text{if } i = 2n+1 \\ \mathbb{Q}_\ell(n-1) & \text{if } i = 2n \\ 0 & \text{else.} \end{cases}$$

Proof. We can write $\mathrm{Poly}^{(1,n-1)}$ as the disjoint union of $\mathbb{A}^n$ and $\mathbb{A}^{n-1} \times (\mathbb{A}^2 - 0)$ where the inclusion of $\mathbb{A}^n$, thought of as a degree $n-1$ polynomial sends $\xi \mapsto (0, \xi)$ while the inclusion of the latter is given by thinking of $\mathbb{A}^2 - 0$ as a degree 1 nonzero polynomial l and the inclusion sends $(l, \xi') \mapsto (l, l\xi')$. The statement on weights follows because we understand the weights of affine space and $\mathbb{A}^2 - 0$. This gives the claim for $\mathrm{Poly}^{(1,n-1)}$, and the claim for $\mathrm{Poly}^{(1,n-1)} \times \mathbb{A}^1$ follows from the Kunneth formula. □

Corollary 7.2.5. *We have*

$$H^i_{c,\text{ét}}(U_n, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(n+4) & \text{if } i = 2n+8 \\ \mathbb{Q}_\ell(n+3) & \text{if } i = 2n+7 \\ \mathbb{Q}_\ell(n+2)^{\oplus 2} & \text{if } i = 2n+5 \\ \mathbb{Q}_\ell(n+1)^{\oplus 2} & \text{if } i = 2n+4 \\ \mathbb{Q}_\ell(n) & \text{if } i = 2n+2 \\ \mathbb{Q}_\ell(n-1) & \text{if } i = 2n+1 \\ 0 & \text{else.} \end{cases}$$

Proof. This follows from the Kunneth formula, the above lemma, and the fact that the compactly supported cohomology of $\mathbb{A}^2 - 0$ is $\mathbb{Q}_\ell(2)$ in degree 4 and $\mathbb{Q}_\ell$ in degree 1. □

Now, in order to show the cohomology of $\text{Poly}_1^{(2,n)}$ is as claimed, it is enough to show $H^\bullet_{c,\text{ét}}(U_n, \mathbb{Q}_\ell) = H^\bullet_{c,\text{ét}}(\mathbb{G}_m, \mathbb{Q}_\ell) \otimes H^\bullet_{c,\text{ét}}(\text{Poly}_1^{(2,n)}, \mathbb{Q}_\ell)$. We will deduce this from Leray Hirsch.

To apply Leray Hirsch, it is enough to construct a map $f_n : U_n \to \mathbb{G}_m$ so that on any fiber of $\pi_n : U_n \to \text{Poly}_1^{(2,n)}$ we have that $f_n|_{\pi_n^{-1}(t)} : \pi_n^{-1}(t) \simeq \mathbb{G}_m \to \mathbb{G}_m$ is raising to some fixed nonzero power. Indeed, consider the map which, on U_n sends a pair $(l, (m, g)) \mapsto \text{Res}(m, g)$. The set of points in the fiber of π_n $(l, (m, g))$ consists of those pairs $\left(\alpha^{-1} l, (\alpha m, \alpha g)\right)$ for $\alpha \in \mathbb{G}_m$, and so under the resultant map, this sends $\alpha \mapsto \alpha^n \text{Res}(m, g)$. Therefore the resulting map from $\mathbb{G}_m \to \mathbb{G}_m$ is given by raising to the nth power on fibers. Therefore, Leray-Hirsch Lemma 7.2.1 implies, $H^\bullet_{c,\text{ét}}(U_n, \mathbb{Q}_\ell) = H^\bullet_{c,\text{ét}}(\mathbb{G}_m, \mathbb{Q}_\ell) \otimes H^\bullet_{c,\text{ét}}(\text{Poly}_1^{(2,n)}, \mathbb{Q}_\ell)$, giving the claim. □

We are now ready to prove the main result of this chapter and complete our computation of the etale cohomology of the resultant 1 hypersurface.

Theorem 7.1.3. Taking cohomology in the exact sequence (7.2.2), and using Lemma 7.2.2 and Lemma 7.2.3, we only need to show the two boundary maps from degrees $2n+2$ to $2n+3$ and $2n-1$ to $2n$ are isomorphisms. If either of them failed to be isomorphisms, then combining (7.2.2) with (7.2.1) shows that the cohomology of $\text{Poly}_1^{(2,n)}$ would fail to be the tensor product of the cohomology of $\mathbb{G}_m$ with the cohomology of something else. So it suffices to show the cohomology of $\text{Poly}_1^{(2,n)}$ decomposes as a tensor product of $\mathbb{G}_m$ and something else. This is essentially a version of Leray-Hirsch for étale cohomology. Namely, in the case of odd n, we know (as explained in the proof of Corollary 7.1.5) the hypersurface is actually a product with the resultant 1 hypersurface, so we are certainly fine in that case.

If we want to also deal with the even case, we can use that there is a $\mathbb{G}_m$ action on the unit resultant open subscheme $\operatorname{Res}^{-1}(\mathbb{A}^1 - 0) \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(n))$, given by scaling q and f. Let Y denote the quotient of the unit resultant open subscheme by this $\mathbb{G}_m$ action. There is also a map $\operatorname{Res}^{-1}(\mathbb{A}^1 - 0) \to \mathbb{G}_m$ given by taking the resultant.

So we get a map $\operatorname{Res}^{-1}(\mathbb{A}^1 - 0) \to Y \times \mathbb{G}_m$. We claim this is a finite flat map. Indeed, if we take such a fiber, and apply the resultant map, it is a map $\mathbb{G}_m \to \mathbb{G}_m$ given by multiplication by the degree of the resultant, which is $n + 2$.

Therefore, $\operatorname{Res}^{-1}(\mathbb{A}^1 - 0) \to Y \times \mathbb{G}_m$. induces an isomorphism on $\mathbb{Q}_\ell$ cohomology, by Leray-Hirsch Lemma 7.2.1, as we wished to show. □

Part II

Low degree covers and elliptic curves

Chapter 8

Classical geometry and Selmer spaces

In this chapter, we describe three related constructions deriving from classical geometry, which relate to Selmer groups of elliptic curves. We start by setting notation in § 8.2 and introducing some definitions in § 8.3. In § 8.4, we connect the 2-Selmer space to 2-torsion line bundles on trigonal curves (i.e., degree 3 covers of $\mathbb{P}^1$). This is an incarnation of the Recillas correspondence. In § 8.5, we connect a certain sub-locus of the 3-Selmer space to 3-torsion lines bundles on hyperelliptic curves. Finally, in § 8.6, we connect a certain sub-locus of the 2-Selmer space to 2-torsion line bundles on hyperelliptic curves. All of these constructions have strong roots in classical geometry; see Remark 8.4.5, Remark 8.5.11, and Remark 8.6.6 relating them to solving quartic equations, cubic equations, and equations with Galois group D_8. They can also all be understood in terms of fixed loci of automorphisms of elliptic curves.

8.1 Story of the project

We now take some time to describe how we came to discover the material of this section. It began at the end of my first year in grad school. I was interested in understanding low degree covers of $\mathbb{P}^1$, which prompted me to learn about the classical Recillas correspondence and its connection to solving the quartic equation. This prompted me to teach a class at Mathcamp that summer on solving the quartic equation.

I asked Ravi for help finding a project, and he suggested reading the self-contained 2-page proof of [vG05, Theorem 6.2]. This led me to realize there was a close connection between 2-Selmer groups of elliptic curves and the Recillas correspondence. Namely, torsors for the 2-torsion of an elliptic curve could be thought of approximately as 2-Selmer elements, and their resolvent cubic was the nontrivial 2-torsion of the elliptic curve. This led me to understand § 8.4, and the later sections § 8.5 and § 8.6 could be viewed as analogous constructions corresponding to elliptic curves with extra automorphisms.

I was able to use this idea to compute the average size of 2-Selmer groups over function fields in a geometric setting, and gave an area exam talk on it. After my talk, Zev Rosengarten asked if it might be possible to generalize this to computing average size of n-Selmer groups. This led me to the beautiful paper of Ellenberg-Venkatesh-Westerland, and I began thinking about whether it was possible to use their topological techniques to approach the question of n-Selmer groups of elliptic curves. I visited Wisconsin several months later for a conference, and talked to Jordan Ellenberg, who suggested it might even be interesting first to compute the average size of n-Selmer groups after first taking a large q limit. I worked out the details of this approach, and wrote a preprint, which used, in a crucial way, the material of this chapter.

Just after I finished writing the preprint, I realized there was a serious hole. I had attempted to apply [Hal08, Theorem 6.4] to my setting. However, that only applied for quadratic twist families. I had shown it applied on an open locus of the relevant constructible sheaf, but unfortunately that open locus was the empty set. Fortunately, I had sent the draft to Anand Deopurkar, who, without even being aware there was an error, immediately pointed me to the paper [dJF11]. This had proven the relevant monodromy result that I could use in place of [Hal08, Theorem 6.4].

As a result, I was able to fix the error, but the contents of this chapter, which I very much enjoyed, were no longer needed in the paper [Lan21a]. It would have been a shame if the classical geometry appearing in this chapter continued to be hidden away, and so I am pleased this thesis is able to provide a home for it.

8.2 Summary of various spaces introduced

For the reader's convenience, in Figure 8.1, we collect notation for the various spaces we introduce throughout this chapter.

8.3 Reviewing the definition of the Selmer space

Here, we briefly recall the construction of the Selmer space and related spaces introduced in [Lan21a, §3].

Notation	Description	Location defined
$\mathcal{E}[n]_B^d$	The sheaf of relative n-torsion of elliptic curves over an open of $\mathscr{W}_B^d \times_B \mathbb{P}_B^1$	§8.3
$\mathscr{T}_B^d$	The moduli of trigonal curves in the Hirzebruch surface $\mathbb{F}_{2d}$	Definition 8.4.1
$\mathscr{U}\mathscr{T}_B^d$	The universal family of trigonal curves over $\mathscr{T}_B^d$	Definition 8.4.1
$\mathscr{H}_B^d$	The moduli of hyperelliptic curves with a map to $\mathbb{P}^1$	Definition 8.5.1
$\mathscr{U}\mathscr{H}_B^d$	The universal family of hyperelliptic curves over $\mathscr{H}_B^d$	Definition 8.5.1
$\mathscr{W}_B^{\boxed{3}\,d}$	The locus in $\mathscr{W}_B^d$ of elliptic curves possessing a geometric order 3 automorphism	Definition 8.5.2
$\mathscr{U}\mathscr{W}_B^{\boxed{3}\,d}$	The universal family of weierstrass models over $\mathscr{W}_B^{\boxed{3}\,d}$	Definition 8.5.2
$\mathscr{Z}_B^d$	The disjoint union $\mathscr{U}\mathscr{H}_B^d \coprod \mathbb{P}^1_{\mathscr{H}_B^d}$	Definition 8.5.5
$\mathscr{Z}_B^{\mathrm{sm},d}$	The smooth locus of $\mathscr{Z}_B^d$ over $\mathbb{P}^1_{\mathscr{H}_B^d}$	Definition 8.5.5
$\mathscr{W}_B^{\boxed{4}\,d}$	The locus in $\mathscr{W}_B^d$ possessing an order 4 automorphism	Definition 8.6.1
$\mathscr{U}\mathscr{W}_B^{\boxed{4}\,d}$	The universal family of Weierstrass models over $\mathscr{W}_B^{\boxed{4}\,d}$	Definition 8.6.1

Figure 8.1: Notation introduced.

The space of Weierstrass equations

Throughout this section, we work relatively over a scheme B on which 2 is invertible. As in [Lan21a, Definition 3.1], define $\mathbb{P}_B^1 := \operatorname{Proj}_B \mathscr{O}_B[s,t]$. Form the affine space,

$$\mathbb{A}_B^{12d+3} := \operatorname{Spec}_B \mathscr{O}_B[a_{2,0}, a_{2,1} \dots, a_{2,2d}, a_{4,0}, \dots, a_{4,4d}, a_{6,0} \dots, a_{6,6d}].$$

For $i \in \{1,2,3\}$, define $a_{2i}(s,t) := \sum_{j=0}^{2id} a_{2i,j} t^j s^{2id-j}$. Let $\mathscr{W}_B^d \subset \mathbb{A}_B^{12d+3}$ denote the open subscheme parameterizing those points such that the Weierstrass equation

$$y^2 z = x^3 + a_2(s,t)x^2 z + a_4(s,t)xz^2 + a_6(s,t)z^3$$

is a minimal Weierstrass equation.

The universal Weierstrass equation

As in [Lan21a, Definition 3.1], one can construct a family of minimal Weierstrass models $\mathscr{U}\mathscr{W}_B^d$ over $\mathbb{P}^1 \times \mathscr{W}_B^d$ as the subscheme of

$$\operatorname{Proj}_{\mathbb{P}_B^1 \times_B \mathscr{W}_B^d} \operatorname{Sym}^\bullet \left(\mathscr{O}_{\mathbb{P}_B^1 \times_B \mathscr{W}_B^d} \oplus \mathscr{O}_{\mathbb{P}_B^1 \times_B \mathscr{W}_B^d}(-2d) \oplus \mathscr{O}_{\mathbb{P}_B^1 \times_B \mathscr{W}_B^d}(-3d) \right)$$

cut out by the equation

$$y^2 z = x^3 + a_2(s,t)x^2 z + a_4(s,t)xz^2 + a_6(s,t)z^3.$$

An open subset

As in [Lan21a, Definition 3.9], let $\mathscr{W}^{\circ d}_B \subset \mathscr{W}^d_B$ denote the open subscheme over which $\mathscr{U}\mathscr{W}^d_B \to \mathscr{W}^d_B$ is smooth. Let $\mathscr{U}\mathscr{W}^{\circ d}_B := \mathscr{U}\mathscr{W}^d_B \times_{\mathscr{W}^d_B} \mathscr{W}^{\circ d}_B$.

The Selmer space

As in [Lan21a, Definition 3.3], denote by f and g the projection maps

$$\mathscr{U}\mathscr{W}^d_B \xrightarrow{f} \mathbb{P}^1_B \times_B \mathscr{W}^d_B \xrightarrow{g} \mathscr{W}^d_B.$$

Assuming further that $2n$ is invertible on B. Define the *n-Selmer sheaf over B of height d* as $\mathcal{S}e\ell^d_{n,B} := R^1 g_*(R^1 f_* \mu_n)$. Define the *n-Selmer space over B of height d,* denoted $\mathrm{Sel}^d_{n,B}$ as the algebraic space representing the sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules $\mathcal{S}e\ell^d_{n,B}$. Let

$$\mathrm{Sel}^{\circ d}_{n,B} := \mathrm{Sel}^d_{n,B} \times_{\mathscr{W}^d_B} \mathscr{W}^{\circ d}_B.$$

For $j : U \subset \mathbb{P}^1_B \times_B \mathscr{W}^d_B$ the locus over which $\mathscr{U}\mathscr{W}^d_B \to \mathbb{P}^1_B \times_B \mathscr{W}^d_B$ is smooth, and $f' : \mathscr{U}\mathscr{W}^d_B \times_{\mathscr{W}^d_B} U \to U$, we let $\mathcal{E}[n]^d_B$ denote the sheaf $R^1 f'_* \mu_n$, which is the sheaf of relative n-torsion

8.4 The 2-Selmer space and 2-torsion on trigonal curves

We now relate the 2-Selmer space to a certain cover of a locus of trigonal curves. To do this, we start by defining our locus of trigonal curves. At this point, it will be useful to recall our notation for Hirzebruch surfaces from § 1.4.1.

Definition 8.4.1. Throughout this definition we work relatively over a base B, where B is a scheme with 2 invertible.

Consider the projective bundle

$$\mathrm{Proj}_{\mathbb{P}^1_B \times_B \mathscr{W}^d_B} \mathrm{Sym}^\bullet \left(\mathscr{O}_{\mathbb{P}^1_B \times_B \mathscr{W}^d_B} \oplus \mathscr{O}_{\mathbb{P}^1_B \times_B \mathscr{W}^d_B}(-2d) \right). \tag{8.4.1}$$

Let z denote a generator of the first summand and x denote a generator of the second summand. Let $\widehat{\mathscr{U}\mathscr{T}^d_B}$ denote the subscheme cut out of the bundle in (8.4.1) by the ideal sheaf generated by $x^3 + a_2(s,t)x^2 z + a_4(s,t)xz^2 + a_6(s,t)z^3$. Let $\mathscr{T}^d_B \subset \mathscr{W}^d_B$ denote the open subscheme over which $\widehat{\mathscr{U}\mathscr{T}^d_B}$ is smooth, and let $\mathscr{U}\mathscr{T}^d_B := \mathscr{T}^d_B \times_{\mathscr{W}^d_B} \widehat{\mathscr{U}\mathscr{T}^d_B}$. This is the universal family of trigonal curves on $\mathbb{F}_{2d}$.

Lemma 8.4.2. *Let B be a base with 2 invertible on B. There is an isomorphism $\phi : \mathscr{W}^{\circ d}_B \simeq \mathscr{T}^d_B$.*

Proof. Note that we have an open immersion $\mathscr{T}^d_B \hookrightarrow \mathscr{W}^d_B$ by construction. We want to see this maps isomorphically to $\mathscr{W}^{\circ d}_B$. We just need to check that, a curve of the form $x^3 + a_2(s,t)x^2z + a_4(s,t)xz^2 + a_6(s,t)z^3$ is smooth if and only if the associated elliptic surface $y^2z = x^3 + a_2(s,t)x^2z + a_4(s,t)xz^2 + a_6(s,t)z^3$ is smooth. Indeed, this can be checked on k points, for k a field. The curve can be realized as the fixed locus of an order 2 automorphism of the surface sending $y \mapsto -y$ and hence is smooth by [CGP15, Proposition A.8.10(2)], as 2 is invertible on k. Conversely, if the curve is smooth, we wish to show the associated surface is smooth. The quotient of the elliptic surface by the order 2 automorphism $y \mapsto -y$ realizes it as a double cover of a Hirzebruch surface, branched over the union of the curve and the image of the identity section. Therefore, the elliptic surface is smooth away from the branch locus of this map. Further, since the ramification locus of this quotient map is a smooth effective Cartier divisor, it follows from the slicing criterion for regularity that the surface is smooth over the ramification divisor as well. □

The next main result is the following relation between $\mathscr{U}\mathscr{T}^d_B$ and the $\mathrm{Sel}^d_{2,B}$.

Proposition 8.4.3. *Let $\phi : \mathscr{W}^{\circ d}_B \simeq \mathscr{T}^d_B$ be as in Lemma 8.4.2, let $\mathscr{U}\mathscr{T}^d_B \xrightarrow{h} \mathscr{T}^d_B$ and $\mathscr{U}\mathscr{W}^{\circ d}_B \xrightarrow{f} \mathbb{P}^1_B \times \mathscr{W}^{\circ d}_B \xrightarrow{g} \mathscr{T}^d_B$ denote the natural projection maps. Then, there is an isomorphism $\mathcal{S}e\ell^d_{2,B}|_{\mathscr{W}^{\circ d}_B} \simeq R^1 h_* \mu_2$ compatible with ϕ.*

Proof assuming Lemma 8.4.4 below. In this proof, all objects will be implicitly restricted from $\mathscr{W}^d_B$ to $\mathscr{W}^{\circ d}_B$. For instance, we write $\mathcal{E}[2]^d_B$ to denote the restriction of $\mathcal{E}[2]^d_B$ from the open set of $\mathbb{P}^1_B \times \mathscr{W}^d_B$ over which $\mathscr{U}\mathscr{W}^d_B$ is smooth to the open set of $\mathbb{P}^1_B \times_B \mathscr{W}^{\circ d}_B$ where $\mathscr{U}\mathscr{W}^{\circ d}_B$ is smooth. We let U denote the open set of $\mathbb{P}^1_B \times_B \mathscr{W}^{\circ d}_B$ over which $\mathscr{U}\mathscr{W}^{\circ d}_B$ is smooth. We define f' as the natural map $f' : \mathscr{U}\mathscr{W}^{\circ d}_B|_U \to U$. Observe that the map h factors as $\mathscr{U}\mathscr{T}^d_B \xrightarrow{\pi} \mathbb{P}^1_B \times_B \mathscr{T}^d_B \xrightarrow{g} \mathscr{T}^d_B$. We claim there is an exact sequence of sheaves on $\mathbb{P}^1_B$

$$0 \longrightarrow R^1 f_* \mu_2 \xrightarrow{\alpha} \pi_* \mu_2 \longrightarrow \mu_2 \longrightarrow 0 \tag{8.4.2}$$

where the second map is the norm map (i.e., the adjoint to $\mu_2 \to \pi^* \mu_2$). To actually construct this sequence and verify it is exact will take some work, and we check this in Lemma 8.4.4. For now, let's complete the proof assuming the existence of this sequence.

Pushing (8.4.2) forward along g to $\mathscr{W}^{\circ d}_B \simeq \mathscr{T}^d_B$, we note that $R^1 g_* \mu_2 = 0$: the vanishing can be checked on stalks, which are 0 due to proper base change and the fact that all fibers of

g are $\mathbb{P}^1$. Since the map $g_*\pi_*\mu_2 \to g_*\mu_2$ is surjective, we obtain an isomorphism

$$R^1 g_* \left(R^1 f_* \mu_2\right) \simeq R^1 g_* (\pi_* \mu_2) . \tag{8.4.3}$$

Since π is a finite map, its higher derived pushforwards vanish, so the composition of functors spectral sequence for $g_* \circ \pi_*$ produces an isomorphism $R^1 g_* (\pi_* \mu_2) \simeq R^1 h_* \mu_2$. Composing this isomorphism with (8.4.3), we obtain the isomorphism $\mathcal{S}el^d_{2,B} \simeq R^1 g_* \left(R^1 f_* \mu_2\right) \simeq\simeq R^1 g_*(\pi_* \mu_2) \simeq R^1 h_* \mu_2$. □

To finish the proof of Proposition 8.4.3, we now prove Lemma 8.4.4.

Lemma 8.4.4. *The sequence* (8.4.2) *exists and is exact.*

Proof. There will be two steps; construction of the map α and verification of exactness.

Step 1: Construction of α

Using the Weil pairing and observing that the 2-torsion of the family of smooth elliptic curves $\mathscr{U}\mathscr{W}^{\circ d}_B|_U \to U$ is canonically identified with $\mathcal{E}[2]^d_B$, we obtain a map $\mathcal{E}[2]^d_B \times \mathcal{E}[2]^d_B \to \mu_2$. Identifying $\mathcal{E}[2]^d_B$ with the relative scheme representing it, we note that it is the disjoint union of two schemes, one of which is given by the identity section and hence isomorphic to the scheme $U \subset \mathbb{P}^1_B \times_B \mathscr{W}^{\circ d}_B$. The other of which is a degree three finite étale cover $T \to U$. In fact, it follows from the identification of the nontrivial 2-torsion in an elliptic curve $y^2 z = x^3 + axz^2 + bz^3$ with the vanishing locus of $x^3 + ax + b$ that T is realized as the pullback

$$\begin{array}{ccc} T & \xrightarrow{j^{\mathscr{U}\mathscr{T}^d_B}} & \mathscr{U}\mathscr{T}^d_B \\ \downarrow{\scriptstyle \pi^U} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & \mathbb{P}^1_B \times_B \mathscr{W}^{\circ d}_B. \end{array} \tag{8.4.4}$$

Hence, we can write $\mathcal{E}[2]^d_B \simeq U \coprod T$. Therefore, we obtain a composite map $(\mathcal{E}[2]^d_B)_T := \mathcal{E}[2]^d_B \times T \to \mathcal{E}[2]^d_B \times \mathcal{E}[2]^d_B \to \mu_2$. Since we also have a structure map $(\mathcal{E}[2]^d_B)_T \to T$, we obtain a map $(\mathcal{E}[2]^d_B)_T \to (\mu_2)_T$ of sheaves on T. As Weil restriction is right adjoint to base change, we obtain a map $\mathcal{E}[2]^d_B \to \pi^U_* \mu_2$ of sheaves on U. Therefore, we obtain a map $j_*\mathcal{E}[2]^d_B \to j_*\pi^U_* \mu_2 \simeq \pi_* j_*^{\mathscr{U}\mathscr{T}^d_B} \mu_2 \simeq \pi_*\mu_2$. As explained in [FLR20, (2.4) and (2.5)] in the proof of [FLR20, Proposition 2.6] we have a natural map $R^1 f_* \mu_2 \to j_*\mathcal{E}[2]^d_B$. Altogether, we obtain the map $R^1 f_* \mu_2 \to j_*\mathcal{E}[2]^d_B \to \pi_*\mu_2$ of (8.4.2).

Step 2: Verification of exactness

We next verify exactness. To check exactness, we argue that it suffices to check exactness over points $y \in U$. Let $x \in \mathscr{W}^d_B$ be a geometric point. Let $U_x \subset \mathbb{P}^1_x$ denote the pullback

$U_x = U \times_{\mathbb{P}^1_B \times_B \mathscr{W}^d_B} x$, and let f'_x, π_x, π^U_x and j_x denote the base changes of f', π, π^U and j to x. Over U_x, the sequence (8.4.2) is, by proper base change, identified with

$$0 \longrightarrow R^1(f'_x)_*\mu_2 \longrightarrow (\pi_x)_*\mu_2 \longrightarrow \mu_2 \longrightarrow 0. \tag{8.4.5}$$

We have a natural isomorphisms

$$R^1(f'_x)_*\mu_2 \simeq \mathscr{E}_x[2] = (j_x)_*(\mathcal{E}[2]^d_B|_{U_x}) \simeq (j_x)_*(R^1(f^U_x)_*\mu_2).$$

So, we now have obtained that (8.4.5) is the pushforward along j_x of the sequence

$$0 \longrightarrow R^1(f^U_x)_*\mu_2 \longrightarrow (\pi^U_x)_*\mu_2 \longrightarrow \mu_2 \longrightarrow 0. \tag{8.4.6}$$

As the norm map is surjective, and pushing forward is always left exact, in order to verify exactness of (8.4.5), it suffices to verify exactness of (8.4.6). Overall, this tells us that we only need to verify exactness of (8.4.2) over points $y \in U$. So, now consider a geometric $y \in U$. Restricting (8.4.2) to y, we obtain the sequence

$$0 \longrightarrow E_y[2] \longrightarrow (\pi_y)_*\mu_2 \longrightarrow \mu_2 \longrightarrow 0 \tag{8.4.7}$$

that we wish to check is exact. By construction, if we let p, q, r denote the three non-identity geometric points of $E_y[2]$, $(\pi_y)_*\mu_2$ is isomorphic to $(\mu_2)_p \times (\mu_2)_q \times (\mu_2)_r$, where $(\mu_2)_z$ is a copy of μ_2 indexed by the point z. Letting $e^y_2(a, b)$ denote the Weil pairing over y of the points a and b, the first map of (8.4.7) is explicitly obtained by sending a point $z \mapsto (e_2(z, p), e_2(z, q), e_2(z, r))$. Exactness of this sequence is then elementary to check, as it corresponds to explicit maps $\mu_2^2 \to \mu_2^3 \to \mu_2$. A thorough proof can be found in [Li19, Lemma 2.7]. Note that Li works over $\mathbb{Q}$ in this verification, but the same proof works over fields of characteristic not 2. □

We now make remarks on various connections to Proposition 8.4.3.

Remark 8.4.5. In these remarks, we let E be an elliptic curve over $k(t)$ corresponding to a point of $\mathscr{W}^{\circ d}_k$, $\mathscr{E}$ its Néron model over $\mathbb{P}^1_k$, and X its minimal regular proper model. We let $T \subset X$ denote the projective completion of the nontrivial 2-torsion of E in X.

1. The isomorphism $\mathcal{S}el^d_{2,k} \simeq R^1h_*\mu_2$ of Proposition 8.4.3 essentially recovers the classical Recillas correspondence relating tetragonal curves (i.e., degree 4 covers of $\mathbb{P}^1$) to trigonal curves (degree 3 covers of $\mathbb{P}^1$) [Rec74] (see also [Don92, §2.4]. Indeed, on stalks, the isomorphism becomes $H^1(\mathbb{P}^1, \mathscr{E}[2]) \simeq H^1(T, \mu_2)$. The right hand side

corresponds to certain degree 4 covers of $\mathbb{P}^1$ by taking the projective completion of the $\mathscr{E}[2]$ torsor, while the left hand side corresponds to degree 2 finite étale covers of the trigonal curve T.

We note that the construction we have described is specific to trigonal curves of extreme Maroni invariant, i.e., it is important there is a section of the $\mathbb{P}^1$ bundle in which the trigonal curve which is disjoint from the trigonal curve. This disjoint section becomes the identity section in the elliptic surface when we take the branched double cover. Of course, the Recillas correspondence exists more generally, and does not require the existence of such a disjoint section.

2. When one starts with the Recillas correspondence and passes to the level of function fields, one obtains a bijection between certain degree 4 extensions M_4 of $k(t)$ and certain degree 2 extensions E of degree 3 extensions M_3 of $k(t)$. This is the classical method used to solve quartic equations, and M_3 is the "resolvent cubic" of M_4. One can express this in terms of the Galois groups of M_4 and M_3 by noting there is an exact sequence of groups

$$0 \longrightarrow K_4 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow 0. \tag{8.4.8}$$

Here we identify S_4 with both the Galois group of the Galois closure of M_4 and the Galois closure of E. We identify S_3 with the Galois group of the Galois closure of M_3. In terms of our setup, we can relate K_4 to the order 4 group $E[2]$, S_3 to the degree 3 extension defined by the Galois closure of $E[2] - \mathrm{id}$ and S_4 to the Galois closure defined by the degree 4 extension corresponding to the 2-Selmer element.

3. It is no coincidence that $\mathscr{E}[2]$ is the fixed locus of the hyperelliptic involution on the Néron model. The hyperelliptic involution generates a group μ_2 which acts on X, the minimal regular proper model of E. Let $[X/\mu_2]$ denote the stack quotient, and let $\pi : T \to \mathbb{P}^1$ denotes the projection. Then via rather involved spectral sequence arguments (somewhat similar to those in [vG05, §6]) one finds $H^2([X/\mu_2], \mathbb{G}_m) \simeq H^1(\mathbb{P}^1, R^1\pi_*\mu_2) \simeq H^1(\mathbb{P}^1, \mathscr{E}[2])$. In this way, the Brauer group of $[X/\mu_2]$ is related to the fixed locus of the hyperelliptic involution in X, and is further identified with the 2-Selmer group of E.

4. Interestingly, the proof of Proposition 8.4.3 explains the Recillas construction in terms of the Weil pairing on an elliptic curve.

5. This relation to the Recillas construction has already appeared in [Vak01, §4], where

Vakil is implicitly examining $\mathrm{Sel}^1_{2,k}$. It has appeared for higher d in [Deo19, §3] (see especially [Deo19, Proposition 3.1] and its proof).

8.5 The 3-Selmer space and 3-torsion on hyperelliptic curves

Similarly to our construction from § 8.4, we next carry out a construction to relate the 3-Selmer space to a cover of a space of hyperelliptic curves. There are two additional complications that did not show up in § 8.4: First, our construction only applies over a certain subset of the 3-Selmer space, corresponding to those elliptic curves with an order 3 automorphism (or equivalently, those of j-invariant 0). Second, the cover of the space of hyperelliptic curves we construct injects into the 3-Selmer space, but is not an isomorphism. We now introduce the relevant moduli spaces.

Definition 8.5.1. Over a fixed base B with 6 invertible, consider the affine space $\mathbb{A}^{2g+3}_{[b_0,\dots,b_{2g+2}]}$. Let $\Delta \subset \mathbb{A}^{2g+3}_{[b_0,\dots,b_{2g+2}]}$ denote the closed subscheme given by the vanishing of the discriminant of the polynomial $\sum_{i=0}^{2g+2} b_i x^i$ (i.e., the set of points $(b_0, \dots, b_{2g+2})$ where $\sum_{i=0}^{2g+2} b_i x^i$ has some repeated root). Define $\mathscr{H}^g_B := \mathbb{A}^{2g+3}_{[b_0,\dots,b_{2g+2}]} - \Delta$ as the *moduli space of genus g framed hyperelliptic curves over B*. Over $\mathbb{P}^1_{[s,t]} \times_B \mathscr{H}^g_B$ construct the projective bundle

$$\mathrm{Proj}_{\mathbb{P}^1_{[s,t]}\times_B \mathscr{H}^g_B} \mathrm{Sym}^\bullet \left(\mathscr{O}_{\mathbb{P}^1_{[s,t]}\times_B \mathscr{H}^g_B} \oplus \mathscr{O}_{\mathbb{P}^1_{[s,t]}\times_B \mathscr{H}^g_B}(-3d) \right). \tag{8.5.1}$$

Let z denote the generator of the first summand and y denote the generator of the second summand. Let $\mathscr{U}\mathscr{H}^g_B$ denote the subscheme of (8.5.1) cut out by the ideal generated by $y^2 = \sum_{i=0}^{2g+2} b_i s^i t^{2g+2-i} z^2$. Then, $\mathscr{U}\mathscr{H}^g_B$ is the *universal family of framed hyperelliptic curves over* $\mathscr{H}^g_B$.

Definition 8.5.2. For B a scheme with 6 invertible, let $\mathscr{W}^{\boxed{3}\,d}_B \subset \mathscr{W}^d_B$ denote the locally closed subscheme with reduced structure corresponding to those curves defined by Weierstrass equations of the form $y^2 z = x^3 + b(s,t) z^3$ for $b(s,t)$ a homogeneous degree $6d$ polynomial with square-free discriminant (i.e., $b(s,t)$ has no repeated roots over any geometric point). Note that this can be viewed as an open in the subset of $\mathscr{W}^d_B$ parameterizing those elliptic curves which geometrically have some order 3 automorphism. Define $\mathscr{U}\mathscr{W}^{\boxed{3}\,d}_B := \mathscr{W}^{\boxed{3}\,d}_B \times_{\mathscr{W}^d_r} \mathscr{U}\mathscr{W}^d_B$.

Remark 8.5.3. For later relating this to the Selmer space, it will be useful to observe that $\mathscr{W}^{\boxed{3}\,d}_k \subset \mathscr{W}^{\circ d}_k$. Indeed, all Weierstrass models corresponding to points of $\mathscr{W}^{\boxed{3}\,d}_k$ will have $6d$ singular geometric fibers corresponding to the $6d$ geometric roots of the polynomial $b(s,t)$ from Definition 8.5.2. It follows from Tate's algorithm [Sil94, IV.9, Tate's Algorithm

9.4] that all points of $\mathscr{W}_k^{\boxed{3}d}$ have $6d$ fibers of type II additive reduction. It follows that $\mathscr{W}_k^{\boxed{3}d} \subset \mathscr{W}^{\circ d}_k$.

We now connect $\mathscr{H}_k^{3d-1}$ and $\mathscr{W}_k^{\boxed{3}d}$.

Lemma 8.5.4. *For B a scheme with 6 invertible, there is an isomorphism $\psi : \mathscr{H}_k^{3d-1} \simeq \mathscr{W}_k^{\boxed{3}d}$.*

Proof. This map is given quite explicitly on points by sending the hyperelliptic curve $y^2 = f(t)z^2$ to the elliptic surface $y^2z = x^3 + f(t)z^3$. Note that both $\mathscr{H}_B^{3d-1}$ and $\mathscr{W}_B^{\boxed{3}d}$ are the complement of the discriminant locus in a projective space of dimension $2g + 2$. □

Our main goal is to prove Proposition 8.5.7, which relates $\mathrm{Sel}^d_{3,B}$ to the three torsion in the universal family of hyperelliptic curves. In order to do so, we next define a scheme relating hyperelliptic curves to the relative 3-torsion over $\mathscr{W}_B^{\boxed{3}d}$.

Definition 8.5.5. Define $\mathscr{Z}_B^d := \mathbb{P}^1_{\mathscr{H}_B^{3d-1}} \coprod \mathscr{U}\mathscr{H}_B^{3d-1}$ and let $\mathscr{Z}_B^{\mathrm{sm},d}$ denote the smooth locus of $\mathscr{Z}_B^d$ over $\mathbb{P}^1_B \times_B \mathscr{H}_B^{3d-1}$. This is represents an étale sheaf, that we call by the same name.

Here is the relation between $\mathscr{Z}_B^d$ and the 3-torsion over $\mathscr{W}_B^{\boxed{3}d}$:

Lemma 8.5.6. *There is a natural inclusion $\mathscr{Z}_B^{sm,d} \hookrightarrow R^1 f_* \mu_3$ under the identification $\mathscr{W}_B^{\boxed{3}d} \simeq \mathscr{H}_B^{3d-1}$ of Lemma 8.5.4.*

Proof. We claim there is closed immersion $\mathscr{Z}_B^d \hookrightarrow \mathscr{U}\mathscr{W}_B^{\boxed{3}d}$ over $\mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{3}d}$. In terms of equations, $\mathscr{Z}_B^d \to \mathscr{U}\mathscr{W}_B^{\boxed{3}d}$ is cut out by the equation $x^3 = y^2z - bz^3$ in a certain projective bundle. We note that $\mathscr{Z}_B^d$ can also be viewed as the fixed locus of any geometric order 3 automorphism of $\mathscr{U}\mathscr{W}_B^{\boxed{3}d}$ over $\mathscr{W}_B^{\boxed{3}d}$ (which may only be defined after adding in cube roots of unity to B).

Next, we claim the smooth locus of $\mathscr{Z}_B^{\mathrm{sm},d}$ over $\mathbb{P}^1_{\mathscr{W}_B^{\boxed{3}d}}$ is contained in the smooth locus of $\mathscr{U}\mathscr{W}_B^{\boxed{3}d}$ over $\mathbb{P}^1_{\mathscr{W}_B^{\boxed{3}d}}$, which is identified with $\mathrm{Pic}^0_{\mathscr{U}\mathscr{W}_B^{\boxed{3}d}/\mathscr{W}_B^{\boxed{3}d}}$. This containment can be seen via the slicing criterion for smoothness as $\mathscr{U}\mathscr{W}_B^{\boxed{3}d}$ is a cyclic triple cover of a projective bundle over $\mathbb{P}^1_{\mathscr{W}_B^{\boxed{3}d}}$ (the bundle defined in Definition 8.5.1), branched over $\mathscr{Z}_B^d$.

Under the identification of $\mathrm{Pic}^0_{\mathscr{U}\mathscr{W}_B^{\boxed{3}d}/\mathscr{W}_B^{\boxed{3}d}}$ as the identity component of $\mathrm{Pic}_{\mathscr{U}\mathscr{W}_B^{\boxed{3}d}/\mathscr{W}_B^{\boxed{3}d}} \simeq R^1 f_* \mathbb{G}_m$, we obtain an inclusion $\mathscr{Z}_B^{\mathrm{sm},d} \to R^1 f_* \mathbb{G}_m$. Here, by abuse of notation, we use f to denote the projection map $\mathscr{U}\mathscr{W}_B^{\boxed{3}d} \to \mathbb{P}^1_{\mathscr{W}_B^{\boxed{3}d}}$ (which is the base change of the map

$\mathscr{UW}_B^d \to \mathbb{P}^1_{\mathscr{W}_B^d}$ which we were also calling f). Note that $R^1 f_* \mu_3$ is the kernel of multiplication by 3 on $R^1 f_* \mathbb{G}_m$, since $f_* \mathbb{G}_m \simeq \mathbb{G}_m$, and multiplication by 3 on $f_* \mathbb{G}_m \to f_* \mathbb{G}_m$ is surjective.

To check $\mathscr{Z}_B^{\mathrm{sm},d} \to R^1 f_* \mathbb{G}_m$ factors through $R^1 f_* \mu_3$, we only need to check its image is killed by multiplication by 3. Further, since everything is pulled back from $B = \operatorname{Spec} \mathbb{Z}[1/6]$, we may assume we are working in this universal case where B is reduced. By reducedness considerations, and commutation of the formation all sheaves with base change, we can check this over geometric points in $\mathbb{P}^1_{\mathscr{W}_B^{\boxed{3}d}}$. The statement is trivial over any point x for which $f^{-1}(x)$ has bad reduction, as for such an x, $(\mathscr{Z}_B^{\mathrm{sm},d})$ factors through the identity section of $\mathscr{UW}_B^{\boxed{3}d}$. So, we only need to check the statement over geometric points t where $f^{-1}(t)$ is a smooth elliptic curve E_t over t. In this case $f^{-1}(t)$ lies in the kernel of the degree 3 isogeny $t \mapsto \iota_x(t) - \iota_x^2(t)$ for ι an order 3 automorphism of E_x. So, $f^{-1}(t) \subset E_t[3]$. This yields the map $\mathscr{Z}_B^{\mathrm{sm},d} \hookrightarrow R^1 f_* \mu_3$. $\square$

Our main result of this section is the following comparison between the 3-Selmer space and cyclic $\mathbb{Z}/3\mathbb{Z}$ covers of hyperelliptic curves. We prove it assuming Lemma 8.5.8 and Lemma 8.5.10 proven below.

Let $h : \mathscr{UH}_B^{3d-1} \to \mathscr{H}_B^{3d-1} \to \mathscr{W}_B^{\boxed{3}d}$ denote the composition of the structure map for the universal family of framed hyperelliptic curves as in Definition 8.5.1 with the isomorphism ψ of Lemma 8.5.4. Let $\mathscr{UW}_B^{\boxed{3}d} \xrightarrow{f} \mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{3}d} \xrightarrow{g} \mathscr{W}_B^{\boxed{3}d}$ be the natural maps as in Definition 8.5.2.

Proposition 8.5.7. *With notation as above, there is an injection of sheaves* $R^1 h_* (\mathbb{Z}/3\mathbb{Z}) \hookrightarrow \mathcal{S}el_{3,B}^d$ *compatible with the isomorphism* $\psi : \mathscr{H}_B^{3d-1} \simeq \mathscr{W}_B^{\boxed{3}d}$ *of Lemma 8.5.4.*

Proof assuming Lemma 8.5.8 and Lemma 8.5.10 below. Let $\pi : \mathscr{UH}_B^{3d-1} \to \mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{3}d}$ denote the structure map, so that $g \circ \pi = h$ (with g and h defined directly before the statement of this proposition). In Lemma 8.5.8 below, we prove there is an exact sequence

$$0 \longrightarrow \mathscr{Z}_B^{\mathrm{sm},d} \xrightarrow{\beta} \pi_* (\mathbb{Z}/3\mathbb{Z}) \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 0 \qquad (8.5.2)$$

with the map $\pi_* (\mathbb{Z}/3\mathbb{Z}) \to \mathbb{Z}/3\mathbb{Z}$ given by the norm map. Now, push (8.5.2) forward along g. We obtain an exact sequence

$$g_* \pi_* (\mathbb{Z}/3\mathbb{Z}) \to g_* (\mathbb{Z}/3\mathbb{Z}) \to R^1 g_* \mathscr{Z}_B^{\mathrm{sm},d} \to R^1 g_* (\pi_* (\mathbb{Z}/3\mathbb{Z})) \to R^1 g_* (\mathbb{Z}/3\mathbb{Z}).$$

Noting that $R^1 g_* (\mathbb{Z}/3\mathbb{Z}) = 0$ and that the map $g_* \pi_* (\mathbb{Z}/3\mathbb{Z}) \to g_* (\mathbb{Z}/3\mathbb{Z})$ is surjective, we obtain that $R^1 g_* \mathscr{Z}_B^{\mathrm{sm},d} \simeq R^1 g_* (\pi_* (\mathbb{Z}/3\mathbb{Z}))$ is an isomorphism. Since π is a finite map, the composition of functors spectral sequence for $g \circ \pi = h$ yields an isomorphism $R^1 g_* (\pi_* (\mathbb{Z}/3\mathbb{Z})) \simeq R^1 h_* (\mathbb{Z}/3\mathbb{Z})$, and therefore

$$R^1 g_* \mathscr{Z}_B^{\mathrm{sm},d} \simeq R^1 h_* (\mathbb{Z}/3\mathbb{Z}) . \tag{8.5.3}$$

To conclude the proof, we only need realize $R^1 g_* \mathscr{Z}_B^{\mathrm{sm},d}$ as a subsheaf of $R^1 g_*(R^1 f_* \mu_3) \simeq \mathcal{S}el^d_{3,B}$. Using the map $\mathscr{Z}_B^{\mathrm{sm},d} \to R^1 f_* \mu_3$ from Lemma 8.5.6, define $\mathscr{Q}$ as the quotient sheaf

$$0 \longrightarrow \mathscr{Z}_B^{\mathrm{sm},d} \longrightarrow R^1 f_* \mu_3 \longrightarrow \mathscr{Q} \longrightarrow 0. \tag{8.5.4}$$

Pushing this forward along g_* and using that $g_* R^1 f_* \mu_3 \to g_* \mathscr{Q}$ is surjective by Lemma 8.5.10, we obtain the desired injection

$$R^1 h_*(\mathbb{Z}/3\mathbb{Z}) \simeq R^1 g_* \mathscr{Z}_B^{\mathrm{sm},d} \hookrightarrow R^1 g_* \left(R^1 f_* \mu_3\right) \simeq \mathcal{S}el^d_{3,B}.$$

□

We now tie up the two loose ends to finish the proof of Proposition 8.5.7.

Lemma 8.5.8. *The sequence* (8.5.2) *exists and is exact.*

Remark 8.5.9. One's initial guess for the definition of β from (8.5.2) might be to imitate the proof of Lemma 8.4.4 and try taking β to be the map induced by the Weil pairing on the 3-torsion. However, if one tried to to this, one would end up with the 0 map! We need to define the map differently.

Proof. Retaining notation as in the proof of Proposition 8.5.7, we need to define β and then show (8.5.2) is exact.

Step 1: Construction of β

By the adjunction property of Weil restriction, to construct β, it suffices to construct a map $\mathscr{Z}_B^{\mathrm{sm},d} \times_{\mathbb{P}^1_{\mathscr{H}_B^{3d-1}}} \mathscr{U}\mathscr{H}_B^{3d-1} \to (\mathbb{Z}/3\mathbb{Z})_{\mathscr{U}\mathscr{H}_B^{3d-1}}$. For this, write $\mathscr{Z}_B^{\mathrm{sm},d} =$

$\mathbb{P}^1_{\mathscr{H}_B^{3d-1}} \coprod \mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1}$ so that the fiber product becomes

$$\begin{aligned}
\mathscr{Z}_B^{\mathrm{sm},d} \times_{\mathbb{P}^1_{\mathscr{H}_B^{3d-1}}} \mathscr{U}\mathscr{H}_B^{3d-1} &\simeq \left(\mathbb{P}^1_{\mathscr{H}_B^{3d-1}} \coprod \mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1}\right) \times_{\mathbb{P}^1_{\mathscr{H}_B^{3d-1}}} \mathscr{U}\mathscr{H}_B^{3d-1} \\
&\simeq \mathscr{U}\mathscr{H}_B^{3d-1} \coprod \left(\mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1} \times_{\mathbb{P}^1_{\mathscr{H}_B^{3d-1}}} \mathscr{U}\mathscr{H}_B^{3d-1}\right) \\
&\simeq \mathscr{U}\mathscr{H}_B^{3d-1} \coprod \left(\mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1} \coprod \mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1}\right).
\end{aligned}$$

This has a natural embedding into $(\mathbb{Z}/3\mathbb{Z})_{\mathscr{U}\mathscr{H}_B^{3d-1}}$ where the first $\mathscr{U}\mathscr{H}_B^{3d-1}$ goes to the component of $(\mathbb{Z}/3\mathbb{Z})_{\mathscr{U}\mathscr{H}_B^{3d-1}}$ indexed by 0, while the two copies of $\mathscr{U}\mathscr{H}_B^{\mathrm{sm},3d-1}$ go to the components of $(\mathbb{Z}/3\mathbb{Z})_{\mathscr{U}\mathscr{H}_B^{3d-1}}$ indexed 1 and 2. This constructs the maps in (8.5.2).

Step 2: Exactness

We now check the sequence is exact. For this, by commutation with base change, we can assume $B = \mathbb{Z}[1/6]$ and hence is reduced, so we can check exactness at any geometric point $t \in \mathbb{P}^1_{\mathscr{U}\mathscr{H}_B^3}$. Over points t with $f^{-1}(t)$ singular (hence isomorphic to $\mathbb{G}_a$) the sequence becomes $\{id\} \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$, with the latter map being the identity, and hence is exact. Over smooth geometric points t, the sequence is identified with $\mathbb{Z}/3\mathbb{Z} \to (\mathbb{Z}/3\mathbb{Z})^2 \to \mathbb{Z}/3\mathbb{Z}$, with the first map given by sending $w \mapsto (w, -w)$ and the second via addition. Therefore, the sequence is still exact at such points. □

To complete the proof of Proposition 8.5.7, we just need to prove Lemma 8.5.10. This boils down to a concrete computation involving the Galois representation associated to the 3-torsion in elliptic curves.

Lemma 8.5.10. *With notation as defined in the proof of Proposition 8.5.7, the map* $g_* R^1 f_* \mu_3 \to g_* \mathscr{Q}$ *obtained by pushing forward* (8.5.4) *is surjective.*

Proof. Retain the notation as defined in the proof of Proposition 8.5.7. We can restrict to a geometric point $t \in \mathscr{W}_B^{\boxed{3}\,d}$. For E_t the corresponding elliptic curve and $\mathscr{E}_t$ its Néron model over $\mathbb{P}^1_t$, the stalks of $g_* R^1 f_* \mu_3$ and $g_* \mathscr{Q}$ over t are then given by $H^0(\mathbb{P}^1_t, \mathscr{E}_t[3])$ and $H^0(\mathbb{P}^1_t, \mathscr{E}_t[3]/(\mathscr{Z}_B^{\mathrm{sm},d})_t)$ respectively. Let $K := K(\mathbb{P}^1_t)$ denote the function field of $\mathbb{P}^1_t$. To check whether these sheaves have sections, we can further pullback along the generic point $\operatorname{Spec} K \to \mathbb{P}^1$.

The existence of such a section can be rephrased in terms of the Galois representation associated to $E_t[3]$, as we now explain. Let

$$\rho^{\mathrm{tor}}_{E_t[3]} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$$

denote the Galois representation associated to the $\mathbb{Z}/3\mathbb{Z}$ sheaf $E_t[3]$ over K. If we identify $(E_t[3])_{\overline{K}} \simeq (\mathbb{Z}/3\mathbb{Z})^2$ we can then choose an element e_1 in $(\mathbb{Z}/3\mathbb{Z})^2$ generating the image of $((\mathscr{Z}_B^{\mathrm{sm},d})_t)_{\overline{K}}$. Let e_2 be a complementary basis vector so that $(E_t[3])_{\overline{K}} = (\mathbb{Z}/3\mathbb{Z})\, e_1 \oplus (\mathbb{Z}/3\mathbb{Z})\, e_2$. In order to prove $H^0(\mathbb{P}^1, \mathscr{E}_t[3]/(\mathscr{Z}_B^{\mathrm{sm},d})_t) = 0$, it is enough to show that the action $\rho^{\mathrm{tor}}_{E_t[3]}$ of $\mathrm{Gal}(\overline{K}/K)$ permutes the two affine lines $(\mathbb{Z}/3\mathbb{Z})\, e_1 + e_2$ and $(\mathbb{Z}/3\mathbb{Z})\, e_1 + 2e_2$. Verifying this will then imply the desired surjection $g_* R^1 f_* \mu_3 \to g_* \mathscr{Q}$ on fibers.

Because we are assuming E_t has a 3-torsion point, the Galois representation preserves $\mathrm{Span}\,(e_1)$ and hence factors through the group of upper triangular matrices. So, for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, $\rho^{\mathrm{tor}}_{E_t[3]}(\sigma)$ must be of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Note that the composition of the determinant with $\rho^{\mathrm{tor}}_{E_t[3]}$ identified with the $\bmod 3$ cyclotomic character. Further, the $\bmod 3$ cyclotomic character is trivial as K contains all cube roots of unity, being a function field over an algebraically closed field (recall the point t we are checking the surjection over is a geometric point). So, we have $d = a$ because the determinant of $\rho^{\mathrm{tor}}_{E_t[3]}$ is 1.

We need to show there is some σ as above for which $d = 2$ (which then permutes $(\mathbb{Z}/3\mathbb{Z})\, e_1 + e_2$ and $(\mathbb{Z}/3\mathbb{Z})\, e_1 + 2e_2$). Hence, it is equivalent to show there is some σ for which $a = 2$. Because the hyperelliptic curves corresponding to points in $\mathscr{H}_B^d$ are geometrically irreducible, the elements e_1 and $2e_1$ must be interchanged by some $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Therefore, there is some σ for which $a = 2$, implying $d = 2$ and $g_* R^1 f_* \mu_3 \to g_* \mathscr{Q}$ is surjective. □

We now make some remarks on connections to Proposition 8.5.7.

Remark 8.5.11. In these remarks, we let E be an elliptic curve over $k(t)$, corresponding to a point of $\mathscr{W}_k^{\boxed{3}\,d}$, with an order 3 automorphism generating a group G. We let $\mathscr{E}$ its Néron model over $\mathbb{P}^1_k$ and X its minimal regular proper model. We let $\mathbb{P}^1 \coprod H \subset X$ denote the fixed locus of the order 3 automorphism and $\mathbb{P}^1 \coprod H^{\mathrm{sm}} \subset \mathscr{E}$ denote the smooth locus of $\mathbb{P}^1 \coprod H$ over $\mathbb{P}^1$.

1. The isomorphism (8.5.3) is a globalization of the expression $H^1(\mathbb{P}^1, \mathbb{P}^1 \coprod H^{\mathrm{sm}}) \simeq H^1(H, \mathbb{Z}/3\mathbb{Z})$. This yields a degree 3 variant of the degree 4 Recillas construction from Remark 8.4.5(1). Namely, the left hand side corresponds to a $\mathbb{P}^1 \coprod H^{\mathrm{sm}}$ torsor which is a degree 3 cover of $\mathbb{P}^1$, while the right hand side corresponds to a degree 3 finite étale cover of a hyperelliptic curve. The correspondence between them is that the latter cover is the Galois closure of the former trigonal curve.

2. When one translates the above construction to the level of function fields, one obtains a bijection between certain degree 3 extensions M_3 of $k(t)$ and certain degree 3

extensions E of degree 2 extensions M_2 of $k(t)$. This is related to how one classically solves the cubic equation, and M_2 is the "discriminant" of M_3.

3. As in Remark 8.4.5(3), via involved spectral sequence arguments, for $\pi : H \to \mathbb{P}^1$ the projection, we have $H^2([X/G], \mathbb{G}_m) \simeq H^1(\mathbb{P}^1, R^1\pi_* G) \simeq H^1(\mathbb{P}^1, \mathbb{P}^1 \coprod H^{\mathrm{sm}})$. This relates the Brauer group of $[X/G]$ to the fixed locus of the order three automorphism of X and the 3-Selmer group of E.

8.6 The 2-Selmer space and 2-torsion on hyperelliptic curves

We now perform a similar construction to that of § 8.4 but for elliptic curves with order 4 automorphisms, as opposed to order 2 automorphisms (i.e., the hyperelliptic involution). We will be a bit more terse in this subsection than § 8.4 and § 8.5 as many of the constructions are similar but simpler.

Definition 8.6.1. For B a scheme with 2 invertible, define $\mathscr{W}_B^{\boxed{4}\,d} \subset \mathscr{W}_B^d$ as the locally closed subscheme with reduced structure corresponding to those curves defined by Weierstrass equations of the form $y^2 z = x^3 + a(s,t)xz^2$, for $a(s,t)$ a homogeneous degree $4d$ square-free polynomial. Note that $\mathscr{W}_B^{\boxed{4}\,d}$ can be viewed as a locally closed subset of $\mathscr{W}_B^d$ parameterizing elliptic curves with an order 4 automorphism, or equivalently, a nontrivial 2-torsion point. Define $\mathscr{UW}_B^{\boxed{4}\,d} := \mathscr{W}_B^{\boxed{4}\,d} \times_{\mathscr{W}_B^d} \mathscr{UW}_B^d$.

Remark 8.6.2. As in Lemma 8.5.4 there is an isomorphism $\mathscr{W}_B^{\boxed{4}\,d} \simeq \mathscr{H}_B^{2d-1}$ given by sending an elliptic surface of the form $y^2 z = zx(x^2 + a(s,t)z^2)$ to the hyperelliptic curve $y^2 = a(s,t)z^2$.

Let $f : \mathscr{UW}_B^{\boxed{4}\,d} \to \mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{4}\,d}$ denote the projection map, and let $j : U \to \mathscr{W}_B^{\boxed{4}\,d}$ denote the inclusion of the open subscheme on which f is smooth. Observe that $j_*\mathcal{E}[2]_B^d|_{\mathscr{UW}_B^{\boxed{4}\,d}}$ is represented on the small étale site of $\mathbb{P}^1_B \times \mathscr{UW}_B^{\boxed{4}\,d}$ by the disjoint union of the smooth locus of $\mathscr{UH}_B^{2d-1} \to \mathscr{H}_B^{2d-1}$ with two copies of $\mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{4}\,d}$ (corresponding to the identity section and non-trivial 2-torsion point). In particular, $j_*\mathcal{E}[2]_B^d|_{\mathscr{UW}_B^{\boxed{4}\,d}}$ commutes with base change. In this way, $j_*\mathcal{E}[2]_B^d$ is identified with the smooth locus of $V(zx(x^2 + a(s,t)z^2))$ over $\mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{4}\,d}$.

Remark 8.6.3. Any elliptic curve E_x associated to a point $x \in \mathscr{W}_k^{\boxed{4}\,d}$ is represented by an equation $y^2 z = x^3 + a(s,t)xz^2$ with $a(s,t)$ square-free. Suppose $a(s,t)$ factors as a product of r irreducible factors. By Tate's algorithm [Sil94, IV, Lemma 9.5(a)], such a curve has precisely r fibers of bad reduction type III, corresponding to the r factors of $a(s,t)$. All other fibers have good reduction. In particular, the component group is $\Phi_{E_x} \simeq \Phi_{E_x}^{\mathrm{geom}} \simeq (\mathbb{Z}/2)^r$.

We next aim to prove the analog of Proposition 8.4.3 over $\mathscr{W}_B^{\boxed{4}\,d}$. For this, we will need the following analog of Lemma 8.4.4.

Lemma 8.6.4. *With notation as in Remark 8.6.2, $\pi : \mathscr{U}\mathscr{H}_B^{2d-1} \coprod \mathbb{P}^1_{\mathscr{H}_B^{2d-1}} \to \mathbb{P}^1_{\mathscr{H}_B^{2d-1}}$ denote the natural projection map. There is an exact sequence*

$$0 \longrightarrow j_*\mathcal{E}[2]_B^d \xrightarrow{\gamma} \pi_*(\mu_2) \longrightarrow \mu_2 \longrightarrow 0 \tag{8.6.1}$$

which is compatible with base change.

Proof. The map $\pi_*\mu_2 \to \mu_2$ is the norm map. The construction of γ here is completely analogous to the construction of α from (8.4.2) using the Weil pairing on the 2-torsion. Verification of exactness is also analogous, by arguing one can restrict to an open in $\mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{4}\,d}$ over which the 2-torsion is finite, and then verifying the result by a direct calculation on stalks. Finally, $j_*\mathcal{E}[2]_B^d$ is representable, hence commutes with base change, as mentioned in Remark 8.6.2. □

We are now ready to prove the analog of Proposition 8.4.3.

Proposition 8.6.5. *Let $h : \mathscr{U}\mathscr{H}_B^{2d-1} \times \mathbb{P}^1_{\mathscr{H}_B^{2d-1}} \to \mathscr{H}_B^{2d-1}$ denote the projection. Under the identification $\mathscr{W}_B^{\boxed{4}\,d} \simeq \mathscr{H}_B^{2d-1}$ from Remark 8.6.2, there is an isomorphism $\mathcal{S}el_{2,B}^d|_{\mathscr{W}_B^{\boxed{4}\,d}} \simeq R^1h_*(\mu_2)$.*

Proof. Let $\mathscr{U}\mathscr{W}_B^{\boxed{4}\,d} \xrightarrow{f} \mathbb{P}^1_B \times_B \mathscr{W}_B^{\boxed{4}\,d} \xrightarrow{g} \mathscr{W}_B^{\boxed{4}\,d}$ and $\pi : \mathscr{U}\mathscr{H}_B^{2d-1} \coprod \mathbb{P}^1_{\mathscr{H}_B^{2d-1}} \to \mathbb{P}^1_{\mathscr{H}_B^{2d-1}}$ denote the natural projection maps. Pushing forward the exact sequence (8.6.1) along g and using surjectivity of the norm map and $R^1g_*\mu_2 = 0$, we obtain the isomorphism $\mathcal{S}el_{2,B}^d = R^1g_*\mathcal{E}[2]_B^d \simeq R^1g_*(\pi_*\mu_2)$. Then, we have an isomorphism $R^1g_*(\pi_*\mu_2) \simeq R^1(g\circ\pi)_*\mu_2$ from the Leray spectral sequence, since π is finite. Further, as $g\circ\pi$ is the disjoint union of an identity map on $\mathbb{P}^1_B$ and the map $h : \mathscr{U}\mathscr{H}_B^{2d-1} \times \mathbb{P}^1_{\mathscr{H}_B^{2d-1}} \to \mathscr{H}_B^{2d-1}$, we find $R^1(g\circ\pi)_*\mu_2 \simeq R^1h_*\mu_2$. Composing the above maps, we find $\mathcal{S}el_{2,B}^d \simeq R^1h_*\mu_2$. □

We make some remarks relating to Proposition 8.6.5.

Remark 8.6.6. Throughout, we let E be an elliptic curve over $k(t)$, corresponding to a point of $\mathscr{W}_k^{\boxed{4}\,d}$, with an order 4 automorphism which generates a group G. Such a curve necessarily has a marked 2-torsion point fixed by G. We assume not all 2-torsion points of

E are defined over $k(t)$. We let $\mathscr{E}$ denote its Néron model over $\mathbb{P}^1_k$ and X its minimal regular proper model. We let $H \subset X$ denote the degree 2 irreducible cover of $\mathbb{P}^1$ contained in the fixed locus of G.

1. The isomorphism from Proposition 8.6.5 recovers the "bigonal construction" [Don92, §2.3]. Indeed, on fibers it can be expressed as $H^1(\mathbb{P}^1, \mathscr{E}[2]) \simeq H^1(H, \mu_2)$ which gives a correspondence between degree 4 covers of $\mathbb{P}^1$ corresponding to the projective closures of torsors for $\mathscr{E}[2]$ and degree 2 covers of H.

2. When one translates the previous part to function fields, one obtains a bijection between certain $(\mathbb{Z}/2)^2$-Galois extensions M_4 of $k(t)$ and certain degree 2 extensions E of degree 2 extensions M_2 over $k(t)$.

3. As in Remark 8.4.5(3), for $\pi : H \to \mathbb{P}^1$ the projection, we have $H^2([X/G], \mathbb{G}_m) \simeq H^1(\mathbb{P}^1, R^1\pi_* G) \simeq H^1(\mathbb{P}^1, \mathscr{E}[2])$. This relates the fixed locus of the order 4 automorphism of X to its 2-Selmer group.

Chapter 9

Stacky heights

In this chapter, we investigate a notion of heights on stacks which will be introduced in a forthcoming paper of Ellenberg Satriano and Zureick-Brown. Specifically, given a vector bundle on a stack, they define a corresponding height function. With their definition, they observe that in characteristic 0 or characteristic more than 3, the stacky height associated to the Hodge bundle on $\overline{\mathscr{M}}_{1,1}$ (the moduli stack of stable elliptic curves) agrees with the Faltings height. They observe that the Hodge bundle does not induce the Faltings height on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3, and it even fails to induce a height function which is Northcott (for which there are finitely many points of bounded height). In this chapter, we show there are in fact no vector bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3 inducing the Faltings height. Nevertheless, we show there are vector bundles inducing a Northcott height function.

We first introduce our main results in more detail in §9.2. We then prove there are no vector bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3 inducing the Faltings height in §9.3, §9.4, and §9.5. Following this, we prove there are Northcott height functions on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3 in § 9.6 and § 9.7. Finally, we characterize indecomposable representations of S_3 in characteristic 3 in §9.8.

9.1 Story of the project

My interest in this question arose after an enlightening talk by Jordan Ellenberg at Stanford. Jordan described the notion of heights on stacks he was writing a paper on, and described a question he and his coauthors were unable to solve about the possible heights on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. During the talk, this sounded like an interesting question, and I immediately had an idea for how to approach it. In the following weeks, I thought some more about the question, and came up with the approach we describe later.

After later sending my solution to Jordan's question to him, he observed it was unfortunate no such vector bundle existed, since it meant the theory they introduce sometimes fails to capture the desired notion of height in small characteristic. However, he was curious whether something could be salvaged, and whether there at least exist Northcott vector bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. This led me to the second part of this chapter, showing that such Northcott vector bundles do exist.

9.2 Introduction to stacky heights

Throughout this chapter, we work over a fixed perfect field k of characteristic 3. Let $\overline{\mathscr{M}}_{1,1}$ denote the Deligne-Mumford moduli stack of stable elliptic curves curves over k. Given a finite extension K over $k(t)$ and an elliptic curve $E \to \operatorname{Spec} K$, there is a Faltings height on that elliptic curve, which we define as follows.

Definition 9.2.1. Let C be the regular proper geometrically connected curve whose generic point is $\operatorname{Spec} K$ and let $f : X \to C$ denote the minimal proper regular model of an elliptic curve $E \to \operatorname{Spec} K$. Then the *Faltings height of E* is given by $\deg f_* \omega_{X/C}$. This Faltings height is also given by $\frac{1}{12} \deg(\Delta_{X/C})$ where $\Delta_{X/C}$ is the discriminant of the relative elliptic surface, viewed as a section of $H^0(C, f_*\omega_{X/C}^{\otimes 12})$. Thinking of $E \to \operatorname{Spec} K$ as a K point of $\overline{\mathscr{M}}_{1,1}$, given by $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, we denote this Faltings height by $\operatorname{ht}(x)$.

On the other hand, given a vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$, and a K point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ (corresponding to an elliptic curve $E \to \operatorname{Spec} K$) Ellenberg, Satriano, and Zureick-Brown [ESZB19] define of notion of height associated to x and $\mathscr{V}$, notated $\operatorname{ht}_{\mathscr{V}}(x)$, see Definition 9.2.13. Suppose k' is a field of characteristic not 2 or 3, and let ω denote the Hodge bundle over $(\overline{\mathcal{M}}_{1,1})_{k'}$. That is, $\omega := f_*\omega_{\mathscr{E}/(\overline{\mathcal{M}}_{1,1})_{k'}}$ for $f : \mathscr{E} \to (\overline{\mathcal{M}}_{1,1})_{k'}$ the universal generalized elliptic curve. Then [ESZB19] show that, for K a finite extension of $k'(t)$, and $x : \operatorname{Spec} K \to (\overline{\mathcal{M}}_{1,1})_{k'}$ a point, $\operatorname{ht}_\omega(x) = \operatorname{ht}(x)$, with the latter notion of Faltings height as defined in Definition 9.2.1. However, as [ESZB19] observe, for k a field of characteristic 3, it is no longer true that $\operatorname{ht}_\omega(x) = \operatorname{ht}(x)$ for all $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$. Moreover, they show there is no line bundle $\mathscr{L}$ on $\overline{\mathscr{M}}_{1,1}$ for which $\operatorname{ht}_{\mathscr{L}}(x) = \operatorname{ht}(x)$. This leads them to the following question:

Question 9.2.2. Is there some vector bundle $\mathscr{V}$ (necessarily of rank more than 1) on $\overline{\mathscr{M}}_{1,1}$ so that $\operatorname{ht}_{\mathscr{V}}(x) = \operatorname{ht}(x)$ for every $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ over a field of characteristic 2 or 3?

In this note, we show that the answer is "no" when k is a perfect field of characteristic 3. More precisely, we have the following:

Theorem 9.2.3. *Let* $\overline{\mathscr{M}}_{1,1}$ *denote the Deligne-Mumford stack of stable elliptic curves over a perfect field* k *of characteristic* 3. *There is no vector bundle* $\mathscr{V}$ *on* $\overline{\mathscr{M}}_{1,1}$ *for which* $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ *for all points* $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, *where* K *is a finite extension of* $k(t)$.

We deduce this from Theorem 9.4.4 at the end of § 9.4.

Despite Theorem 9.2.3, all hope for stacky heights is not lost. Recall that a height function satisfies the Northcott property if there are finitely many points of bounded height.

Theorem 9.2.4. *Let* $k = \mathbb{F}_q$ *be a finite field of characteristic* 3. *For* $x : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$, *the vector bundle* $\mathscr{V}$ *on* $\overline{\mathscr{M}}_{1,1}$ *defined later in Notation 9.6.1 satisfies the Northcott property.*

We prove this at the end of § 9.6. This result leaves open the following question.

Question 9.2.5. Is there some vector bundle $\mathscr{W}$ on $\overline{\mathscr{M}}_{1,1}$ and some integer n so that $n\,\mathrm{ht}(x) = \mathrm{ht}_{\mathscr{W}}(x)$?

9.2.6 Idea of the proof

The idea of the proof of Theorem 9.2.3 is to show that any $\mathscr{V}$ which induces the correct local stacky height for places of type III reduction necessarily induces the incorrect local stacky height for places of type IV reduction. Using that k has characteristic 3, we note that places of type IV and III reduction map to the same point in $\overline{\mathscr{M}}_{1,1}$, and we can use the local structure of coarse moduli spaces to identify $\mathscr{V}$ with a G representation, for G the dicyclic group of order 12, which is the geometric automorphism group of an elliptic curve of j invariant 0. Then, by analyzing these representations, and the constraints placed on them by elliptic curves with type IV reduction, we see that the local stacky height induced by $\mathscr{V}$ must agree with that of the Hodge bundle.

Another perspective on the idea of the proof is as follows. In order to have the correct local height at places of type III, the vector bundle $\mathscr{V}$ is forced agree with the height induced by a line bundle of order 4 in a neighborhood of the point of j-invariant 0. Therefore such a vector bundle cannot detect nontrivial stacky heights associated to wildly ramified curves or cyclic cubic twists.

The structure of this note is as follows: We next prove a general result saying that local stacky heights on Deligne-Mumford stacks can be computed étale locally on the target coarse moduli space in § 9.3. We then reduce Theorem 9.2.3 to the statement about local stacky heights Theorem 9.4.4 at the end of § 9.4. We prove Theorem 9.4.4 at the end of § 9.5.

We then prove Theorem 9.2.4 in § 9.6. Finally, in § 9.8, we classify indecomposable representations of S_3 in characteristic 3.

Remark 9.2.7 (Speculation)**.** It would be very interesting if it were possible to show there is no bundle $\mathscr{V}$ such that there exists some rational number r with $\operatorname{ht}_{\mathscr{V}}(x) = r \operatorname{ht}(x)$. One possible strategy for doing so would be to use a similar argument as given here with two not too difficult additional ingredients: an understanding of the representations of the dicyclic group of order 12 in characteristic 3 (see §9.8 for a related analysis of the representations of S_3 in characteristic 3) and an understanding of what the stacky heights associated to a vector bundle on $\overline{\mathscr{M}}_{1,1}$ look like over its generic point.

In fact, I would guess, because the local Faltings height (see Definition 9.4.1) cannot always agree with the local stacky height (defined in Notation 9.3.1), we cannot even achieve $\operatorname{ht}_{\mathscr{V}}(x) = r \operatorname{ht}(x) + O(1)$ for some constant r. My reason for guessing this is that it seems possible to construct elliptic curves over function fields with arbitrarily many places of bad additive reduction of a specified isomorphism type in the strict henselization, and no other places of additive reduction.

Remark 9.2.8 (Speculation)**.** Maybe the vector bundle which is the pushforward from the 2-torsion cover almost capture this property, except it "completely misses quadratic twists." I think it should be possible with some work to exactly compute the heights associated to this vector bundle via using its relation to quotients of a semistable Weierstrass model over an extension field. Maybe there is some way to add back in the quadratic twists (perhaps using the 4-torsion and 2-torsion similarly to the way we do later).

9.2.9 Review of the definition of heights on stacks

We now recall the definition of heights on stacks introduced in [ESZB19].

Definition 9.2.10. Let k be a field, let C be a regular proper integral curve over k, and let $K := K(C)$. Let $\mathscr{X}$ be an algebraic stack over k and $x : \operatorname{Spec} K \to \mathscr{X}$ be a K-point. A tuning stack $\mathscr{C}$ for x is a normal algebraic stack $\mathscr{C}$ with finite diagonal together with a map $\overline{x} : \mathscr{C} \to \mathscr{X}$ extending x so that $\pi : \mathscr{C} \to C$ is a birational coarse space map.

Remark 9.2.11. Although we will not need it, one can also extend Definition 9.2.10 to the number field case as follows. Let L be a number field, $B = \operatorname{Spec} \mathscr{O}_L$, and let $\mathscr{X}$ be an algebraic stack over B. Let K/L be a finite extension of number fields and $x : \operatorname{Spec} K \to \mathscr{X}$ be a K point. A tuning stack $\mathscr{C}$ for x is then a normal algebraic stack $\mathscr{C}$ with finite diagonal together with a map $\overline{x} : \mathscr{C} \to \mathscr{X}$ extending x so that $\pi : \mathscr{C} \to \operatorname{Spec} \mathscr{O}_K$ is a birational coarse space map.

Remark 9.2.12. This definition of tuning stack differs slightly from that of [ESZB19], in that we do not require $\mathscr{X}$ to be a stack over C. In particular, if $\mathscr{X}$ is a stack over a base k, we may discuss tuning stacks and heights of points associated to function fields of transcendence degree 1 over k. We also restrict ourselves to the function field setting.

We now recall the notion of stacky height.

Definition 9.2.13. Let $\mathscr{X}$ be a proper algebraic stack over a field k with finite diagonal, let C be a smooth proper geometrically connected curve over k with function field $K(C)$. Let $\mathscr{V}$ be a vector bundle on $\mathscr{X}$ and $x \in \mathscr{X}(K(C))$. If $\mathscr{C}$ is a tuning stack for x and $\overline{x}, \pi$ are the corresponding maps defined in Definition 9.2.10, then we define the *height of x with respect to $\mathscr{V}$* as

$$\mathrm{ht}_{\mathscr{V}}(x) := -\deg_C\left(\pi_*\overline{x}^*\mathscr{V}^\vee\right).$$

We also define the *stable height of x with respect to $\mathscr{V}$* as

$$\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(x) := -\deg_{\mathscr{C}}\left(\overline{x}^*\mathscr{V}^\vee\right).$$

To make sense of this definition, one has to check various properties, such as that this notion is independent of the choice of tuning stack. These are verified in [ESZB19].

One can also define heights in the number field case, but then one has to use metrized line bundles and Arakelov heights. We will only be concerned with the function field case, and so do not discuss this further.

Remark 9.2.14. It seems to us there are many stacks appearing in arithmetic geometry which are universally closed but not separated, and it would be interesting to extend the notion of heights on stacks to the case where stacks are non-separated. We suggest one possible way to extend the definition is by defining $\mathrm{ht}_{\mathscr{V}}(x)$ to be $\min_{\overline{x}:\mathscr{C}\to\mathscr{X}} \mathrm{ht}_{\mathscr{V}}(\overline{x})$, where the minimum is taken over all tuning stacks $\overline{x} : \mathscr{C} \to \mathscr{X}$ as defined in Definition 9.2.13. This definition seems to work well at least in a few examples I tried, including $\mathscr{X} = B\mathbb{G}_m$, (this does not have finite diagonal, but still has a coarse moduli space, and the definitions make sense here,) Hurwitz stacks, and Selmer stacks.

9.3 Local Heights are local on the target coarse moduli space

As a preliminary step to proving our main result, we first check one can compute local stacky heights on the strict henselization over the coarse moduli space of the target stack. The upshot of this is that we can compute local heights on Deligne-Mumford stacks in terms of representations, using the local structure theorem for Deligne Mumford stacks. The main result is Proposition 9.3.2, and it is probably obvious to experts. But I decided to write out the details for completeness. To state our upcoming results, we introduce some terminology for our setup.

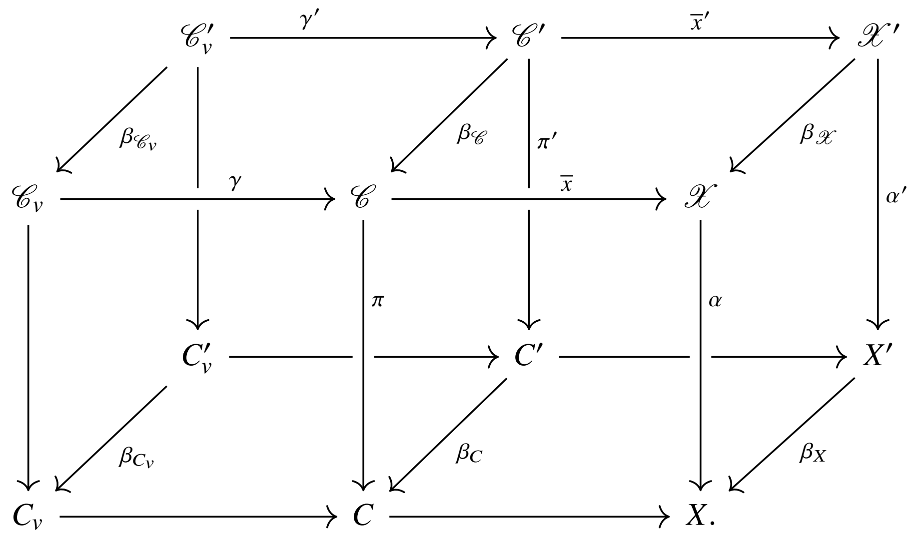
Notation 9.3.1. Let L be a field and let K either be a number field or a finite extension of $L(t)$ with regular model C (which we take to be the spectrum of the ring of integers in the case K is a number field and a regular proper curve over L when K is a function field). Suppose $x : \operatorname{Spec} K \to \mathscr{X}$ is a K point of a stack $\mathscr{X}$ locally of finite presentation and with finite diagonal, so that $\mathscr{X}$ possess a coarse moduli space $\alpha : \mathscr{X} \to X$. Let $\mathscr{C}$ denote the universal tuning stack associated to x (defined as the relative normalization of some extension $U \to \mathscr{X}$ of $x : \operatorname{Spec} K \to \mathscr{X}$, for U an open subscheme of the regular proper curve whose function field is $\operatorname{Spec} K$) so that we have a diagram

$$\begin{array}{ccccc} \operatorname{Spec} K & \longrightarrow & \mathscr{C} & \xrightarrow{\overline{x}} & \mathscr{X} \\ & \searrow & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \alpha} \\ & & C & \longrightarrow & X. \end{array} \qquad \text{(with } x = \overline{x} \circ (\operatorname{Spec} K \to \mathscr{C})\text{)} \tag{9.3.1}$$

Let v denote a closed point of C and suppose that the image of v in X is s. Let $\beta_X : X' \to X$ denote an étale map from a local scheme X' whose closed point maps to s. Suppose β_X has degree d over v, i.e., $\beta_X^{-1}(s)$ has degree d over $\kappa(s)$.

Let C_v denote the localization of C at v and let $\mathscr{X}' := X' \times_X \mathscr{X}, \mathscr{C}' := X' \times_X \mathscr{C}, C' := X' \times_X C$. Let $C'_v := C' \times_C C_v, \mathscr{C}'_v := \mathscr{C}' \times_C C_v, \mathscr{C}_v := \mathscr{C} \times_C C_v$. Let $\beta_{\mathscr{X}} : \mathscr{X}' \to \mathscr{X}, \beta_C : C' \to C, \beta_{\mathscr{C}} : \mathscr{C}' \to \mathscr{C}, \beta_{C_v} : C'_v \to C_v, \beta_{\mathscr{C}_v} : \mathscr{C}'_v \to \mathscr{C}_v$ denote the base change of β_X to $\mathscr{X}, C, \mathscr{C}, C_v$, and $\mathscr{C}_v$ respectively. Let $\gamma : \mathscr{C}_v \to \mathscr{C}$ denote the inclusion and let

$\gamma' : \mathscr{C}'_v \to \mathscr{C}'$ denote the base change to $\mathscr{C}'$ as in the following diagram of fiber cubes



In this setting, for $\mathscr{V}$ a vector bundle on $\mathscr{X}$ and v a place of K, we recall the definition of the *local stacky height* associated to x and $\mathscr{V}$ at the place v as

$$\delta_{\mathscr{V};v}(x) := \deg(\operatorname{coker}\left(\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee \to \gamma^*\overline{x}^*\mathscr{V}^\vee\right)).$$

Proposition 9.3.2. *With notation as in Notation 9.3.1,*

$$d\delta_{\mathscr{V};v}(x) = \deg\operatorname{coker}\left(\gamma'^*\pi'^*\pi'_*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee \to \gamma'^*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee\right). \tag{9.3.2}$$

If $\mathscr{X}$ is additionally Deligne-Mumford, we may assume that $\mathscr{X}' = [V'/G]$ for $V' \to X'$ a Galois étale finite cover and G the automorphism group of any geometric point of $\mathscr{X}$ over v. In particular, if $\mathscr{X}$ is Deligne-Mumford we may assume $\beta^*_{\mathscr{X}}\mathscr{V}$ corresponds to a G-representation over $\mathscr{O}_{V'}$,

Remark 9.3.3. The right hand side of (9.3.2) may seem daunting, but it could reasonably be called the local degree over v associated to $\beta^*_{\mathscr{X}}\mathscr{V}^\vee$. The practical application of the above lemma is then that the local stacky height $\delta_{\mathscr{V};v}(x)$ may therefore be computed in terms of a corresponding local stacky height on $[V'/G]$, which can be described in terms of a G representation. There are many techniques for computing these local stacky heights, especially developed in papers by Yasuda. See, for example, [Yas17] and references therein.

We now record the main consequence of Remark 9.3.3 which we will need.

Lemma 9.3.4. *With notation as in Notation 9.3.1, suppose v is a closed point of C which has d points $v_1, \ldots, v_d$ lying over it in C'. Then, for any $1 \le i \le d$, we have $\delta_{\mathscr{V};v}(\overline{x}) = \delta_{\beta^*_{\mathscr{X}};v_i}(\overline{x}')$.*

Proof. By Proposition 9.3.2, since the right hand side of (9.3.2) is equal to the sum of the local degrees $\sum_{i=1}^d \delta_{\beta^*_{\mathscr{X}}\mathscr{V},v_i}$, we find $d\delta_{\mathscr{V};v}(x) = \sum_{i=1}^d \delta_{\beta^*\mathscr{V},v_i}(x')$. Additionally, as $C' \to C$ is Galois, all the points $v_1, \ldots, v_d$ have the same local degree. Hence, for any i, $\delta_{\beta^*\mathscr{V},v_i}(x') = \frac{1}{d}(d\delta_{\mathscr{V};v}(x)) = \delta_{\mathscr{V};v}(x)$. □

Having seen what it is good for, let us now prove Proposition 9.3.2.

Proof of Proposition 9.3.2. The second statement regarding Deligne-Mumford stacks is a consequence of the local structure of Deligne Mumford stacks [Ols16, Theorem 11.3.1]. The last statement is a restatement of the correspondence between vector bundles on $[V'/G]$ and G-representations on $\mathscr{O}_{V'}$.

So, we only need to check (9.3.2). Observe that by the definition of local stacky height, we have

$$\delta_{\mathscr{V};v}(x) = \deg(\operatorname{coker}\left(\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee \to \gamma^*\overline{x}^*\mathscr{V}^\vee\right)),$$

where we recall the degree on the right hand side can be defined in terms of the corresponding degree on a finite flat cover.

Therefore, since $\beta_{\mathscr{C}_v}$ has degree d, right exactness of tensor products implies the pullback of the above cokernel $\operatorname{coker}\left(\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee \to \gamma^*\overline{x}^*\mathscr{V}^\vee\right)$ under $\beta_{\mathscr{C}_v}$ has degree equal to

$$\deg\operatorname{coker}\left(\beta^*_{\mathscr{C}}\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee \to \beta^*_{\mathscr{C}}\gamma^*\overline{x}^*\mathscr{V}^\vee\right) = d\delta_{\mathscr{V};v}(x).$$

So, it suffices to demonstrate an isomorphism

$$\begin{aligned}
&\operatorname{coker}\left(\beta^*_{\mathscr{C}_v}\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee \to \beta^*_{\mathscr{C}_v}\gamma^*\overline{x}^*\mathscr{V}^\vee\right)\\
&\simeq \operatorname{coker}\left(\gamma'^*\pi'^*\pi'_*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee \to \gamma'^*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee\right)
\end{aligned}$$

This follows because pushforwards and pullbacks commute with flat base change, as we now spell out. Indeed, we have natural isomorphisms

$$\begin{aligned}
\beta^*_{\mathscr{C}_v}\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee &\simeq \gamma'^*\beta^*_{\mathscr{C}}\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee\\
&\simeq \gamma'^*\pi'^*\beta^*_C\pi_*\overline{x}^*\mathscr{V}^\vee\\
&\simeq \gamma'^*\pi'^*\pi'_*\beta^*_{\mathscr{C}}\overline{x}^*\mathscr{V}^\vee\\
&\simeq \gamma'^*\pi'^*\pi'_*\overline{x}'^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee
\end{aligned}$$

Similarly, we have $\beta^*_{\mathscr{C}_v}\gamma^*\overline{x}^*\mathscr{V}^\vee \simeq \gamma'^*\beta^*_{\mathscr{C}}\overline{x}^*\mathscr{V}^\vee \simeq \gamma'^*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee$. These isomorphisms fit in a diagram

$$\begin{array}{ccc} \beta^*_{\mathscr{C}_v}\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{V}^\vee & \longrightarrow & \gamma'^*\pi'^*\pi'_*\overline{x}'^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee \\ \downarrow & & \downarrow \\ \beta^*_{\mathscr{C}_v}\gamma^*\overline{x}^*\mathscr{V}^\vee & \longrightarrow & \gamma'^*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{V}^\vee, \end{array} \tag{9.3.3}$$

We show this commutes next in Lemma 9.3.5, which induces the claimed isomorphism on cokernels of the vertical maps above. □

The following lemma is presumably obvious, but it took me a few hours to actually verify, so I decided to write it out.

Lemma 9.3.5. *The diagram* (9.3.3) *commutes, even when $\mathscr{V}^\vee$ is replaced by an arbitrary sheaf $\mathscr{G}$.*

Proof. Throughout the proof, we replace $\mathscr{V}^\vee$ by a general sheaf $\mathscr{G}$. Observe (9.3.3) fits in a sequence of diagrams

$$\begin{array}{ccccccc} \beta^*_{\mathscr{C}_v}\gamma^*\pi^*\pi_*\overline{x}^*\mathscr{G} & \longrightarrow & \gamma'^*\beta^*_{\mathscr{C}}\pi^*\pi_*\overline{x}^*\mathscr{G} & \longrightarrow & \gamma'^*\pi'^*\pi'_*\beta_{\mathscr{C}}\overline{x}^*\mathscr{G} & \longrightarrow & \gamma'^*\pi'^*\pi'_*\overline{x}'^*\beta^*_{\mathscr{X}}\mathscr{G} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \beta^*_{\mathscr{C}_v}\gamma^*\overline{x}^*\mathscr{G} & \longrightarrow & \gamma^*\beta^*_{\mathscr{C}}\overline{x}^*\mathscr{G} & \longrightarrow & \gamma^*\beta^*_{\mathscr{C}}\overline{x}^*\mathscr{G} & \longrightarrow & \gamma'^*(\overline{x}')^*\beta^*_{\mathscr{X}}\mathscr{G}. \end{array} \tag{9.3.4}$$

Let $\mathscr{F} := \overline{x}^*\mathscr{G}$. The left and right squares of the above diagram clearly commute, so it suffices to show

$$\begin{array}{ccc} \beta^*_{\mathscr{C}}\pi^*\pi_*\mathscr{F} & \longrightarrow & \pi'^*\pi'_*\beta^*_{\mathscr{C}}\mathscr{F} \\ & \searrow \quad \swarrow & \\ & \beta^*_{\mathscr{C}}\mathscr{F} & \end{array} \tag{9.3.5}$$

commutes. Expanding the top map as a composition of two maps, we want to check commutativity of

$$\begin{array}{ccc} & \pi'^*\beta_C\pi_*\mathscr{F} & \\ \swarrow & & \searrow \\ \beta^*_{\mathscr{C}}\pi^*\pi_*\mathscr{F} & & \pi'^*\pi'_*\beta^*_{\mathscr{C}}\mathscr{F} \\ \searrow & & \swarrow \\ & \beta^*_{\mathscr{C}}\mathscr{F}. & \end{array} \tag{9.3.6}$$

with the upper right map given by the composition of the base change map with π'^*. Applying the (π'^*, π'_*) adjunction, we want to check commutativity of

$$\begin{tikzcd} & \beta_C^* \pi_* \mathscr{F} \arrow[dl] \arrow[dr] & \\ \pi'_* \beta_{\mathscr{C}}^* \pi^* \pi_* \mathscr{F} \arrow[dr] & & \pi'_* \beta_{\mathscr{C}}^* \mathscr{F} \arrow[dl] \\ & \pi'_* \beta_{\mathscr{C}}^* \mathscr{F}. & \end{tikzcd} \tag{9.3.7}$$

The bottom right map is the identity, and hence we may collapse it and apply the $(\beta_C^*, (\beta_C)_*)$ adjunction to reduce to verifying commutativity of

$$\begin{tikzcd} \pi_* \mathscr{F} \arrow[rr, "f"] \arrow[dr] & & (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \pi^* \pi_* \mathscr{F} \arrow[dl] \\ & (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \mathscr{F}. & \end{tikzcd} \tag{9.3.8}$$

To ease this verification, we can break the above diagram into two diagrams which we will verify commutativity of separately. We will rewrite the above diagram as

$$\begin{tikzcd} \pi_* \mathscr{F} \arrow[r, "h"] \arrow[d] \arrow[dr, "f"] & (\pi_* \pi^*) \pi_* \mathscr{F} \arrow[d, "g"] \\ (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \mathscr{F} & (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \pi^* \pi_* \mathscr{F} \arrow[l] \end{tikzcd} \tag{9.3.9}$$

Where the map f is the same one as in (9.3.8) and the map g is induced by the composition of $\pi^* \pi_*$ with the base change map.

We now check the upper right triangle and outer square of (9.3.9) commute, which will imply the lower triangle commutes, which is (9.3.8).

To check the outer square of (9.3.9) commutes, observe that h has a section ϕ given by the counit of the natural adjunction

$$\begin{tikzcd} \pi_* \mathscr{F} \arrow[d] & (\pi_* \pi^*) \pi_* \mathscr{F} \arrow[l, "\phi"'] \arrow[d, "g"] \\ (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \mathscr{F} & (\beta_C)_* \pi'_* \beta_{\mathscr{C}}^* \pi^* \pi_* \mathscr{F}. \arrow[l] \end{tikzcd} \tag{9.3.10}$$

Note that (9.3.10) commutes by functoriality of the adjunction map. Since $\phi \circ h = \mathrm{id}$

since ϕ and h correspond to the unit and counit of the (π^*, π_*) adjunction, it follows that the outer square of (9.3.9) also commutes.

So, to complete the proof, we just need to check the upper right triangle of (9.3.9) commutes. Letting $\mathscr{H} := \pi_* \mathscr{F}$, we want to show commutativity of

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\qquad\qquad} & \pi_* \pi^* \mathscr{H} \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle g'} & \\ & (\beta_C)_* * \pi'_* \beta^*_{\mathscr{C}} \pi^* \mathscr{H} & \end{array} \tag{9.3.11}$$

where f' is induced by f and g' is induced by g. That is, g' is the composition of π^* with the base change map. Expanding out the definition of the base change map gives that g' is realized as the composition of the natural adjunction $\pi_* \pi^* \mathscr{H} \to \pi_* (\beta_{\mathscr{C}})_* \beta^*_{\mathscr{C}} \pi^*$ with the identification $\pi_* (\beta_{\mathscr{C}})_* \beta^*_{\mathscr{C}} \pi^* = (\beta_C)_* \pi'_* \beta^*_{\mathscr{C}} \pi^*$. In other words, we which to verify commutativity of the diagram

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{f'} & (\beta_C)_* \pi'_* \beta^*_{\mathscr{C}} \pi^* \mathscr{H} \\ \downarrow & & \downarrow \\ \pi_* \pi^* \mathscr{H} & \longrightarrow & \pi_* (\beta_{\mathscr{C}})_* \beta^*_{\mathscr{C}} \pi^* \mathscr{H}. \end{array} \tag{9.3.12}$$

Now, let $\mu = \pi \circ \beta_{\mathscr{C}}$ and let $\nu = \beta_C \circ \pi'$. The composition $\mathscr{H} \to \pi_* \pi^* \mathscr{H} \to \pi_* (\beta_{\mathscr{C}})_* \beta^*_{\mathscr{C}} \pi^* \mathscr{H}$ is given by the adjunction $\mathscr{H} \to \mu_* \mu^* \mathscr{H}$. On the other hand, the map f' is induced by the natural map $\mathscr{H} \to \nu_* \mu^* \mathscr{H}$ which is coming from the adjunction for ν via the identification $\mu = \nu$. Therefore, both compositions in (9.3.12) are given by the adjunction $\mathscr{H} \to \mu_* \mu^* \mathscr{H}$ and the diagram commutes. □

9.4 The reduction to local heights

In this section, we explain how to reduce Theorem 9.2.3 to another result about local heights. For this, we have to review the sense in which the Faltings height in the sense of Definition 9.2.1 can be decomposed into local Faltings heights and stable Faltings height.

Recall that in [ESZB19] the authors introduce a notion of a local stacky height. In our context, this agrees with the notion of local stacky height we defined in Notation 9.3.1. For K a finite extension of $k(t)$ and v a closed point of the proper regular model of K over k and $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, we let $\delta_{\mathscr{V};v}(x)$ denote the local stacky height associated to $\mathscr{V}$ and x at v, as defined in Notation 9.3.1. Further, [ESZB19] define a notion of stable stacky height $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(x)$, see Definition 9.2.13 which satisfies the relation $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(x) + \sum_v \delta_{\mathscr{V};v}(x) = \mathrm{ht}_{\mathscr{V}}(x)$.

We next recall the analogous notion of local and stable Faltings height associated to elliptic curves.

Definition 9.4.1. Given a point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ define the *stable Faltings height* of x as follows: For each closed point v, let $\operatorname{Spec} L \to \operatorname{Spec} K$ be a finite extension over which x acquires semistable reduction at v. Define the local stable Faltings height of x at v, notated $\mathrm{ht}_v^{\mathrm{st}}(x)$, to be $\mathrm{ht}_v^{\mathrm{st}}(x) := \frac{1}{12}\sum_{w|v}\frac{1}{\deg(L/K)}\deg(\Delta_w)$, for Δ_w the discriminant of w restricted to the local ring at w, for w ranging over closed points of L over v. Define the *stable Faltings height* by $\mathrm{ht}^{\mathrm{st}}(x) := \sum_v \mathrm{ht}_v^{\mathrm{st}}(x)$.

Also, define the *local Faltings height* of $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ at a closed point v, notated $\mathrm{ht}_v(x)$, to be $\frac{1}{12}(\deg \Delta_v) - \mathrm{ht}_v^{\mathrm{st}}(x)$.

Using uniqueness of semistable models, it is not hard to see the above notion of stable Faltings height is well defined, and can be computed explicitly in terms of the discriminant at various closed points v.

Example 9.4.2. For example, when the fiber at a k-rational closed point v has reduction type I_n^*, we have $\deg \Delta_v = n + 6$. Then, $\mathrm{ht}_v^{\mathrm{st}}(x) = \frac{1}{12}n$ (using that char $k = 3$ so there is no wild inertia at such closed points) while $\mathrm{ht}_v(x) = 1/2$.

In order to prove our main result, we will need to examine elliptic curves with extra automorphisms. The next remark describes them.

Remark 9.4.3. Recall that there are only 2 possibilities for the geometric automorphism group of an elliptic curve in characteristic 3. It is either $\mathbb{Z}/2\mathbb{Z}$ or the dicyclic group of order 12, which we denote G. This dicyclic group G of order 12 is a split extension (i.e., semidirect product) of $\mathbb{Z}/4\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$, where the generator of $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{Z}/3\mathbb{Z}$ by the nontrivial automorphism. For the remainder, we fix a splitting to identify $\mathbb{Z}/4\mathbb{Z}$ as a subgroup of G.

It will be useful to note that the center of G is $\mathbb{Z}/2\mathbb{Z}$, which can be identified as an index 2 subgroup of $\mathbb{Z}/4\mathbb{Z}$ for any choice of splitting $\mathbb{Z}/4\mathbb{Z} \to G$. The quotient of G by its central $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to S_3.

Further, a semistable elliptic curve has geometric automorphism group G if and only if its j invariant is 0.

The key to proving Theorem 9.2.3 is the following:

Theorem 9.4.4. *Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points x : Spec $K \to \overline{\mathscr{M}}_{1,1}$, for K ranging over finite extensions of $k(t)$. Let $\beta : [V'/G] \to \overline{\mathscr{M}}_{1,1}$ be an étale Galois map over the point of j-invariant 0 as in Proposition 9.3.2, with*

G as in Remark 9.4.3. Then, $\beta^*\mathscr{V}$ is a vector bundle on $[V'/G]$ corresponding to a G representation which is a sum of line bundles. Moreover, under the correspondence between line bundles on $[V'/G]$ and 1-dimensional representations of G, all but one of these line bundles is trivial. Moreover, the local height $\delta_{\mathscr{V};v}(x)$ at any place v is an integral multiple of $\frac{1}{4}$.

We prove this at the end of §9.5.

9.4.5 Deducing Theorem 9.2.3

We next verify Theorem 9.2.3 assuming Theorem 9.4.4. It suffices to exhibit a point $x : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ which has a place v for which $\delta_{\mathscr{V};v}$ is not an integral multiple of $\frac{1}{4}$.

Example 9.4.6. The magma code

```
F<t> :=FunctionField(GF(3));
E := EllipticCurve([0,t,0,t^4,t^2]);
LocalInformation(E);
```

shows the elliptic curve $y^2 = x^3 + tx^2z + t^4xz^2 + t^2z^3$, has a unique place of additive reduction is (t). Moreover, it shows that this curve has reduction type IV and discriminant of valuation 5 at (t).

Lemma 9.4.7. *Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, for K a finite extension of $k(t)$. Any point $y : y : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ which has a unique place v of additive reduction (i.e., a unique place with nontrivial local height) gives an example of a point for which $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$.*

Proof. Let $K/k(t)$ denote an extension on which y acquires semistable reduction and let $z : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ denote the corresponding point. Because $\mathrm{ht}_{\mathscr{V}}(z) = \mathrm{ht}(z)$, and both $\mathrm{ht}_{\mathscr{V}}(z) = \mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(z) = \deg(K/k(t))\,\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(y)$ and $\mathrm{ht}(z) = \mathrm{ht}^{\mathrm{st}}(z) = \deg(K/k(t))\,\mathrm{ht}^{\mathrm{st}}(y)$ we conclude $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(y) = \mathrm{ht}^{\mathrm{st}}(y)$. Because we are assuming $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(y) + \delta_{\mathscr{V};v}(y) = \mathrm{ht}_{\mathscr{V}}(y) = \mathrm{ht}(y) = \mathrm{ht}^{\mathrm{st}}(y) + \mathrm{ht}_v(y)$ and we have shown $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(y) = \mathrm{ht}^{\mathrm{st}}(y)$, we conclude $\delta_{\mathscr{V};v}(y) = \mathrm{ht}_v(y)$. □

We are ready to deduce Theorem 9.2.3 from Theorem 9.4.4.

Theorem 9.4.4 implies Theorem 9.2.3. The curve of Example 9.4.6 has a unique place $v = (t)$ of additive reduction and so we have $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ by Lemma 9.4.7. Observe $\mathrm{ht}_v(x) = \frac{5}{12}$ because this curve has discriminant of valuation 5 at v, by Example 9.4.6. We conclude $\delta_{\mathscr{V};v}(x)$ is not an integral multiple of $\frac{1}{4}$, contradicting Theorem 9.4.4. □

Remark 9.4.8. We could have also taken the following alternate means to deducing Theorem 9.2.3 from Theorem 9.4.4: Observe that $\mathbb{Z}/3\mathbb{Z}$ is identified with the commutator of G. Therefore, any line bundle induces the trivial representation on $\mathbb{Z}/3\mathbb{Z}$. Hence, any point of $\overline{\mathscr{M}}_{1,1}$ factoring through $B(\mathbb{Z}/3\mathbb{Z})$ at the point of $\overline{\mathscr{M}}_{1,1}$ corresponding to j-invariant 0 has trivial local height, using Theorem 9.4.4. Hence, any nontrivial cubic twist has trivial local height, but we can easily construct cubic twists of the form $y^2 = x^3 - x + f(t)$ with nontrivial local Faltings height. As a concrete example, we can take $f(t) = t + t^4$.

9.5 Computing heights on $\overline{\mathscr{M}}_{1,1}$

So, to conclude the proof of Theorem 9.2.3, it remains to prove Theorem 9.4.4. We do so at the end of this section, after some preliminaries about the stack $\overline{\mathscr{M}}_{1,1}$ and representations of the dicyclic group of order 12, which is the geometric automorphism group of an elliptic curve with j invariant 0 over the perfect characteristic 3 field k. To begin, we make an observation explaining why it is harmless to pass to an algebraic closure.

Remark 9.5.1. For $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, observe that both notions of $\mathrm{ht}_v(x)$ and $\delta_{\mathscr{V};v}(x)$ are stable under base change along separable field extensions k' of k. Namely, $\mathrm{ht}_v(x) = \sum_{w|v} \mathrm{ht}_w(x')$ for w ranging over closed points of $K \otimes_k k'$ and x' the base change of x to k'. Similarly, $\delta_{\mathscr{V};v}(x) = \sum_{w|v} \delta_{\mathscr{V};w}(x')$. Because we are assuming k is perfect, it follows that the algebraic closure of k is separable over k. Hence, for the purpose of computing local heights, we are free to assume our base field k is algebraically closed for the remainder of this section.

Remark 9.5.2. If we have an elliptic curve with reduction $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*$, or IV^* at v, it has potentially good reduction, with corresponding j invariant 0. Therefore, if $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ is a map with one of the above reduction types at v, we must have that v maps to the point of $\overline{\mathscr{M}}_{1,1}$ lying over $j = 0$ in the coarse moduli space, for $j : \overline{\mathscr{M}}_{1,1} \to \mathbb{P}^1$ the coarse moduli space of $\overline{\mathscr{M}}_{1,1}$.

We next introduce some notation used heavily in the remainder of the proof.

Notation 9.5.3. We will assume $k = \overline{k}$ (using Remark 9.5.1 at various points to justify this assumption). Suppose we have some vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$, a point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, and a place v of K so that $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$. Then, by the local structure for Deligne-Mumford stacks (see the final part of Proposition 9.3.2), we may choose an étale map $X' \to \overline{\mathscr{M}}_{1,1}$ for X' a local scheme with closed point mapping to $j = 0$, so that X' possesses a finite étale Galois cover $V' \to X'$ with $[V'/G] \simeq \overline{\mathscr{M}}_{1,1} \times_{\mathbb{P}^1} X' =: \mathscr{X}'$, with G as in Remark 9.4.3.

Let $\mathscr{V}'$ denote the restriction of $\mathscr{V}$ to $\mathscr{X}'$. Then, a vector bundle on $\mathscr{X}'$ is the same as a G-representation over V', so $\mathscr{V}'$ can be described as a G-representation over V'. By increasing the extension $X' \to \overline{\mathscr{M}}_{1,1}$ if necessary, we may freely assume that V' is in fact a local ring, using that finite rings over strictly henselian rings decompose as a product of local rings. Hence, we may assume V' is a dvr. Note that regularity of V' follows because it is a G-cover of the regular $\overline{\mathscr{M}}_{1,1} = [V'/G]$. Let $\rho : G \to \mathrm{GL}(\mathscr{V}|_{V'})$ denote the corresponding G-representation.

Now, recall that in Remark 9.4.3, we chose a splitting $\mathbb{Z}/4\mathbb{Z} \to G$. Fix a generator 1 of $\mathbb{Z}/4\mathbb{Z}$ in G. This acts diagonalizably on $\mathscr{V}|_{V'}$. When we restrict ρ to a $\mathbb{Z}/4\mathbb{Z}$ representation, we obtain a decomposition $\rho|_{\mathbb{Z}/4\mathbb{Z}} \simeq \oplus_{i=0}^{3} \chi_i^{b_i}$ where χ_i are the four characters of $\mathbb{Z}/4\mathbb{Z}$ given by $\chi_i(1) = \zeta^i$ for ζ a fixed primitive 4th root of unity.

We can now characterize the b_i appearing in the above decomposition of $\rho|_{\mathbb{Z}/4\mathbb{Z}}$.

Lemma 9.5.4. *Suppose $\mathscr{V}$ satisfies $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ for some place v at which x has type* III *reduction. Under the above decomposition of $\rho|_{\mathbb{Z}/4\mathbb{Z}} \simeq \oplus_{i=0}^{3} \chi_i^{b_i}$ from Notation 9.5.3, after possibly modifying our choice of generator* 1 *for $\mathbb{Z}/4\mathbb{Z}$, we have $b_1 = 1$ and $b_2 = b_3 = 0$.*

Proof. Recall that by Remark 9.5.1 we may assume k is algebraically closed. From the explicit computation of the discriminant for an elliptic curve $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ with a closed point v of type III reduction, we know $\mathrm{ht}_v(x) = \frac{1}{4}$. Therefore, we must have $\delta_{\mathscr{V};v}(x) = 1/4$. Let C_v be the strict henselization of the local ring of the proper regular curve over k whose function field is $\operatorname{Spec} K$. Note that the map $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ induces a map $C_v \to V' \to \overline{\mathscr{M}}_{1,1}$. Because v acquires semistable reduction after a degree 4 cyclic extension, the induced G action on $\mathscr{V}|_{C_v}$ factors through the $\mathbb{Z}/4\mathbb{Z}$ quotient of G. By the assumption that k is algebraically closed, if $\beta : [V'/G] \to \overline{\mathscr{M}}_{1,1}$ has degree d, then our point v has d preimages $v_1, \ldots, v_d$, and so Lemma 9.3.4 applies to tell us that for any i, $\delta_{\beta^*\mathscr{V},v_i}(x') = \delta_{\mathscr{V},v}(x) = \frac{1}{4}$.

On the other hand, by [WY15, Lemma 4.3], $\delta_{\beta^*\mathscr{V},v_i}(x') = \frac{1}{4}(b_1 + 2b_2 + 3b_3)$. So, we must have $\frac{1}{4}(b_1 + 2b_2 + 3b_3) = \frac{1}{4}$. This implies $b_1 = 1$ and $b_2 = b_3 = 0$ as the b_i are nonnegative integers. □

The following lemma is a special case of a standard fact about group representations, used in the proof of complete reducibility over characteristic 0 fields.

Lemma 9.5.5. *Any G representation V splits as a direct sum $V_1 \oplus V_2$ where a generator of $\mathbb{Z}/4\mathbb{Z}$ acts on V_1 with eigenvalues* ± 1 *and acts on V_2 with eigenvalues $\pm\zeta$, for ζ a primitive* 4*th root of unity.*

Proof. This follows from a standard averaging trick. Namely, let $\alpha \in \mathbb{Z}/4\mathbb{Z}$ be the square of a generator. Then, define V_1 and V_2 as the ± 1 eigenspaces and $\pm\zeta$ eigenspaces of a generator of $\mathbb{Z}/4\mathbb{Z}$, respectively. We want to realize V_1 and V_2 as subrepresentations of V. For this, observe $\rho|_{V_1}(g) = \frac{1}{2}(\rho(g) + \rho(\alpha)\cdot\rho(g))$ and $\rho|_{V_2}(g) = \frac{1}{2}(\rho(g) - \rho(\alpha)\cdot\rho(g))$ because $\rho(\alpha)|_{V_1} = 1$ and $\rho(\alpha)|_{V_2} = -1$. Centrality of α in G then implies these restrictions are G-subrepresentations. For example,

$$\begin{aligned}\rho|_{V_1}(g)\cdot\rho|_{V_1}(h) &= \frac{1}{2}(\rho(g)+\rho(\alpha)\cdot\rho(g))\frac{1}{2}(\rho(h)+\rho(\alpha)\cdot\rho(h))\\ &= \frac{1}{4}(\rho(g)\rho(h)+\rho(\alpha g)\rho(\alpha h)+\rho(\alpha g)\rho(h)+\rho(g)\rho(\alpha h))\\ &= \frac{1}{2}(\rho(g)\rho(h)+\rho(\alpha)\rho(gh))\\ &= \rho|_{V_1}(gh)\end{aligned}$$

□

Combining the above, we now deduce a very strong constraint on the structure of $\mathscr{V}$.

Corollary 9.5.6. *Suppose $\mathscr{V}$ is a rank n vector bundle on $\overline{\mathscr{M}}_{1,1}$ satisfying $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ for some point $x : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ and some place v at which x has type* III *reduction. Then, with notation as in Notation 9.5.3, $\mathscr{V}|_{V'}$ splits as the direct sum of an $(n-1)$-dimensional trivial representation and a 1-dimensional irreducible G-representation factoring through $\mathbb{Z}/4\mathbb{Z}$ with the generator acting by ζ.*

Proof. Lemma 9.5.4 tells us that a generator of $\mathbb{Z}/4\mathbb{Z}$ has a single ζ eigenvalue and $n-1$ eigenvalues equal to 1. By Lemma 9.5.5, the representation splits as a direct sum of a 1 dimensional representation where the generator of $\mathbb{Z}/4\mathbb{Z}$ acts by ζ and an $n-1$ dimensional representation where the generator acts trivially. It suffices to show that the whole G action is trivial on this $n-1$ dimensional representation. Because the generator of $\mathbb{Z}/4\mathbb{Z}$ acts trivially, its square also acts trivially, and so the G representation factors through the action of $G/(\mathbb{Z}/2\mathbb{Z}) \simeq S_3$. Additionally, the generator of $\mathbb{Z}/4\mathbb{Z}$ maps to a transposition in S_3, and we are assuming this transposition acts trivially. However, any S_3 representation in which the transposition acts trivially is a trivial representation, since transpositions generate S_3. Therefore, this $n-1$ dimensional subrepresentation is the trivial representation. □

We now use the above constraints on $\mathscr{V}$ combined with the following example of an elliptic curve having type III reduction at a unique place in order to deduce Theorem 9.4.4.

Example 9.5.7. The magma code

```
F<t> :=FunctionField(GF(3));
E := EllipticCurve([0,t+t^2,0,t+t^2 +t^3,t^2+t^4+t^5]);
LocalInformation(E);
```

shows that the elliptic curve $y^2 = x^3 + (t + t^2)x^2z + (t + t^2 + t^3)xz^2 + (t^2 + t^4 + t^5)z^3$ has a unique place of additive reduction (t). Further, at (t), this curve has reduction type III and discriminant of valuation 3.

Lemma 9.5.8. *Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, for K a finite extension of $k(t)$. The point $y : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ of Example 9.5.7 (base changed from $\mathbb{F}_3$ to k) gives an example of a point for which $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ at the place (t) of additive type* III *reduction.*

Proof. Let $v = (t)$ denote the place of $k(t)$. Note this is the unique place of $k(t)$ at which y has additive reduction, by Example 9.5.7. The lemma then follows from Lemma 9.4.7. □

We can now prove the main result, Theorem 9.4.4.

Proof of Theorem 9.4.4. Retain notation from Notation 9.5.3. By Lemma 9.5.8, the hypothesis of Corollary 9.5.6 holds, and so $\mathscr{V}|_{V'}$ splits as the direct sum of an $(n-1)$-dimensional trivial representation and a 1-dimensional irreducible G-representation factoring through $\mathbb{Z}/4\mathbb{Z}$ with the generator acting by ζ, as desired.

To conclude the final statement that every local height is a multiple of $\frac{1}{4}$ we may apply Lemma 9.3.4 to reduce to the case of computing the local height for $\beta^*\mathscr{V}$ on $[V'/G]$. By our description of $\beta^*\mathscr{V}$ above, we know it is pulled back from a vector bundle on $[V'/(\mathbb{Z}/4\mathbb{Z})]$, and so it suffices to show every local height on a stack of the form $[V'/(\mathbb{Z}/4\mathbb{Z})]$ is an integral multiple of $\frac{1}{4}$. This follows from the computation in [WY15, Lemma 4.3]. □

9.6 Proving the existence of Northcott bundles

In this section, we prove Theorem 9.2.4 by showing there exist Northcott bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. We now introduce some idiosyncratic notation, which will be in place for the remainder of this section. For the remainder of this section, we assume the face field k is a finite field of characteristic 3.

Notation 9.6.1. Over $\overline{\mathscr{M}}_{1,1}$ we have a universal genus 1 curve $\overline{\mathscr{M}}_{1,2}$ and the smooth locus of the projection $\overline{\mathscr{M}}_{1,2} \to \overline{\mathscr{M}}_{1,1}$ is a relative group scheme whose n-torsion we denote $\mathscr{M}_{1,2}[n]$. Let $\overline{\mathscr{M}}_{1,1}(4)$ denote the normalization of $\overline{\mathscr{M}}_{1,1}$ in $\mathscr{M}_{1,2}[4]$, so $f : \overline{\mathscr{M}}_{1,1}(4) \to \overline{\mathscr{M}}_{1,1}$ is a degree 16 representable finite flat morphism. (In fact, $\overline{\mathscr{M}}_{1,1}(4)$ is just the usual

Deligne-Rapoport compactification [DR73] of elliptic curves with a 4-torsion point.) Let $\mathscr{V} := f_* \mathscr{O}_{\overline{\mathscr{M}}_{1,1}(4)}$.

Fix a point $x : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ corresponding to an elliptic curve over $k(t)$. By the valuative criterion of properness, there exists some finite extension M over $k(t)$ such that if Y denotes the proper regular curve over k whose function field is M, then $\operatorname{Spec} M \to \operatorname{Spec} k(t) \xrightarrow{x} \overline{\mathscr{M}}_{1,1}$ extends to a morphism $Y \to \overline{\mathscr{M}}_{1,1}$. After possibly replacing M by a larger extension, we assume $M/k(t)$ is Galois with Galois group G. Let $L := \operatorname{Spec} k(t) \times_{\overline{\mathscr{M}}_{1,1}} \overline{\mathscr{M}}_{1,1}(4)$, let $N := \operatorname{Spec} M \times_{\overline{\mathscr{M}}_{1,1}} \overline{\mathscr{M}}_{1,1}(4)$ and let $X := Y \times_{\overline{\mathscr{M}}_{1,1}} \overline{\mathscr{M}}_{1,1}(4)$. Note that G acts on N via its action on M over $k(t)$, and similarly acts on X via its induced action of Y. Let $\mathscr{P} := [Y/G]$ and let $\mathscr{T} := [X/G]$. Let $\pi : \mathscr{P} \to \mathbb{P}^1$ denote the coarse space morphism. Similarly, let $\pi' : \mathscr{T} \to T$ denote the coarse space of $\mathscr{T}$.

Summarizing the above using diagrams, we have:

$$\begin{array}{ccccc} \operatorname{Spec} N & \xrightarrow{y'} & \operatorname{Spec} L & \xrightarrow{x'} & \overline{\mathscr{M}}_{1,1}(4) \\ \downarrow{\scriptstyle f_M} & & \downarrow{\scriptstyle f_{k(t)}} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} M & \xrightarrow{y} & \operatorname{Spec} k(t) & \xrightarrow{x} & \overline{\mathscr{M}}_{1,1} \end{array} \tag{9.6.1}$$

$$\begin{array}{ccccc} X & \xrightarrow{\overline{y}'} & \mathscr{T} & \xrightarrow{\overline{x}'} & \overline{\mathscr{M}}_{1,1}(4) \\ \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f_{\mathscr{P}}} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\overline{y}} & \mathscr{P} & \xrightarrow{\overline{x}} & \overline{\mathscr{M}}_{1,1} \end{array} \tag{9.6.2}$$

$$\begin{array}{ccccc} \operatorname{Spec} L & \longrightarrow & \mathscr{T} & \xrightarrow{\pi'} & T \\ \downarrow & & \downarrow{\scriptstyle f_{\mathscr{P}}} & & \downarrow{\scriptstyle f_{\mathbb{P}^1}} \\ \operatorname{Spec} k(t) & \longrightarrow & \mathscr{P} & \xrightarrow{\pi} & \mathbb{P}^1_k. \end{array} \tag{9.6.3}$$

We will freely use the names of maps as labeled in the above diagrams.

The idea of the proof of Theorem 9.2.4 is to relate the stacky height to the discriminant of an associated cover of $\mathbb{P}^1$ by the relative 4-torsion, and then use the fact that there are only finitely many covers of $\mathbb{P}^1$ of fixed degree and bounded discriminant. The following is the key to the proof, and we shall come back to proving it in the final section, after showing why it implies our main result.

Proposition 9.6.2. *For k a finite field of characteristic 3, there are finitely many elliptic curves with the same associated cover $f_{\mathbb{P}^1} : T \to \mathbb{P}^1_k$. Moreover, there are finitely many elliptic curves with the same the birational class of $f_{\mathbb{P}^1} : T \to \mathbb{P}^1_k$.*

Before giving the proof, in order to apply the machinery of [ESZB19], we need the following straightforward but important fact. We use the following definition of tuning stack we gave in Definition 9.2.10, which differs slightly from that of [ESZB19].

Lemma 9.6.3. *$\mathscr{P}$ is a tuning stack for x.*

Proof. The group G acts on Y and the map $Y \to \overline{\mathscr{M}}_{1,1}$ is G invariant because it is G-invariant on the generic point and $\overline{\mathscr{M}}_{1,1}$ is separated. Therefore, we obtain a map $[Y/G] \to \overline{\mathscr{M}}_{1,1}$. By construction, $[Y/G]$ is birational to $\mathbb{P}^1$. Further, $[Y/G]$ is a normal Deligne-Mumford stack with finite diagonal because it is a quotient of a normal scheme by a constant group. □

The following explicit description of the height associated to $\mathscr{V}$ will allow us to easily verify the Northcott property. We use $\omega_{f_{\mathbb{P}^1}}$ to denote the relative dualizing sheaf associated to the map $f_{\mathbb{P}^1}$.

Lemma 9.6.4. $\mathrm{ht}_{\mathscr{V}}(x) = \frac{1}{2}\deg \omega_{f_{\mathbb{P}^1}}$.

Proof. By definition of heights and Lemma 9.6.3, $\mathrm{ht}_{\mathscr{V}}(x) = -\deg\left(\pi_* \overline{x}^* f_* \mathscr{O}_{\overline{\mathscr{M}}_{1,1}(4)}\right)$. Since f is finite, it satisfies cohomology and base change, and so we have

$$\begin{aligned}
\mathrm{ht}_{\mathscr{V}}(x) &= -\deg\left(\pi_* \overline{x}^* f_* \mathscr{O}_{\overline{\mathscr{M}}_{1,1}(4)}\right)\\
&= -\deg\left(\pi_* (f_{\mathscr{P}})_* \overline{x}'^* \mathscr{O}_{\overline{\mathscr{M}}_{1,1}(4)}\right)\\
&= -\deg\left(\pi_* (f_{\mathscr{P}})_* \mathscr{O}_{\mathscr{T}}\right)\\
&= -\deg\left((f_{\mathbb{P}^1})_* \pi'_* \mathscr{O}_{\mathscr{T}}\right)\\
&= -\deg\left((f_{\mathbb{P}^1})_* \mathscr{O}_T\right).
\end{aligned}$$

Note above we are also using that $\pi'_* \mathscr{O}_{\mathscr{T}} = \mathscr{O}_T$, by the Keel-Mori theorem.

Now, using Riemann-Roch for vector bundles on $\mathbb{P}^1$,

$$\deg((f_{\mathbb{P}^1})_* \mathscr{O}_T) = \chi((f_{\mathbb{P}^1})_* \mathscr{O}_T) - \mathrm{rk}(f_{\mathbb{P}^1})_* \mathscr{O}_T = \chi(\mathscr{O}_T) - 16 = -g + 1 - 16.$$

Then, Riemann Hurwitz shows

$$2g - 2 = 16(-2) + \deg \omega_{f_{\mathbb{P}^1}}.$$

So, combining the above two expressions yields

$$\deg(f_{\mathbb{P}^1})_* \mathscr{O}_T = \frac{-1}{2}\deg \omega_{f_{\mathbb{P}^1}},$$

proving the claim. □

Proof of Theorem 9.2.4, assuming Proposition 9.6.2. Let us now explain why the above proves the Northcott property for $\mathscr{V}$. We want to show there are only finitely many elliptic curves of height at most N. By Proposition 9.6.2, it is enough to show there are finitely many corresponding curves $f_{\mathbb{P}^1} : T \to \mathbb{P}^1$ such that the corresponding height is at most N. Using Lemma 9.6.4, it is enough to show there are only finitely many birational classes of degree 16 covers $T \to \mathbb{P}^1$ of discriminant degree at most $N/2$. Taking their normalization (noting that if T has discriminant degree $\leq N/2$ then the same will be true of its normalization), it suffices to show there are finitely many degree 16 covers. I believe this is a standard fact, but in any case it follows from finite-typedness of the Hurwitz stack of degree 16 covers (with smooth source of bounded genus) of $\mathbb{P}^1$. □

9.7 Bounding the number of elliptic curves with fixed 4-torsion

To conclude the proof of Theorem 9.2.4 showing there are Northcott bundles on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3, it remains to prove Proposition 9.6.2. The idea will be to show that the 2-torsion already characterizes the elliptic curve up to quadratic twists, and then the 4 torsion differentiates between most quadratic twists. Recall that we say two elliptic curves $y^2 = x^3 + A_1x^2 + A_2x + A_3$ and $y^2 = x^3 + B_1x^2 + B_2x + B_3$ are quadratic twists of each other if the latter is isomorphic to $(h/g)y^2 = x^3 + A_1x^2 + A_2x + A_3$ for $h, g \in k[t] - \{0\}$. Recall we are taking $k = \mathbb{F}_q$ to be a finite field.

Lemma 9.7.1. *Let $Q \subset T$ denote the union of connected components corresponding to the closure of the nontrivial 2-torsion via the natural inclusion $E[2] - E[1] \hookrightarrow E[4]$. So Q is a degree 3 cover of $\mathbb{P}^1$. Then, two points $a, b : \mathbb{F}_q(t) \to \overline{\mathscr{M}}_{1,1}$ with associated nontrivial 2-torsion schemes $Q_a \to \mathbb{P}^1$ and $Q_b \to \mathbb{P}^1$ have Q_a is birational to Q_b over $\mathbb{P}^1$ if and only if a and b are related via a quadratic twist.*

Proof. The points a and b have associated minimal Weierstrass equations $y^2z = x^3 + A_1x^2z + A_2xz^2 + A_3z^3$ and $y^2z = x^3 + B_1x^2z + B_2xz^2 + B_3z^3$. Because T is birational to $\mathscr{T}$, we obtain that Q_a is birational to the curve $x^3 + A_1x^2z + A_2xz^2 + A_3z^3$ and similarly Q_b is birational to the curve $x^3 + B_1x^2z + B_2xz^2 + B_3z^3$. If $\deg A_i = 2id_a$ and $\deg B_i = 2id_b$ then Q_c (for $c \in \{a, b\}$) naturally lives in the Hirzebruch surface $\mathrm{Proj}_{\mathbb{P}^1}(\mathscr{O}(2d_c) \oplus \mathscr{O}(4d_c))$ with the associated quotients onto $\mathscr{O}(2d_c)$ and $\mathscr{O}(4d_c)$ coordinates corresponding to the variables x and z. Now, just as elliptic curves have a minimal Weierstrass form, such trigonal curves have a minimal form (realized as a curve of extreme Maroni invariant on a Hirzebruch surface). Given a curve $x^3 + C_1x^2z + C_2xz^2 + C_3z^3$, the aforementioned minimal form is obtained by factoring out any factor which appears to order i in C_i. Then, supposing the minimal form for Q_a is obtained by factoring out f^i from A_i and the minimal form for Q_b is obtained by factoring out g^i from B_i, a and b are related by a quadratic twist by f/g. Conversely, if a and b are related by a quadratic twist, their associated 2-torsion schemes will be birational. □

Lemma 9.7.2. *If $a, b : \operatorname{Spec} \mathbb{F}_q(t) \to \overline{\mathscr{M}}_{1,1}$ correspond to two elliptic curves related by a quadratic twist, then for fixed a there are only finitely many b such that the associated degree* 16 *covers $T_a \to \mathbb{P}^1$ and $T_b \to \mathbb{P}^1$ are birationally isomorphic over $\mathbb{P}^1$.*

Proof. Suppose a corresponds to a quadratic twist of b by some rational function h/g where $h, g \in \mathbb{F}_q[x]$. With Q_a and Q_b the corresponding 2-torsion curves as in the Lemma 9.7.1, we see from Lemma 9.7.1 that Q_a is birational to Q_b over $\mathbb{P}^1$. Let us call this birational morphism β. Suppose we have an birational isomorphism $\alpha : T_a \simeq T_b$ over $\mathbb{P}^1$. Then, there are canonical "reduction $\bmod 2$" maps $\pi_a : T_a \to Q_a$ and $\pi_b : T_b \to Q_b$. These fit into a square (all compatible with the given projections to $\mathbb{P}^1$)

$$\begin{array}{ccc} T_a & \xrightarrow{\alpha} & T_b \\ \downarrow{\scriptstyle \pi_a} & & \downarrow{\scriptstyle \pi_b} \\ Q_a & \xrightarrow{\beta} & Q_b. \end{array} \tag{9.7.1}$$

Replacing T_a and T_b by the normalizations of their Galois closures T'_a and T'_b, and Q_a and Q_b by the normalizations of their Galois closures Q'_a and Q'_b, we obtain $\pi'_a : T'_a \to Q'_a$ and $\pi'_b : T'_b \to Q'_b$.

Suppose the cubic extension associated to Q_a and Q_b geometrically factors over the function field of $\mathbb{P}^1$ as $(x - s_1)(x - s_2)(x - s_3)$, for $s_i \in \overline{\mathbb{F}_q(t)}$. We claim that the subscheme of $V(h)$ disjoint from $V(s_i - s_j)$ can be recovered from the ramification locus of the map π'_a. This follows from the fact that if Q_a corresponds to the cubic extension which geometrically factors as $(x - s_1)(x - s_2)(x - s_3)$, then the field extension associated to the 4-torsion is generated by $\sqrt{h(s_i - s_j)}$ for $1 \le i \le j \le 3$, see [Yel15, Proposition 3.1].

Therefore, the only way α can exist compatibly with the projections to $\mathbb{P}^1$ is when h agrees with g, up to some element which is a factor of differences of the s_i. In other words, if the two resulting curves have T_a and T_b which are birational over $\mathbb{P}^1$, then h/g lies in a finite set given by factors of $s_i - s_j$ and scalars. □

Proof of Proposition 9.6.2. If two points a and b have associated T_a and T_b which are birational, then their associated 2-torsion schemes Q_a and Q_b as in Lemma 9.7.1 are also birational. By Lemma 9.7.1, the elliptic curves corresponding to a and b are then related by a quadratic twist, and Lemma 9.7.2 then implies the proposition. □

9.8 Representations of S_3 in characteristic 3

In this section, we classify indecomposable representations of S_3 over an arbitrary field of characteristic 3. While it was not strictly needed in our analysis of stacky heights on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3 above, it is intimately related, and would likely be useful in case someone was interested in further analyzing exactly which height functions exist.

We work over a fixed field k of characteristic 3. Let triv denote the trivial representation of S_3 over k, let sgn denote the sign representation over k, let perm denote the natural permutation representation on the set $\{1, 2, 3\}$ with corresponding basis e_1, e_2, e_3 over k, and let std denote the subrepresentation of perm over k. given by the hyperplane $e_1 + e_2 + e_3 = 0$.

Remark 9.8.1. For future convenience, it will be useful to have the following explicit descriptions of std and perm in characteristic 3. Let $\tau \in S_3$ denote a transposition and $\sigma \in S_3$ denote a 3-cycle. Then we can choose a basis for the 2-dimensional representation $\mathrm{std} : S_3 \to \mathrm{Aut}(V)$ so that it is given by

$$\mathrm{std}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathrm{std}(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Similarly, we can choose a basis for the 3-dimensional representation $\mathrm{perm} : S_3 \to \mathrm{Aut}(V)$ so that it is given by

$$\mathrm{perm}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathrm{perm}(\sigma) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This can be verified directly, but also follows from the proofs of Lemma 9.8.4 and Lemma 9.8.5 below.

Then, we have the following characterization of indecomposable representations over $\mathbb{F}_3$. Surely this is known before, but I asked some people and couldn't find a reference, so I decided to work out a proof. The proof is completed at the end of this section, after some preliminary lemmas. The entire proof is elementary. The idea is just to multiply out matrices and see what constraints the defining relations of S_3 yield on the resulting representations.

Theorem 9.8.2. *Let k be a field of characteristic* 3*. The* 6 *representations*

$$\mathrm{triv}, \mathrm{std}, \mathrm{perm}, \mathrm{sgn}, \mathrm{sgn} \otimes \mathrm{std}, \mathrm{sgn} \otimes \mathrm{perm}$$

define pairwise non-isomorphic representation. Any indecomposable S_3 representation over k lies in the above list.

Lemma 9.8.3. *The only irreducible representations of S_3 over k, up to isomorphism, are* sgn *and* triv*. In particular, every indecomposable representation has a filtration by copies of* sgn *and* triv*.*

Proof. By Brauer's theorem [Ser77, §18.2, Corollary 3], there are as many irreducible representations as conjugacy classes in S_3 of order prime to 3. Note that this uses the field k contains mth roots of unity, where m is the lcm of the orders of elements of S_3 which are prime to p. For S_3, the resulting m is 2, and indeed any field of characteristic 3 contains ± 1. There are two such conjugacy classes, hence only 2 representations. Therefore, triv and sgn are the only irreducible S_3 representations over k. □

Lemma 9.8.4. *The only* 2*-dimensional indecomposable representations of S_3 over k, up to isomorphism are* std *and* sgn ⊗ std

Proof. Let $\rho : S_3 \to \mathrm{Aut}(V)$ be a 2-dimensional indecomposable S_3 representation over k. By Lemma 9.8.3, V must contain a subrepresentation isomorphic to either sgn or triv. After possibly tensoring with sgn, we may assume it contains one isomorphic to triv. We then want to show that such a representation must be the standard representation.

Let τ denote a transposition and σ denote a 3-cycle. Observe that τ cannot act trivially, because then the S_3 representation would be trivial, as S_3 has no proper normal subgroups containing τ. Therefore, after choosing an appropriate basis, we may assume τ acts as

$$\rho(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We may also choose the basis so that $\rho(\sigma)$ is upper triangular, necessarily with 1's on the diagonal as it has order 3. By rescaling the second basis vector, we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This shows that up to isomorphism, there is a unique indecomposable 2-dimensional S_3 representation with a subrepresentation isomorphic to triv. Since std is one such representation, it must be std, as can also be verified directly. □

Lemma 9.8.5. *The only 3-dimensional indecomposable representations of S_3 over k, up to isomorphism are* perm *and* $\mathrm{sgn} \otimes \mathrm{perm}$

Proof. Let $\rho : S_3 \to \mathrm{Aut}(V)$ be a 3-dimensional indecomposable S_3 representation over k. As in the proof of Lemma 9.8.4, it suffices to show that, up to isomorphism, there is a unique 3-dimensional indecomposable S_3 representation containing triv as a subrepresentation.

By Lemma 9.8.3, we know ρ must contain a 2-dimensional indecomposable subrepresentation, which is isomorphic to std by Lemma 9.8.4. Therefore, with notation as in Remark 9.8.1, and using the construction of the std representation given there, we can assume

$$\rho(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & x \end{pmatrix}, \rho(\sigma) = \begin{pmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where $x \in \{1, -1\}$ and $a, b \in k$. By multiplying out matrices, the constraint that $\tau\sigma\tau^{-1} = \sigma^2$ yields the relations $-xb = 2b$ and $xa = 2a + b$. The first constraint forces $x = 1$ and the second then yields $a = -b$. Note that $a \neq 0$ as the representation is indecomposable. By rescaling the third basis vector, we may therefore assume $a = 1$ and so the representation can be taken to be given by that called perm in Remark 9.8.1. □

Lemma 9.8.6. *There are no 4-dimensional indecomposable representations of S_3 in characteristic 3.*

Proof. Let $\rho : S_3 \to \mathrm{Aut}(V)$ be a 4-dimensional S_3 representation over k. By Lemma 9.8.3, after tensoring with sgn, we may assume ρ contains a subrepresentation isomorphic to triv. By Lemma 9.8.5, if ρ were indecomposable, it would necessarily contain a 3-dimensional subrepresentation isomorphic to perm. Therefore, with notation as in Remark 9.8.1, we can assume any 4-dimensional representation ρ is given in the form

$$\rho(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \rho(\sigma) = \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

with $x \in \{1, -1\}$ and $a, b, c \in k$. The constraint that $\rho(\sigma)^3 = \mathrm{id}$ implies $c = 0$. The constraint that $\rho(\tau)\rho(\sigma)\rho(\tau^{-1}) = \rho(\sigma)^2$ yields the two relations $-xb = 2b$ and $xa = 2a + b$. Therefore, $x = 1$ and $a = -b$. Hence, if ρ were not indecomposable, after possibly rescaling the last basis vector, we can assume ρ is of the form

$$\rho(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rho(\sigma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

However, this representation is reducible. Namely, let e_i denote the ith basis vector in the above matrix representation. Then, ρ has a subrepresentation spanned by e_1, e_2, and e_3 which is isomorphic to perm and a complementary copy of triv spanned by $e_3 - e_4$. □

Proof of Theorem 9.8.2. The statement for representations of dimension at most 4 holds true by Lemma 9.8.3, Lemma 9.8.4, Lemma 9.8.5, and Lemma 9.8.6. So, it suffices to prove that S_3 has no indecomposable representations of dimension $d \geq 5$. But, by Lemma 9.8.3, any such representation would necessarily have an indecomposable 4-dimensional subrepresentation. None of these exist by Lemma 9.8.6. □

Chapter 10

Low degree covers

In this chapter, we describe a structure theorem parameterizing Gorenstein degree d covers over a general base and discuss some further related topics. The structure theorem is a generalization of [CE96, Theorem 2.1]. All of this chapter is joint work with Ravi Vakil, and some is also joint with Melanie Wood. It seems this theorem has only been stated over integral noetherian bases in the literature, and we provide the straightforward generalization to arbitrary bases in § 10.2. In the case that the degree of the cover is at most 5, these structure theorems have particularly simple alternate descriptions, and we also recall these in § 10.2. In § 10.3, we use the above mentioned parameterizations to explicitly describe the stack of degree d covers for $d \leq 5$ as a global quotient stack. We note that these descriptions have implicitly been used to count number fields of degree at most 5. Finally, in § 10.5, we describe a canonical pairing on the Casnati Ekedahl resolution. This pairing gives some additional structure to covers of degree 6, which we are interested in understanding better, but does not seem to yield any structure theorem in degree 6 as simple as in the cases of degree at most 5.

10.1 Story of the project

Before continuing with the mathematics of this final chapter, I wanted to take some time to tell the story of how I learned about the material here. It began in a conversation about the Grothendieck ring of varieties I had with Anand Deopurkar. He mentioned to me an interesting open question about generalizing semi-ample Bertini theorems to the Grothendieck ring. (In retrospect, it is good we did not work on this question, as this, and much more, has since been carried out in [BH19].) This question was about generalizing work of Melanie Wood and Ravi Vakil [VW15].

During a visit to Wisconsin in my second year in graduate school, (the selfsame visit mentioned in § 8.1,) I met with Melanie and asked her about the generalizations I was discussing with Anand. Interestingly, I had never discussed these questions with my advisor, Ravi Vakil. After asking her about these, she mentioned there was actually a closely related question she was working on with Ravi, and invited me to join them.

The problem was to count covers of degree ≤ 5 of $\mathbb{P}^1$ in the Grothendieck ring, i.e., compute the classes of Hurwitz stacks parameterizing covers of degree $d \leq 5$ as $g \to \infty$. We are currently working on the paper, and it appears to be in its final stages of preparation, but, as usual with these things, it is anyone's guess as to when it will actually be made public. The material of § 10.2 and § 10.3 are slated to appear in that forthcoming paper.

After learning about Casnati-Ekedahl's parameterizations for covers of arbitrary degree, Ravi suggested thinking about whether they could be used in any way to probe the birational properties of Hurwitz stacks parameterizing covers of $\mathbb{P}^1$ of degree 6. We found a fair amount of structure underlying covers of degree ≥ 6, but not enough to get a grip on the birational properties of the moduli stack. In the remainder of this chapter, we discuss some of this structure we have found.

10.2 Generalizations of Casnati-Ekedahl

There is a structure theorem proven by Casnati-Ekedahl [CE96] which describes a minimal resolution of covers of arbitrary degree, and then gives concrete descriptions of these covers in degrees 3 and 4. A similar description in the degree 5 case is due to Casnati in [Cas96]. However, the main structure theorem on resolutions of finite covers [CE96, Theorem 2.1] (a reformulation of [CE96, Theorem 1.3]) is unfortunately only stated for integral noetherian varieties. In order to use it to parameterize stacks over non-reduced bases, it will be useful to have a version applicable for arbitrary bases. Essentially the same proofs given in [CE96, Theorem 2.1] applies, except that one must replace applications of Grauert's theorem with applications of cohomology and base change. We thank Gianfranco Casnati for confirmation of this. Following this, we explain how the structure theorems for covers of degree $d \leq 5$ appearing in [CE96, Theorem 3.4, Theorem 4.4] and [Cas96, Theorem 3.8] similarly generalize to the non-integral setting. In fact, we upgrade their structure theorems to a bijection in degrees $3 \leq d \leq 5$. We also upgrade Casnati's result in degree 5 in an additional way to deal with all Gorenstein covers, see Remark 10.2.6.

10.2.1 The main structure theorem from Casnati-Ekedahl

We next recall the main structure theorem and give its proof in the more general setting. We first recall some terminology. A finitely locally free surjective map $X \to Y$ of degree d is *Gorenstein* if the scheme theoretic fiber X_y over $\kappa(y)$ for every $y \in Y$ is Gorenstein. For k a field, We also say a subscheme $X \subset \mathbb{P}^n_k$ is *arithmetically Gorenstein* if the homogeneous coordinate ring $\oplus_{i \geq 0} H^0(X, \mathscr{O}_X(i))$ is Gorenstein. For $\mathscr{E}$ a finite locally free sheaf over $\mathscr{O}_Y$ of rank $d+1$, we let $\pi : \mathbb{P}\mathscr{E} \to Y$ denote the corresponding projective bundle $\mathbb{P}\mathscr{E} := \operatorname{Proj} \operatorname{Sym}^\bullet \mathscr{E}$. A projective bundle of relative dimension m over a scheme Y is a scheme $Z \to Y$ isomorphic to $\mathbb{P}\mathscr{E}$ for $\mathscr{E}$ a locally free sheaf on Y of rank $m+1$.

Theorem 10.2.2 (Generalization of [CE96, Theorem 2.1], see also [CN07, Theorem 2.2])**.** *Let X and Y be schemes and let $\rho : X \to Y$ be a finite locally free surjective Gorenstein map of degree d, for $d \geq 3$. Suppose we are given a projective bundle $\pi : \mathbb{P} \to Y$ of relative dimension $d-2$, and an embedding $i : X \to \mathbb{P}$ such that $\rho = \pi \circ i$. Further assume that $\rho^{-1}(y) \subset \pi^{-1}(y) \simeq \mathbb{P}^{d-2}_{\kappa(y)}$ is an arithmetically Gorenstein nondegenerate subscheme for each point $y \in Y$. Any two such tuples $(\mathbb{P}, \pi, i)$ and $(\mathbb{P}', \pi', i')$ are uniquely isomorphic, meaning there is a unique isomorphism $\psi : \mathbb{P} \simeq \mathbb{P}'$ such that $\pi' \circ \psi = \pi$ and $\psi \circ i = i'$. Moreover, for any such triple $(\mathbb{P}, \pi, i)$ with $\rho = \pi \circ i$, the following properties hold.*

(i) Let $\rho^\# : \mathscr{O}_Y \to \rho_* \mathscr{O}_X$ denote the natural map induced by $\rho : X \to Y$ and let $\mathscr{E}^\vee := \operatorname{coker} \rho^\#$. Then, $\mathbb{P} \simeq \mathbb{P}\mathscr{E}$.

(ii) The composition $\phi : \rho^* \mathscr{E} \to \rho^* \rho_* \omega_{X/Y} \to \omega_{X/Y}$ is surjective and the ramification divisor $R \subset X$ satisfies $\mathscr{O}_X(R) \simeq \omega_{X/Y} \simeq \mathscr{O}_X(1) := i^* \mathscr{O}_{\mathbb{P}\mathscr{E}}(1)$.

(iii) There is a sequence $\mathscr{N}_0, \mathscr{N}_1, \ldots, \mathscr{N}_{d-2}$ of finite locally free $\mathscr{O}_{\mathbb{P}}$ sheaves with $\mathscr{N}_0 := \mathscr{O}_{\mathbb{P}\mathscr{E}}$ and an exact sequence

$$0 \longrightarrow \mathscr{N}_{d-2}(-d) \xrightarrow{\alpha_{d-2}} \mathscr{N}_{d-3}(-d+2) \xrightarrow{\alpha_{d-3}} \cdots$$
$$\cdots \xrightarrow{\alpha_2} \mathscr{N}_1(-2) \xrightarrow{\alpha_1} \mathscr{O}_{\mathbb{P}} \longrightarrow \mathscr{O}_X \longrightarrow 0, \tag{10.2.1}$$

unique up to unique isomorphism restricting to the identity map $\mathscr{O}_X \to \mathscr{O}_X$ such that the restriction of (10.2.1) *to the fiber $\mathbb{P}_y := \pi^{-1}(y)$ over y is a minimal free resolution of the structure sheaf of $X_y := \rho^{-1}(y)$. There are finite locally free sheaves $\mathscr{F}_i$ on Y so that $\mathscr{N}_i \simeq \pi^* \mathscr{F}_i$, so $\mathscr{N}_i$ is fiberwise trivial. Further $\mathscr{N}_{d-2}$ is invertible, and, for*

$i = 1, \dots, d-3$, one has

$$\beta_i := \operatorname{rk} \mathscr{N}_i = \operatorname{rk} \mathscr{F}_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1}. \tag{10.2.2}$$

Moreover, $\pi^*\pi_*\mathscr{N}_i \simeq \mathscr{N}_i$ for $0 \leq i \leq d-2$, and $\mathscr{H}om_{\mathscr{O}_{\mathbb{P}}}(\mathscr{N}_i(-i-1), \mathscr{N}_{d-2}(-d)) \simeq \mathscr{N}_{d-2-i}(-d-1-i)$, for $1 \leq i \leq d-3$. Additionally, the formation of $\pi_*\mathscr{N}_\bullet$ commutes with base change on Y.

(iv) If $\mathbb{P}$ is isomorphic to $\mathbb{P}\mathscr{E}'$ for some locally free sheaf $\mathscr{E}'$ on Y and the map $X \to \mathbb{P}\mathscr{E}'$ is induced by a surjection $\rho^*\mathscr{E}' \to \mathscr{L}$, for $\mathscr{L}$ an invertible sheaf on X, then $\mathscr{E}' \simeq \mathscr{E}$ if and only if $\mathscr{N}_{d-2} \simeq \pi^* \det \mathscr{E}'$ in the resolution (10.2.1) *computed with respect to the polarization $\mathscr{O}_{\mathbb{P}\mathscr{E}'}(1)$.*

(v) The pushforward of the map $\alpha_1 : \mathscr{N}_1(-2) \to \mathscr{O}_{\mathbb{P}^1}$ along π induces an injection $\mathscr{F}_1 \to \operatorname{Sym}^2 \mathscr{E}$ and for $d-3 \geq i \geq 1$, the pushforward $\alpha_i : \mathscr{N}_i(-i-1) \to \mathscr{N}_{i-1}(-i)$ along π induces an injection $\mathscr{F}_i \to \mathscr{F}_{i-1} \otimes \mathscr{E}$.

(vi) For any point $y \in Y$, no subscheme $X'_y \subset X_y$ of degree $d-1$ is sent under ρ to a hyperplane of $\pi^{-1}(y)$.

Remark 10.2.3. The statement of Theorem 10.2.2 differs in several ways from the original statement in [CN07, Theorem 2.1]. As pointed out in [CN07, Theorem 2.2] we have added a nondegenerate hypothesis to the statement. We also do not work over noetherian or reduced bases, but to compensate, we have added a finite presentation hypothesis on the map ρ, i.e., we have required that it is locally free. We have also added property (v) and (iv).

Proof. As a first step, we reduce to the case X and Y are noetherian. In view of the asserted uniqueness, by Zariski descent, we may reduce to the case that Y is affine. Because $\rho : X \to Y$ is locally finitely presented as it is finite locally free, we can write Y as a limit of spectra of finite type $\mathbb{Z}$-algebras Y_i. We can then spread out all of the data described in the theorem to some such Y_i, and realize ρ along with all of the above data as the base change of some $\rho_i : X_i \to Y_i$ along a map $Y \to Y_i$, for Y_i finite type over $\operatorname{Spec} \mathbb{Z}$. In particular, we can assume Y and X are noetherian.

The proof of [CE96, Theorem 2.1] in [CE96, Theorem 2.1], is broken up into steps A, B, C, and D. Step A has a minor inaccuracy which we next address. The only generalization needed occurs in step B, while steps C and D go through without change.

Next, the proof of this statement in the key case that $Y = \operatorname{Spec} k$ is a field is given in [CE96, Step A, p. 443]. It appears this has a minor error as it is claimed that, given a finite k-algebra A and a generalized trace map $\eta : A \to k$, i.e., a surjection whose kernel only contains the 0 ideal, one can modify η to assume $\eta(e_0) = 0$ for e_0 the unit of A. This is possible over an algebraically closed field, but is not possible over finite fields, such as in the case $Y = \operatorname{Spec} \mathbb{F}_2$ and $X = \cup_{i=1}^{5} Y$. However, we can still construct the embedding $X \to \mathbb{P}\mathscr{E}$ over k induced by the relative dualizing sheaf $\omega_{X/Y}$ and construct a minimal free resolution for this embedding. All the properties appearing in the theorem may be verified after base change to the algebraic closure of k, at which point the proof appearing in [CE96, Step A, p. 443] goes through. We also note that in the statement of [Sch86, Lemma, p. 119] which is cited in [CE96, Step A, p. 443], the subscheme D there should have degree d and lie in $\mathbb{P}^{d-2}$, as opposed to degree $d-2$ in $\mathbb{P}^{d-1}$. Note that in order to apply [Sch86, Lemma, p. 119], we use the hypothesis that $X \subset \mathbb{P}\mathscr{E}$ is nondegenerate, a hypothesis which was omitted in [CE96, Theorem 2.1]. At this point, (vi) follows from [Sch86, Lemma, p. 119].

Having established the result when $Y = \operatorname{Spec} k$, it remains to carry out the proof for general bases following [CE96, Step B, C, and D, p. 445-447]. In what follows, we next recapitulate the argument for step B [CE96, p. 445], modifying the application of Grauert's theorem to one of cohomology and base change.

Recall the statement of Step B: Suppose there is a factorization $\rho = \pi \circ i$, for $\pi : \mathbb{P} \to Y$ a projective $\mathbb{P}^{d-2}$ bundle and $i : X \to \mathbb{P}$ an embedding with X_y an arithmetically Gorenstein subscheme of $\mathbb{P}_y$ for each $y \in Y$. Then, (10.2.1) exists, is unique up to unique isomorphisms, restricts to a minimal free resolution of $\mathscr{O}_{X_y}$ over each point $y \in Y$, and $\pi^*\pi_*\mathscr{N}_\bullet \simeq \mathscr{N}_\bullet$.

For the remainder of the construction, we only handle the case $d \geq 4$. The case $d = 3$ is quite analogous to the case $d \geq 4$, though significantly easier as the resolution has length 2.

Define maps j_y, i_y as in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathbb{P} & \xleftarrow{j_y} & \mathbb{P}_y \\ & \searrow^{\rho} & \downarrow{\scriptstyle \pi} & & \downarrow \\ & & Y & \xleftarrow{i_y} & y \end{array} \tag{10.2.3}$$

Letting $\mathscr{I}$ denote the ideal sheaf of X in $\mathbb{P}$, we claim that $j_y^*\mathscr{I}$ is the ideal sheaf of X_y in $\mathbb{P}_y$. To see this, we only need to verify that $j_y^*\mathscr{I} \to j_y^*\mathscr{O}_{\mathbb{P}} \to j_y^*\mathscr{O}_X$ is exact. Since $\mathscr{O}_X$ is flat over Y, we will verify more generally that for $\mathscr{H}, \mathscr{G}, \mathscr{F}$ three sheaves on X with $\mathscr{F}$ flat over Y, and an exact sequence $0 \to \mathscr{H} \to \mathscr{G} \to \mathscr{F} \to 0$, the pullback sequence $0 \to j_y^*\mathscr{H} \to j_y^*\mathscr{G} \to j_y^*\mathscr{F} \to 0$ is exact. Indeed, this holds because $R^1 j_y^*\mathscr{F} = \mathscr{T}or_1^{\mathscr{O}_{\mathbb{P}}}(\mathscr{F}, \mathscr{O}_{\mathbb{P}_y}) = \mathscr{T}or_1^{\mathscr{O}_Y}(\mathscr{F}, \kappa(y)) = 0$. Here we are using that $\mathscr{F}$ is flat over Y for the final vanishing and $\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}_y} \simeq \mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)$ for the equality of $\mathscr{T}or$ sheaves.

Next, [CE96, Step A, p. 443] provides a resolution of $\mathscr{I}_{X_y/\mathbb{P}_y} = j_y^* \mathscr{I}$ of the form

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_{\mathbb{P}_y}(-d) \xrightarrow{\alpha_{d-2,y}} \mathscr{O}_{\mathbb{P}_y}(2-d)^{\oplus \beta_{d-3}} \xrightarrow{\alpha_{d-3,y}} \cdots \\ \cdots \xrightarrow{\alpha_{2,y}} \mathscr{O}_{\mathbb{P}_y}(-2) \xrightarrow{\alpha_{1,y}} j_y^*\mathscr{I} \longrightarrow 0. \end{aligned} \tag{10.2.4}$$

We claim $j_y^*\mathscr{I}$ is 3-regular, in the sense of Castelnuovo-Mumford regularity, i.e., $H^i(\mathbb{P}_y, j_y^*\mathscr{I}(3-i)) = 0$ for $i \geq 1$. To verify this, it follows from the definition of regularity that for an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of sheaves with $\mathscr{F}'$ $m+1$-regular and $\mathscr{F}$ m-regular, $\mathscr{F}''$ is also m regular. Using this and the fact that $\mathscr{O}_{\mathbb{P}_y}(-k)$ is k-regular (and hence it is also $k+1$ regular by [FGI+05, Lemma 5.1(b)]), it follows by induction that $\operatorname{im} \alpha_{d-i,y}$ is $d-i+2$ regular. Therefore, $j_y^*\mathscr{I} = \operatorname{im} \alpha_{1,y}$ is 3-regular. By [FGI+05, Lemma 5.1(b)], we obtain $H^1(\mathbb{P}_y, j_y^*\mathscr{I}(n)) = 0$ for $n \geq 2$. Hence, by cohomology and base change, $R^1\pi_*\mathscr{I}(n) = 0$ for $n \geq 2$.

For $\mathscr{F}$ a sheaf, let us denote by $\phi_y^i(\mathscr{F}) : R^i\pi_*\mathscr{F} \otimes \kappa(y) \to H^i(X_y, \mathscr{F}|_{X_y})$ the natural base change map. Then we have seen above that, for $n \geq 2$, $\phi_y^1(\mathscr{I}(n))$ is an isomorphism at all y. Further, $R^1\pi_*\mathscr{I}(n)$ is locally free (and in fact equal to 0) which implies by cohomology and base change that $\phi_y^0(\mathscr{I}(n))$ is an isomorphism for all $n \geq 2$. In other words, the formation of $\pi_*\mathscr{I}(n)$ then commutes with base change on Y. Further, again by cohomology and base change, $\pi_*\mathscr{I}(n)$ is a locally free sheaf when $n \geq 2$ (since the condition from the theorem on cohomology and base change that ϕ_y^{-1} be an isomorphism is vacuously satisfied).

Set $\mathscr{F}_1 := \pi_*\mathscr{I}(2)$ and $\mathscr{N}_1 := \pi^*\mathscr{F}_1$. Let $\alpha_1 : \mathscr{N}_1(-2) \to \mathscr{I}$ denote the evaluation map coming from the adjunction $\pi^*\pi_*\mathscr{I}(2) \otimes \mathscr{O}_{\mathbb{P}}(-2) \to \mathscr{I}(2) \otimes \mathscr{O}_{\mathbb{P}}(-2) \to \mathscr{I}$. As we have shown above, the formation of $\mathscr{F}_1$, and hence $\mathscr{N}_1$, commutes with base change. Further, naturality of the map α_1, coming from the adjunction, also implies $j_y^*(\alpha_1) = \alpha_{1,y}$. Therefore, α_1 is surjective, as its cokernel has empty support.

We next construct sheaves $\mathscr{F}_i$ and $\mathscr{N}_i$ inductively, with $\mathscr{N}_i = \pi^*\mathscr{F}_i$, for $2 \leq i \leq d-3$. Let $\mathscr{A}_1 := \mathscr{I}$. For $i \geq 2$, assume inductively we have constructed the map α_{i-1} and define $\mathscr{A}_i := \ker \alpha_{i-1}$. Analogously to the above verification that $j_y^*\mathscr{I}$ is 3-regular, it follows that $j_y^*\mathscr{A}_i$ is $i+2$ regular. Therefore, by [FGI+05, Lemma 5.1(b)], $H^1(\mathbb{P}_y, j_y^*\mathscr{A}_i(k)) = 0$ for $k \geq (i+2)-1 = i+1$. Analogously to the above case when $i = 1$, it follows from cohomology and base change that $R^1\pi_*\mathscr{A}_i(k) = 0$ for $k \geq i+1$, $\pi_*\mathscr{A}_i(k)$ is locally free for $k \geq i+1$, and the formation of $\pi_*\mathscr{A}_i(k)$ commutes with base change for $k \geq i+1$. Then, set $\mathscr{F}_i := \pi_*\mathscr{A}_i(i+1)$ and $\mathscr{N}_i := \pi^*\mathscr{F}_i$.

We next construct the map $\alpha_i : \mathscr{N}_i \to \mathscr{N}_{i-1}$. Begin with the inclusion $\mathscr{A}_i(i+1) \to \mathscr{N}_{i-1}(1)$ (obtained by twisting the inclusion $\mathscr{A}_i \to \mathscr{N}_{i-1}(-i)$, coming from the definition of

$\mathscr{A}_i$, by $i+1$). Apply $\pi^*\pi_*$ to obtain a map $\pi^*\pi_*\mathscr{A}_i(i+1) \to \pi_*\pi^*\mathscr{N}_{i-1}(1)$. Twist by $-i-1$ which yields the composite map

$$\begin{aligned}\mathscr{N}_i(-i-1) &= (\pi^*\pi_*\mathscr{A}_i(i+1))\,(-i-1)\\ &\to (\pi^*\pi_*\mathscr{N}_{i-1}(1))\,(-i-1)\\ &\simeq (\mathscr{N}_{i-1}\otimes \pi^*\pi_*\mathscr{O}(1))\,(-i-1)\\ &\to \mathscr{N}_{i-1}(-i),\end{aligned} \tag{10.2.5}$$

which we call α_i. Since $\mathscr{N}_i$ commutes with base change, and this map is obtained from adjunction, the formation of α_i also commutes with base change. Also, since pushforward is left exact, we obtain condition (v) in the theorem from the above construction of $\mathscr{F}_i$, provided we show the above construction is the unique such one as in the statement (which will be done later in the proof).

Finally, we similarly construct $\mathscr{F}_{d-2}$, $\mathscr{N}_{d-2}$, and α_{d-2}, assuming we have constructed α_{d-3}. Let $\mathscr{A}_{d-2} := \ker \alpha_{d-3}$. By cohomology and base change, we find $j_y^*\mathscr{A}_{d-2}$ is in fact d-regular (as opposed to only $d-1$ regular, as was the case for $\mathscr{A}_i$ with $i < d-2$). Therefore, by cohomology and base change, we find $R^1\pi_*\mathscr{A}_{d-2}(-d) = 0$ and also that $\pi_*\mathscr{A}_{d-2}(-d)$ is locally free and commutes with base change. We set $\mathscr{F}_{d-2} := \pi_*\mathscr{A}_{d-2}(-d)$ and $\mathscr{N}_{d-2} := \pi^*\mathscr{F}_{d-2}$. Analogously to (10.2.5), there is a canonical map $\alpha_{d-2} : \mathscr{N}_{d-2}(-d) \to \mathscr{N}_{d-3}(-d+2)$ coming from adjunction which commutes with base change. Altogether, we have constructed a complex as in (10.2.1) which commutes with base change on Y and restricts to the minimal free resolution (10.2.4) on each fiber $y \in Y$. It follows from Nakayama's lemma that the complex (10.2.1) is exact, because it is exact when restricted to each fiber over $y \in Y$.

Further, because $\mathscr{N}_i = \pi^*\mathscr{F}_i$, it follows from the projection formula that $\pi_*\mathscr{N}_i \simeq \pi_*(\mathscr{O}_{\mathbb{P}} \otimes \pi^*\mathscr{F}_i) \simeq \pi_*\mathscr{O}_{\mathbb{P}} \otimes \mathscr{F}_i \simeq \mathscr{F}_i$, and so $\pi^*\pi_*\mathscr{N}_i \simeq \mathscr{N}_i$.

We next verify uniqueness of our constructed resolution $\mathscr{N}_\bullet$, up to unique isomorphism. Suppose $\mathscr{M}_\bullet$ is another such minimal free resolution. Over any local scheme $\operatorname{Spec}\mathscr{O}_{y,Y} \subset Y$, there is an isomorphism $\phi_U : \mathscr{N}_\bullet|_{\operatorname{Spec}\mathscr{O}_{y,Y}} \simeq \mathscr{M}_\bullet|_{\operatorname{Spec}\mathscr{O}_{y,Y}}$ by a sheafified version of [Eis95, Theorem 20.2]. Such an isomorphism spreads out to an isomorphism over some affine open $U \subset Y$. Further, this isomorphism is unique up to homotopy by a sheafified version of [Eis95, Lemma 20.3]. We claim there are no nonzero homotopies $s : \mathscr{N}_\bullet|_U \to \mathscr{M}_\bullet|_U$. Indeed, such an homotopy would yield a map $s_i : \mathscr{N}_i|_U \to \mathscr{M}_{i+1}|_U$. We wish to show this map is 0. To check it is 0, it suffices to show it is 0 over each $y \in Y$. Over a point $y \in Y$, this corresponds to a map $\mathscr{O}_{\mathbb{P}_y}(a)^{\oplus b} \to \mathscr{O}_{\mathbb{P}_y}(c)^{\oplus d}$ with $c < a$. It follows that there are no nonzero such maps, so the isomorphism ϕ_U is unique. Hence, by this uniqueness, we obtain via Zariski descent an isomorphism $\phi : \mathscr{N}_\bullet \simeq \mathscr{M}_\bullet$. This isomorphism is unique because it is unique when restricted to each member of an open cover.

This concludes our update to the proof of [CE96, Theorem 2.1] since, as mentioned, the remaining steps C and D given in the proof of [CE96, Theorem 2.1] go through without change. □

The following useful corollary tells us that any two "canonical embeddings" of a Gorenstein cover are related by an automorphism of $\mathbb{P}\mathscr{E}$ coming from $\mathscr{E}$. A special case of this was stated in [CN07, Corollary 2.3], though the proof there seems quite terse.

Corollary 10.2.4. *With notation as in Theorem 10.2.2, suppose we are given $\rho : X \to Y$ and two maps $i_1 : X \to \mathbb{P}\mathscr{E}$ and $i_2 : X \to \mathbb{P}\mathscr{E}$ so that $\rho = \pi \circ i_1 = \pi \circ i_2$ and $\rho^{-1}(y)$ is arithmetically Gorenstein and nondegenerate under both embeddings i_1 and i_2. Assume further that both maps above are induced by surjections from $\rho^*\mathscr{E}$ to an invertible sheaf on X. Then, the unique isomorphism $\psi : \mathbb{P}\mathscr{E} \to \mathbb{P}\mathscr{E}$ taking $i_1(X)$ to $i_2(X)$ is induced by an automorphism of $\mathscr{E}$.*

Proof. For both maps i_1 and i_2, we know the restriction of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ (the line bundle receiving a surjection from $\rho^*\mathscr{E}$ in the statement of this corollary) to X is isomorphic to $\omega_{X/Y}$ by Theorem 10.2.2(ii). Hence, we obtain that the automorphism is induced by some automorphism ϕ of $\pi_*\omega_{X/Y}$, determined up to unit. The maps i_1 and i_2 induce two surjections $q_1, q_2 : \pi_*\omega_{X/Y} \to \mathscr{O}_Y$ with the maps i_1 and i_2 coming via the linear subsystems $\ker(q_1)$ and $\ker(q_2)$. To show we have an induced map between $\ker(q_1)$ and $\ker(q_2)$, which are both abstractly isomorphic to $\mathscr{E}$, it is enough to show that, up to unit, $q_1 = q_2 \circ \phi$. We may verify this locally, and hence assume Y is the spectrum of a local ring. Using Theorem 10.2.2(vi), in both of the maps i_1 and i_2, there is no subscheme of degree $d - 1$ on the closed fiber contained in a hyperplane, and hence the same holds over the whole local scheme Y. We may rephrase this as the condition that the two relative hyperplane sections of $\mathbb{P}\mathscr{E}$ associated to q_1 and q_2 do not meet $i_1(X)$ and $i_2(X)$. Equivalently the two hyperplane sections associated to q_1 and q_2 are nowhere vanishing on X, and therefore related by a unit. By modifying ϕ by this unit, we may assume $q_1 = q_2 \circ \phi$.

Under the above identifications, the image of $\mathscr{E} \to \pi_*\omega_{X/Y}$ is identified with the kernel of the natural map $\pi_*\omega_{X/Y} \to \mathscr{O}_X$ dual to $\rho^\#$. Since this map is also fixed by the resulting automorphism, the automorphism of $\pi_*\omega_{X/Y}$ restricts to an automorphism of $\mathscr{E}$ which induces the resulting automorphism of $\mathbb{P}\mathscr{E}$. □

10.2.5 Low degree structure theorems

We next recall the structure theorems for covers of degrees 3, 4, and 5 from [CE96] and [Cas96]. In these papers, these structure theorems are stated for degree d finite flat surjective maps $X \to Y$ with Y integral and noetherian, and we now develop the generalizations we will need, as described in Remark 10.2.6.

Remark 10.2.6. Surprisingly, the structure theorem in degree 5, Theorem 10.2.15, (and to a lesser extent Theorem 10.2.14 and Theorem 10.2.13) appears to be new. A similar result in degree 5 is proven in [Cas96, Theorem 3.8] but with certain additional restrictions on the covers and sections that Casnati refers to as being "regular." This regularity condition amounts to the assumption that the map $\wedge^2 \mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{E}$ associated to a section $\eta \in \mathscr{H}(\mathscr{E}, \mathscr{F})$ is surjective. Additionally, [Cas96, Theorem 3.8] does not claim there is a bijection between covers and sections up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$, but only gives constructions of maps in both directions. We note similarly that the statements for degrees 3 and 4 appearing in [CE96, Theorem 3.4 and 4.4] also only give the constructions of the maps, but to not claim the bijective statements. Further, in [CE96] and [Cas96] the structure theorems for degrees 3, 4, and 5 are only stated for degree d finite flat surjective maps $X \to Y$ with Y integral and noetherian, whereas ours hold for arbitrary schemes Y.

Although there are other ways of approaching the upcoming proofs, in order to prove the theorems in degrees 4 and 5, it will be convenient to appeal to smoothness of the algebraic stack of degree d Gorenstein covers. To this end, let Covers_d denote the fibered category over $\operatorname{Spec} \mathbb{Z}$ whose S points are finite locally free covers $X \to S$ of degree d with Gorenstein fibers. Morphisms in this fibered category are morphisms of covers.

Lemma 10.2.7. *For $d \leq 5$, Covers_d is a smooth algebraic stack over $\operatorname{Spec} \mathbb{Z}$.*

Proof. The cases $d \leq 3$ follow from [Poo08, Proposition 8.4], so it only remains to deal with the cases $d = 4$ and $d = 5$. The basic input is [CN07, Remark 5.5], which shows that $\mathrm{Hilb}_d^{aG,0}$, the open subset of the Hilbert scheme of degree d subschemes of $\mathbb{P}^{d-2}$ parameterizing arithmetically Gorenstein nondegenerate subschemes, has smooth geometrically irreducible fibers over $\operatorname{Spec} \mathbb{Z}$ when $d \leq 5$. (We note that [CN07] makes a standing assumption that the characteristic is not 2 or 3, but it is not used in [CN07, Remark 5.5] or the results leading to it. However, there is a minor error in the statement of [CN07, Remark 3.5] as when $d = 4$, the dimension of the tangent space in that statement should be 8, which is different from the claimed value of 6.)

We next check $\mathrm{Hilb}_d^{aG,0}$ is in fact smooth over $\operatorname{Spec} \mathbb{Z}$, and not just over each residue field. For this it suffices to check it is flat. Because maps sending all associated points to the generic point of $\operatorname{Spec} \mathbb{Z}$ are flat, it is enough to show $\mathrm{Hilb}_d^{aG,0}$ is integral. Irreducibility follows because there is a dense open parameterizing étale degree d subschemes, which is dense in every fiber, as the fibers are geometrically irreducible. Granting this irreducibility, we then find that every point is regular, using the slicing criterion for regularity, and hence $\mathrm{Hilb}_d^{aG,0}$ is reduced.

Finally, we deduce smoothness and algebraicity of Covers_d over $\mathrm{Spec}\,\mathbb{Z}$ from smoothness of $\mathrm{Hilb}_d^{aG,0}$. By Theorem 10.2.2, the natural map $\mathrm{Hilb}_d^{aG,0} \to \mathrm{Covers}_d$ is a PGL_{d-1} torsor, and so smoothness of the former implies smoothness of the later. □

To introduce notation simultaneously in the cases of degrees $3, 4$, and 5, we use the following notation.

Notation 10.2.8. Let $d \in \{3, 4, 5\}$. Let Y be a scheme. Fix a locally free sheaf $\mathscr{E}$ on Y of rank $d - 1$. If $d = 4$, let $\mathscr{F}$ be a locally free sheaf on Y of rank 2 and if $d = 5$, let $\mathscr{F}$ be a locally free sheaf on Y of rank 5. We use the tuple $(\mathscr{E}, \mathscr{F}_\bullet)$ to denote the pair $(\mathscr{E}, \mathscr{F})$ when $d = 4$ or $d = 5$ and to denote $\mathscr{E}$ when $d = 3$. Define the associated sheaf

$$\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet) := \begin{cases} \mathrm{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee & \text{if } d = 3 \\ \mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E} & \text{if } d = 4 \\ \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee & \text{if } d = 5. \end{cases} \tag{10.2.6}$$

We will often use $\mathscr{H}$ to denote $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ when the data $(\mathscr{E}, \mathscr{F}_\bullet)$ is clear from context. We will see that sections of the above sheaf $\mathscr{H}$ defines subschemes of $\mathbb{P}\mathscr{E}$. When these subschemes have dimension 0 in fibers, we will see they induce degree d locally free covers. The structure theorems for degrees $3, 4$, and 5 essentially say that the resulting covers are in bijection with such sections, up to automorphisms of $(\mathscr{E}, \mathscr{F}_\bullet)$.

10.2.9 The resolutions in low degree

In order to state the structure theorems in degrees $3, 4$, and 5, we now want a way of associating a subscheme of $\mathbb{P}\mathscr{E}$ to a section. We will give a description of this association separately in the cases that $d = 3, 4$, and 5.

In the cases $d = 3, 4$, and 5 (10.2.1) becomes respectively

$$0 \longrightarrow \pi^* \det \mathscr{E}(-3) \xrightarrow{\sigma} \mathscr{O}_{\mathbb{P}} \longrightarrow \mathscr{O}_X \longrightarrow 0, \tag{10.2.7}$$

$$0 \longrightarrow \pi^* \det \mathscr{E}(-4) \xrightarrow{\sigma} \pi^* \mathscr{F}(-2) \longrightarrow \mathscr{O}_{\mathbb{P}} \longrightarrow \mathscr{O}_X \longrightarrow 0, \tag{10.2.8}$$

$$\begin{aligned} 0 \longrightarrow \pi^* \det \mathscr{E}(-5) \longrightarrow \pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3) \xrightarrow{\sigma} \\ \xrightarrow{\sigma} \pi^* \mathscr{F}(-2) \longrightarrow \mathscr{O}_{\mathbb{P}} \longrightarrow \mathscr{O}_X \longrightarrow 0, \end{aligned} \tag{10.2.9}$$

with the rank of the locally free sheaves $\mathscr{E}$ and $\mathscr{F}$ in the degree 3, 4, and 5 cases given in Notation 10.2.8.

10.2.10 The maps Φ_d in low degree

In the above 3 cases, corresponding to degrees 3, 4, and 5 respectively, we have isomorphisms

$$\Phi_3 : H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathscr{E}, \pi^* \det \mathscr{E}^\vee(3)).$$

$$\Phi_4 : H^0(Y, \operatorname{Sym}^2 \mathscr{E} \otimes \mathscr{F}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathscr{E}, \pi^* \mathscr{F}^\vee(2))$$

$$\Phi_5 : H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathscr{E}, \wedge^2 \pi^* \mathscr{F} \otimes \pi^* \det \mathscr{E}^\vee(1)).$$

10.2.11 The maps Ψ_d in low degree

Next, for $d \leq 3 \leq 5$, given a section $\eta \in H^0(Y, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))$, we define an associated scheme $\Psi_d(\eta)$ over Y.

When $d = 3$, we begin with a section $\eta \in H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$, which, via Φ_3 can be viewed as an element of $H^0(\mathbb{P}\mathscr{E}, \pi^* \det \mathscr{E}^\vee(3))$. Such a section corresponds to a map $\mathscr{O}_{\mathbb{P}\mathscr{E}} \to \pi^* \det \mathscr{E}^\vee(3)$, or equivalently a map $\pi^* \det \mathscr{E}(-3) \to \mathscr{O}_{\mathbb{P}\mathscr{E}}$. We let $\Psi_3(\eta)$ denote the support of the cokernel of this map. That is, we define $\Psi_3(\eta) \subset \mathbb{P}\mathscr{E}$ so that on $\mathbb{P}\mathscr{E}$ we have an exact sequence

$$\pi^* \det \mathscr{E}(-3) \longrightarrow \mathscr{O}_{\mathbb{P}\mathscr{E}} \longrightarrow \mathscr{O}_{\Psi_3(\eta)} \longrightarrow 0. \tag{10.2.10}$$

When $d = 4$, given $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})$, define $\Phi_4(\eta)$ to be the subscheme of $\mathbb{P}\mathscr{E}$, considered as the support of the cokernel of the map $\pi^* \mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$ corresponding to $\Phi_4(\eta)$.

Finally, when $d = 5$, given $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \pi^* \det \mathscr{E}^\vee \otimes \mathscr{E})$, from $\Phi_5(\eta)$ we obtain a corresponding alternating map $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3) \to \pi^* \mathscr{F}(-2)$. The five 4×4 Pfaffians of this map determine a map of sheaves $\pi^*\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$, as may be computed locally. Define $\Psi_5(\eta)$ as the support of the cokernel of the map $\pi^*\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$ in $\mathbb{P}\mathscr{E}$.

Definition 10.2.12. Let $d \in \{3,4,5\}$, Y be a scheme, and $(\mathscr{E},\mathscr{F}_\bullet), \mathscr{H}(\mathscr{E},\mathscr{F}_\bullet)$ be sheaves on Y as in Notation 10.2.8. We say $\eta \in H^0(Y, \mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))$ has the *right codimension* at a point $y \in Y$ if the fiber of $\Psi_d(\eta)$ over y has dimension 0. We say η has the *right codimension* if it has the right codimension at every $y \in Y$.

Finally, we are ready to state the low degree structure theorems. The structure theorem in degree 3 is as follows.

Theorem 10.2.13 (Generalization of [CE96, Theorem 3.4])**.** *Fix a scheme Y and a rank 2 locally free sheaf $\mathscr{E}$ on Y. There is a bijection between finite locally free Gorenstein covers $\rho : X \to Y$ of degree 3 such that $\mathscr{E}^\vee \simeq \operatorname{coker}\rho^\#$ and, up to automorphisms of $\mathscr{E}$, sections $\eta \in H^0(Y, \operatorname{Sym}^3\mathscr{E} \otimes \det\mathscr{E}^\vee)$ having the right codimension at every $y \in Y$. The bijection explicitly sends a section $\eta \in H^0(Y, \operatorname{Sym}^3\mathscr{E} \otimes \det\mathscr{E}^\vee)$ to $\Psi_3(\eta) \subset \mathbb{P}\mathscr{E}$.*

The following proof extends that given in [CE96, Theorem 3.4]. We note that there the base is assumed to be reduced and noetherian, and the bijection is not explicitly stated. We outline the proof for the reader's convenience.

Proof. Given such a $\rho : X \to Y$, we obtain from Theorem 10.2.2, a resolution of $\mathscr{O}_{\mathbb{P}\mathscr{E}}$ as in (10.2.7), unique up to unique isomorphism. The map σ in (10.2.7) defines a section $\eta := \Phi_3^{-1}(\sigma) \in H^0(Y, \operatorname{Sym}^3\mathscr{E} \otimes \det\mathscr{E}^\vee)$. Note that the resulting η has the right codimension at every $y \in Y$ because $X \to Y$ is finite by assumption.

Conversely, given η of the right codimension at every $y \in Y$, the resulting sequence (10.2.10) is then left exact as the kernel of $\Phi_3^{-1}(\eta)$ has vanishing support. This presentation shows X is locally finitely presented over Y. Further, X is finite as it is locally of finite presentation, proper, and quasi-finite [Gro66, 8.11.1]. Flatness of $X \to Y$ may be verified locally, in which case it holds as X is cut out of $\mathbb{P}^1_Y$ by a single equation of degree 3 not vanishing on any fibers. Therefore, X is a finite locally free degree 3 cover of Y. Therefore, exactness of (10.2.10) implies $\mathscr{E}^\vee \simeq \operatorname{coker}\rho^\#$ from Theorem 10.2.2(iii) and (iv).

It remains to see that these two maps we have defined establish a bijection. For this, we show the compositions of these maps in both orders are equivalent to the identity map. If we begin with a cover $\rho : X \to Y$, (10.2.7) defines a resolution of $X \to \mathbb{P}\mathscr{E}$ giving X as the vanishing locus $\Psi_3(\eta) \subset \mathbb{P}\mathscr{E}$. To show the other composition is equivalent to the identity, begin with some $\eta \in H^0(Y, \operatorname{Sym}^3\mathscr{E} \otimes \det\mathscr{E}^\vee)$, and let X denote the associated cover $\Psi_3(\eta)$. The bundle $\mathscr{E}^X$ associated to X from Theorem 10.2.2 is then isomorphic to $\mathscr{E}$ using Theorem 10.2.2(iv), as we may view η as a map $\pi^*\det\mathscr{E}(-3) \to \mathscr{O}_{\mathbb{P}\mathscr{E}}$. Upon choosing such an isomorphism $\mathscr{E} \simeq \mathscr{E}^X$, we obtain a section $\eta^X \in H^0(Y, \mathrm{Sym}^3 \mathscr{E}^X \otimes \det(\mathscr{E}^X)^\vee) \simeq H^0(Y, \mathrm{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$. Using Theorem 10.2.2(iv), there is an automorphism of $\mathbb{P}\mathscr{E}$ taking $\Psi_3(\eta)$ to $\Psi_3(\eta^X)$. From Theorem 10.2.2(iv) and the fact that the leftmost term of the resolution (10.2.7) is $\pi^* \det \mathscr{E}(-3)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_* \omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 10.2.4, this automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. Hence, after composing with the automorphism of $\mathscr{E}$, we can assume η and η^X define isomorphic subschemes of $\mathbb{P}\mathscr{E}$, and so differ by a scalar. By composing with an automorphism of $\mathscr{E}$ multiplying by the inverse of this scalar, η and η^X are identified. □

We next verify the structure theorem in degree 4.

Theorem 10.2.14 (Generalization of [CE96, Theorem 4.4])**.** *Fix a scheme Y, a rank 3 locally free sheaf $\mathscr{E}$ on Y, and a rank 2 locally free sheaf $\mathscr{F}$ on Y. There is a bijection between*

1. *finite locally free Gorenstein maps $\rho : X \to Y$ of degree 4 whose associated resolution (10.2.8) has sheaves $\mathscr{E}^X, \mathscr{F}^X$ which are isomorphic to $\mathscr{E}$ and $\mathscr{F}$,*
2. *and, up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$, sections $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E})$ having the right codimension at every $y \in Y$.*

If $\mathscr{E}^X$ and $\mathscr{F}^X$ are sheaves obtained from $\rho : X \to Y$ as above, there is an isomorphism $\det \mathscr{F}^X \simeq \det \mathscr{E}^X$. The bijection explicitly sends a section $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E})$ to $\Psi_4(\eta)$, considered as a subscheme of $\mathbb{P}\mathscr{E}$.

Proof. Beginning with a cover $X \to Y$, we obtain a resolution (10.2.8), and, upon choosing isomorphisms $\mathscr{E}^X \simeq \mathscr{E}$ and $\mathscr{F}^X \simeq \mathscr{F}$, we obtain a section $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E})$ having the right codimension at every $y \in Y$.

To construct the map in the other direction, we wish to verify $V(\Phi_4(\eta))$ is a finitely presented Gorenstein cover of Y. On fibers, $V(\Phi_4(\eta))$ is described as a dimension 0 intersection of two quadrics. Since η has the right codimension at $y \in Y$, it has degree 4 over y. Then, $V(\Phi_4(\eta))$ is Gorenstein because it is a local complete intersection. To deduce flatness, we may first reduce to the case Y is smooth. Indeed, we may work fppf locally on Y, in which case $X \to Y$ is pulled back from a cover $X' \to Y'$ for Y' smooth using Lemma 10.2.7. In this case, since Y is reduced, flatness follows from constancy of the degree (i.e., Hilbert polynomial under the canonical embedding of Theorem 10.2.2) of X over Y.

We next verify that $\det \mathscr{E}^X \simeq \det \mathscr{F}^X$. (In addition to the argument we give next, this can also be deduced using the Koszul complex as in [CE96, Theorem 4.4].) Indeed, using Lemma 10.2.7, any such cover $X \to Y$ is pulled back from the smooth stack Covers_d. It is enough to demonstrate the isomorphism in this universal setting, where $X \to \mathrm{Covers}_d$

is the universal degree d cover. The resulting projective bundle $\mathbb{P}\mathscr{E}^X$ is also smooth over $\operatorname{Spec}\mathbb{Z}$, and therefore the determinant of the codimension 2 substack $\mathscr{O}_X \subset \mathbb{P}\mathscr{E}^X$ vanishes. This follows from normality of Covers_d, which implies that the trivialization of $\det(\mathscr{O}_X)$ away from X extends to the codimension 2 substack $X \subset \mathbb{P}\mathscr{E}^X$. Given this, we find that the $\det(\pi^*\mathscr{F}^X(-2)) \simeq (\det \pi^*\mathscr{E}^X)(-4)$. Since the former has rank 2 and the latter has rank 1, it follows that $\det \pi^*\mathscr{F}^X \simeq \det \pi^*\mathscr{E}^X$ and so $\det \mathscr{F}^X \simeq \det \mathscr{E}^X$.

It remains to prove the compositions of the above maps in both directions are equivalent to the identity. As in the degree 3 case, if we start with a cover, produce the associated section η^X, $\Psi_4(\eta^X)$ is isomorphic to X via the construction. For showing the reverse composition is equivalent to the identity, start with some section η. Let X denote the resulting cover $\Psi_4(\eta)$. Choose identifications $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}^X \simeq \mathscr{F}$ so that we obtain an associated section $\eta^X \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E}) \simeq H^0(Y, (\mathscr{F}^X)^\vee \otimes \operatorname{Sym}^2 \mathscr{E}^X)$. We wish to show η^X is related to η by automorphisms of $\mathscr{E}$ and $\mathscr{F}$. Note also here that any automorphism of $\mathscr{E}$ and $\mathscr{F}$ send η to another section defining an isomorphic cover. Using Theorem 10.2.2, since there is an automorphism of $\mathbb{P}\mathscr{E}$ taking the subscheme $\Psi_4(\eta^X)$ to $\Psi_4(\eta)$. From Theorem 10.2.2(iv) and the fact that the leftmost term of the resolution (10.2.8) is $\pi^* \det \mathscr{E}(-4)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_*\omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 10.2.4, the above automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. By composing with the inverse of this automorphism, we may assume the resulting map is the identity on $\mathbb{P}\mathscr{E}$, and so the automorphism of $\mathbb{P}\mathscr{E}$ is then induced by some scalar automorphism of $\mathscr{E}$. After adjusting this scalar, we may assume it is the identity. Since $\mathscr{F}$ is subsheaf of $\operatorname{Sym}^2 \mathscr{E}$ by Theorem 10.2.2(v), the image of the induced map $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$ is uniquely determined by X, but the precise map is only determined up to automorphism of $\mathscr{F}$. Upon composing with such an automorphism, we may identify not just the images of $\mathscr{F}$ in $\operatorname{Sym}^2 \mathscr{E}$, but further we may identify the maps. Under these identifications, η agrees with η^X, when viewed as maps $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$. □

We next state and prove the analogous structure theorem in degree 5.

Theorem 10.2.15 (Generalization of [Cas96, Theorem 3.8]). *Fix a scheme Y, a rank 4 locally free sheaf $\mathscr{E}$ on Y, and a rank 5 locally free sheaf $\mathscr{F}$ on Y. There is a bijection between*

1. *finite locally free Gorenstein maps $\rho : X \to Y$ of degree 5 whose associated resolution as in (10.2.9) has sheaves $\mathscr{E}^X, \mathscr{F}^X$ which are isomorphic to $\mathscr{E}$ and $\mathscr{F}$, and*

2. *up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$, sections $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$ having the right codimension at every $y \in Y$.*

If $\mathscr{E}^X$ and $\mathscr{F}^X$ are sheaves obtained from $\rho : X \to Y$ as above, there is an isomorphism $\det \mathscr{F}^X \simeq (\det \mathscr{E}^X)^{\otimes 2}$*. The bijection explicitly sends a section $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$ to $\Psi_5(\eta)$, considered as a subscheme of $\mathbb{P}\mathscr{E}$.*

Proof. Beginning with a cover $X \to Y$, we obtain a resolution (10.2.9). Upon choosing isomorphisms $\mathscr{E}^X \simeq \mathscr{E}$ and $\mathscr{F}^X \simeq \mathscr{F}$ we obtain a section $\eta \in H^0(Y, \mathscr{F}^{\otimes 2} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$ having the right codimension at every $y \in Y$. We wish to check next that this section actually lies in $H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$. Viewing this as a map $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^* \mathscr{F}(1)$ via (10.2.9), it is enough to verify the map is alternating locally on the base. Therefore, for this verification, we may assume Y is the spectrum of a local ring and $\mathscr{E}$ is trivial. After this reduction, $X \subset \mathbb{P}\mathscr{E}$ is codimension 3 and arithmetically Gorenstein, and so the Buchsbaum-Eisenbud structure theorem for codimension 3 Gorenstein schemes [BE77, Theorem 2.1(2)] applies. The proof of [BE77, Theorem 2.1(2)] moreover produces a minimal free resolution of $\mathscr{O}_{\mathbb{P}\mathscr{E}} \to \mathscr{O}_X$ which by Theorem 10.2.2 must agree with (10.2.9). Since the map corresponding to $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^* \mathscr{F}(1)$ is alternating in the resolution of [BE77, Theorem 2.1(2)] it follows that $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^* \mathscr{F}(1)$ is also alternating.

To construct the map in the other direction, we wish to verify $V(\Phi_5(\eta))$ is a finitely presented Gorenstein cover of Y. The finite presentation condition follows from the resolution given in (10.2.9). We may check the remaining conditions locally on Y, and hence assume Y is the spectrum of a local ring. Observe that $X \to \mathbb{P}\mathscr{E}$ is arithmetically Gorenstein and of codimension 3, using the assumption that η has the right codimension at each $y \in Y$. Using [BE77, Theorem 2.1(1)], we find that X is Gorenstein and is cut out scheme theoretically by the five 4×4 Pfaffians associated to η, thought of as a map $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^* \mathscr{F}(1)$. On fibers, $V(\Phi_5(\eta))$ is described as the vanishing of the five 4×4 Pfaffians of an alternating linear map. The resolution [BE77, Theorem 2.1(1)], can be identified with on of the form (10.2.9), from which one may calculate that the Hilbert polynomial of every fiber is 5. Therefore, the resulting scheme $V(\Phi_5(\eta))$ is finite and each fiber has degree 5.

To deduce flatness of $\rho : X \to Y$, we may first reduce to the case Y is smooth. Indeed, we may work fppf locally on Y, in which case $X \to Y$ is pulled back from a cover $X' \to Y'$ for Y' smooth using Lemma 10.2.7. In this case, since Y is reduced, flatness follows from constancy of the degree (i.e., Hilbert polynomial under the canonical embedding of Theorem 10.2.2) of X over Y.

We next verify that in this setting $(\det \mathscr{E}^X)^{\otimes 2} \simeq \det \mathscr{F}^X$. Indeed, using Lemma 10.2.7, any such cover $X \to Y$ is pulled back from the smooth stack Covers_d. It is enough to demonstrate the isomorphism in this universal setting, where $X \to \mathrm{Covers}_d$ is the universal degree d cover. It follows from [Cas96, Proposition 3.5] that $(\det \mathscr{E}^X)^{\otimes 2} \simeq \det \mathscr{F}^X$ over a certain dense open substack $\mathscr{U} \subset \mathrm{Covers}_d$ (corresponding to those η inducing a surjective map $\wedge^2 \mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{E}$, a property which Casnati calls "regular") whose complement has codimension 2. This codimension 2 property can be deduced from the fact that the locus where the matrix appearing in [Cas96, (2.2)] has rank ≤ 3 has codimension at least 2. Since Covers_d is smooth, hence normal, we obtain that the isomorphism $(\det \mathscr{E}^X)^{\otimes 2} \simeq \det \mathscr{F}^X$ extends over the complement of $\mathscr{U}$ to an isomorphism over all of Covers_d.

It remains to prove the compositions of the above maps are equivalent to the identity. As in the degree 3 case, if we start with a cover, produce the associated section η^X, $\Psi_5(\eta^X)$ is isomorphic to X via the construction.

For the reverse composition, start with some section η and let X denote the resulting cover $\Psi_5(\eta)$. Choose identifications $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}^X \simeq \mathscr{F}$ so that we obtain an associated section $\eta^X \in H^0(Y, \wedge^2 \mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{E}) \simeq H^0(Y, \wedge^2 (\mathscr{F}^X)^\vee \otimes \det \mathscr{E}^X \to \mathscr{E}^X)$. We wish to show η^X is related to η by automorphisms of $\mathscr{E}$ and $\mathscr{F}$. Note also here that any automorphism of $\mathscr{E}$ and $\mathscr{F}$ send η to another section defining an isomorphic cover. Using Theorem 10.2.2, since there is an automorphism of $\mathbb{P}\mathscr{E}$ taking $\Psi_5(\eta^X)$ to $\Psi_5(\eta)$. From Theorem 10.2.2(iv) and the fact that the leftmost term of the resolution (10.2.8) is $\pi^* \det \mathscr{E}(-5)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_* \omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 10.2.4, this automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. By composing with the inverse of this automorphism, we may η and η^X define the same subscheme of $\mathbb{P}\mathscr{E}$. Hence we may assume the automorphism of $\mathbb{P}\mathscr{E}$ is then induced by some scalar automorphism of $\mathscr{E}$. After adjusting this scalar, we may assume it is the identity. By Theorem 10.2.2(iii), we obtain a unique isomorphism between the two resolutions of X in $\mathbb{P}\mathscr{E}$ (10.2.9) determined by η and η^X. This isomorphism can be specified as a tuple of 5 maps between the nonzero terms of (10.2.9).

We next show we can apply an automorphism of $\mathscr{F}$ so as to assume the map $\pi^* \mathscr{F}(-2) \to \pi^* \mathscr{F}(-2)$ is the identity. By the above identifying the images $X \subset \mathbb{P}\mathscr{E}$ Since $\mathscr{F}$ is subsheaf of $\mathrm{Sym}^2 \mathscr{E}$ by Theorem 10.2.2(v), the image of the induced map $\mathscr{F} \to \mathrm{Sym}^2 \mathscr{E}$ coming from the Pfaffians associated to η is uniquely determined by X, but the precise map is only determined up to automorphism of $\mathscr{F}$. Upon composing with such an automorphism, we may identify not just the images of $\mathscr{F}$ in $\mathrm{Sym}^2 \mathscr{E}$, but further we may identify the maps. Under these identifications, η agrees with η^X, when viewed as maps $\mathscr{F} \to \mathrm{Sym}^2 \mathscr{E}$.

So far, we have constructed a map of the two resolutions (10.2.9) associated to η and η_X. Upon choosing identifications $\mathscr{E} \simeq \mathscr{E}^X$ and $\mathscr{F} \simeq \mathscr{F}_X$ as above, we have enforced that the map of resolutions is given by the identity on the terms $\mathscr{O}_X \to \mathscr{O}_X, \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_{\mathbb{P}}$, and $\pi^* \mathscr{F}(-2) \to \pi^* \mathscr{F}(-2)$. When we write the second nonzero term of (10.2.9) as $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$, we have identified it via Grothendieck duality as pairing with the third nonzero term $\pi^* \mathscr{F}(-3)$ into $\pi^* \mathscr{E}(-5)$, and therefore the induced automorphism $\pi^* \mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$ must respect this duality. In particular, since we have reduced to the case where the automorphism of $\pi^*\mathscr{F}(-2)$ is the identity, we also obtain the induced automorphism of $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$ is the identity. Using Theorem 10.2.2(v) to guarantee that the maps η and η^X from $\mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{F} \otimes \mathscr{E}$ are injective, we obtain the desired identification of η with η^X. □

Finally, we recall a rather elementary criterion for when $\Psi_d(\eta)$ is geometrically connected.

Theorem 10.2.16 (Part of [CE96, Theorem 3.6], [CE96, Theorem 4.5], [Cas96, Theorem 4.4])**.** *Keeping notation as in Notation 10.2.8, assume that Y is a geometrically connected and geometrically reduced projective scheme over a field k. If $h^0(Y, \mathscr{E}^\vee) = 0$, then $\Psi_d(\eta)$ is geometrically connected.*

Proof. The proof is essentially given in [CE96, Theorem 3.6], and we repeat it for the reader's convenience. Let $X := \Psi_d(\eta)$. If $h^0(Y, \mathscr{E}^\vee) = 0$ the exact sequence

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow \rho_*\mathscr{O}_X \longrightarrow \mathscr{E}^\vee \longrightarrow 0 \qquad (10.2.11)$$

induces an isomorphism $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \rho_*\mathscr{O}_X) = H^0(X, \mathscr{O}_X)$. Since Y is geometrically connected and geometrically reduced, we have $h^0(Y, \mathscr{O}_Y) = 1$. From this we find $H^0(X, \mathscr{O}_X) = 1$ as well, and therefore X is necessarily geometrically connected. □

10.3 A presentation of the stack of degree d Gorenstein covers

In this section, we give a description of the stack of degree d Gorenstein covers as a global quotient stack for $3 \le d \le 5$. We now introduce the groups we will be quotienting by.

Definition 10.3.1. Given a scheme Y over a base B and an integer d, let *resolution data* for Y and d denote a tuple of locally free sheaves $(\mathscr{E}, \mathscr{F}_\bullet)$ on Y, where $\mathscr{E}$ is a locally free sheaf of rank $d-1$ and $\mathscr{F}_\bullet$ denotes the sequence $\mathscr{F}_1, \ldots, \mathscr{F}_{\lfloor \frac{d-2}{2} \rfloor}$ where $\operatorname{rk} \mathscr{F}_i = \beta_i$ as in (10.2.2). (We will typically take $B = \operatorname{Spec} k$ for k a field, except in Proposition 10.3.4 where we take $B = \operatorname{Spec} \mathbb{Z}$.)

Let $3 \le d \le 5$, fix a scheme Y over a field, and fix resolution data $(\mathscr{E}, \mathscr{F}_\bullet)$ for a degree d cover of Y. For $\mathscr{G}$ a locally free sheaf on Y, let $\Delta_{\mathscr{G}} := \mathbb{G}_m \to \operatorname{Aut}_{\mathscr{G}/Y}$ denote the map adjoint to the central inclusion $(\mathbb{G}_m \times_B Y) \to \operatorname{Aut}_{\mathscr{G}/Y}$ on Y.

Then, define the *automorphism sheaf* of this resolution data to be the B-scheme

$$\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet} := \begin{cases} \mathrm{Aut}_{\mathscr{E}/Y} & \text{if } d = 3 \\ \mathrm{coker}(\Delta_{\mathscr{E}}, (\Delta_{\mathscr{F}_1})^2) & \text{if } d = 4 \\ \mathrm{coker}((\Delta_{\mathscr{E}})^2, (\Delta_{\mathscr{F}_1})^3) & \text{if } d = 5. \end{cases} \tag{10.3.1}$$

When $d = 4$ or 5, we will often denote $\mathscr{F}_1$ by $\mathscr{F}$.

Remark 10.3.2. Concretely, $\mathscr{E}$ and $\mathscr{F}_\bullet$ in Definition 10.3.1 are (sequences of) sheaves of the following ranks. For $d = 3$, $\mathscr{E}$ is free of rank 2 and $\mathscr{F}_\bullet$ is trivial (i.e., the sequence of sheaves has length 0). When $d = 4$, $\mathscr{E}$ is free of rank 3 and $\mathscr{F}_\bullet = \mathscr{F}$ is free of rank 2. When $d = 5$, $\mathscr{E}$ is free of rank 4 and $\mathscr{F}_\bullet = \mathscr{F}$ is free of rank 5.

We next describe a presentation of the stack parameterizing degree d Gorenstein covers for $3 \leq d \leq 5$. With notation as in Definition 10.3.1, we work over $B = \operatorname{Spec} \mathbb{Z}$. As introduced in the beginning of § 10.2.5, we use Covers_d to denote the fibered category whose S points are finite locally free covers $X \to S$ of degree d with Gorenstein fibers.

Now, for each $3 \leq d \leq 5$, fix free sheaves on $\operatorname{Spec} \mathbb{Z}$, $\mathscr{E}$ and $\mathscr{F}_\bullet$ as in Definition 10.3.1 and Remark 10.3.2. Let $\mathrm{U}_d \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\operatorname{Spec} \mathbb{Z}, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))$ denote the open subscheme functorially parameterizing those sections η so that $\Psi_d(\eta)$ defines a degree d locally free Gorenstein cover, for Ψ_d the maps (depending on $3 \leq d \leq 5$) defined in § 10.2.5.

Definition 10.3.3. For $d \leq 3 \leq 5$, the map Ψ_d induces a map $\mu_d : \mathrm{U}_d \to \mathrm{Covers}_d$. There is a natural action of $\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$ on U_d, induced by the action of $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ on U_d. The map μ_d is invariant under this action, since the resulting abstract degree d cover is unchanged by such re-coordinatizations, we obtain an induced map from the quotient stack $\phi_d : [\mathrm{U}_d/\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}] \to \mathrm{Covers}_d$.

Proposition 10.3.4. *For $3 \leq d \leq 5$, the map ϕ_d defined in Definition 10.3.3 is an isomorphism.*

Proof. We will construct an inverse map using Theorem 10.2.2. Using Theorem 10.2.2, there is an $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ torsor torsor T_d over Covers_d whose S-points parameterize covers $X \to S$ together with specified trivializations $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}_\bullet^X \simeq \mathscr{F}_\bullet$ of the sheaves $\mathscr{E}^X$ and $\mathscr{F}_\bullet^X$ associated to X coming from Theorem 10.2.2. Note here that T_d maps surjectively to Covers_d because for any S point, there is an open cover of S on which these vector bundles become isomorphic to trivial bundles. The structure theorems Theorem 10.2.13, Theorem 10.2.14, and Theorem 10.2.15 then give a section $\eta \in \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$. This induces a map $\mathrm{T}_d \to \mathrm{U}_d$.

We wish to show this induced map is an isomorphism in degree 3 and a $\mathbb{G}_m$ torsor in degrees 4 and 5, where $\mathbb{G}_m$ is the copy of $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}}$ as in Definition 10.3.1 whose quotient yields $\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$. Once we verify this, the structure theorems Theorem 10.2.13 Theorem 10.2.14 Theorem 10.2.15 imply that the composition of this map with ϕ_d is the structure map for the torsor $\mathrm{T}_d \to \mathrm{Covers}_d$. Therefore, it will follow that the resulting map $[\mathrm{U}_d/\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}] \to [\mathrm{T}_d/\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}] \simeq \mathrm{Covers}_d$ is an isomorphism.

First, we verify the map $\mathrm{T}_d \to \mathrm{U}_d$ is invariant under the above mentioned $\mathbb{G}_m$ action in the cases that $d = 4$ and 5. In the degree 4 case, scaling $\mathscr{E}$ by λ and $\mathscr{F}$ by λ^2 scales $\mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E}$ by $\lambda^{-2} \cdot \lambda^2 = 1$. In the degree 5 case, scaling $\mathscr{E}$ by λ^2 and $\mathscr{F}$ by λ^3 scales $\wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee$ by $\lambda^6 \cdot \lambda^2 \cdot \lambda^{2 \cdot -4} = 1$.

Therefore, to conclude the verification, it is enough to show the only elements of $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ fixing a given section are trivial when $d = 3$ and lie in $\mathbb{G}_m$ when $d = 4$ or 5. To start, the map $X \to \mathbb{P}\mathscr{E}$ realizes X as a nondegenerate subscheme of $\mathbb{P}\mathscr{E}$, and therefore the only the trivial element of $\mathrm{PGL}_{\mathscr{E}}$ fixes X as a subscheme of $\mathbb{P}\mathscr{E}$. In the degree 3 case, scaling by λ in the central $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}}$ scales the resulting section by λ, and so only the identity element of $\mathrm{Aut}_{\mathscr{E}}$ preserves the section. This establishes the claim when $d = 3$.

We now consider the cases $d = 4$ and $d = 5$. We are seeking automorphisms of $\mathscr{E}$ and $\mathscr{F}$ preserving a given section $\eta \in \mathrm{U}_d$. We have seen above that any such automorphism must act on $\mathscr{E}$ by some element λ in the central $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}}$. Since we are quotienting by a copy of $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}}$ which maps surjectively to the central $\mathbb{G}_m$ in $\mathrm{Aut}_{\mathscr{E}}$, we may modify our given automorphism so as to assume it is trivial in $\mathrm{Aut}_{\mathscr{E}}$. Note that when $d = 5$, we may have to pass to an fppf cover so as to extract a square root of λ. We may now assume the automorphism is trivial on $\mathscr{E}$ and wish to show it is also trivial on $\mathscr{F}$. However, the given section η induces an injective map $\mathscr{F} \to \mathrm{Sym}^2 \mathscr{E}$, realizing $\mathscr{F}$ as a subsheaf of $\mathrm{Sym}^2 \mathscr{E}$ by Theorem 10.2.2(v). Since we are assuming the automorphism acts as the identity on $\mathscr{E}$ and it preserves this inclusion, it must also act as the identity on $\mathscr{F}$. □

10.4 Resolution of the stack of simply branched covers

Using Theorem 10.2.2, given any finite flat Gorenstein cover $X \to Y$ of schemes degree d, we obtain a canonical embedding $X \to \mathbb{P}\mathscr{E}$ and a resolution of X in $\mathbb{P}\mathscr{E}$. Because this is functorial, we also obtain a resolution in the case that X and Y are stacks. In particular, there is a universal resolution associated to the universal degree d cover of Covers_d. We can also restrict this to various opens of Covers_d. For example, if we restrict this to the point of Covers_d corresponding to a d étale cover, this cover restricts to the map $BS_{d-1} \to BS_d$ induced by the inclusion $S_{d-1} \to S_d$ as the stabilizer of the last point, and we can describe an embedding $BS_{d-1} \to \mathbb{P}\mathscr{E}$ and a resolution in terms of vector bundles with an S_d action, or equivalently, S_d representations. See [Wil13, Chapter 5 and 6] for a description of the resolution in these terms. In particular, the relevant representations are given in [Wil13, Lemma 142].

Definition 10.4.1. There is also a larger open substack of Covers_d consisting of simply branched covers. This stack has two points, the point corresponding to degree d étale covers and the point corresponding to simply branched covers (a degree $d-2$ étale scheme and a copy of the dual numbers). Maps to this stack correspond to simply branched covers, and we denote this open substack of Covers_d parameterizing simply branched covers by SB_d.

Throughout this section, to simplify matters, we work over a base where 2 is invertible. We first describe the linear algebraic data needed to specify a vector bundle on SB_d in § 10.4.2. The resolution over the generic fiber has previously been established. In this case, the vector bundles correspond to the irreducible S_d representations appearing in [Wil13, Lemma 142]. In order to specify the resolution over all of SB_d, we will need to describe the resolution over the special fiber, and then how to glue the special fiber to the generic fiber. The special fiber itself has automorphism group which is $B\mathbb{G}_m \times BS_{d-2}$, at least away from characteristic not 2. The copy of $B\mathbb{G}_m$ is identified with the automorphisms of the dual numbers, while BS_{d-2} is the automorphisms of the degree $d-2$ étale subscheme of the universal family over the special fiber. In § 10.4.3, we describe the $\mathbb{G}_m$ part of the representation over the special fiber in degree 3. In § 10.4.4, we generalize this to describe the $\mathbb{G}_m$ part of the representation over the special fiber in arbitrary degree. Then in § 10.4.5, we add in the S_{d-2} action so as to describe the whole resolution over the special fiber. At the end, we spell out the resolution explicitly in low degrees. Finally, in § 10.4.6, we explain how to glue the resolution over the special fiber to that over the generic fiber to obtain a resolution over all of SB_d.

The writing in this section will be somewhat looser than in other previous parts of the thesis, and we will focus on describing the resolutions leaving more details to the reader.

10.4.2 Descent data for describing SB_d

To simplify matters, throughout this chapter, we will work over a base with 2 invertible. In this case, we have an fppf cover of SB_d by $[\mathbb{A}^1/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ acts by weight 2. The cover is induced by the degree d cover given by $d-2$ disjoint copies of $\mathbb{A}^1$ together with a copy of the degree 2 cover $\operatorname{Spec} k[y] \to \operatorname{Spec} k[x], x \mapsto y^2$, which is ramified to degree 2 over the origin. We can describe vector bundles on $[\mathbb{A}^1/\mathbb{G}_m]$ as vector spaces V with a decomposition $V_{\mathrm{even}} \oplus V_{\mathrm{odd}}$ where both have a filtration: the kth part of the filtration of V_{even} corresponds to the subspace where $\mathbb{G}_m$ acts by weight at least $2k$, while on V_{odd} it corresponds to the subspace where $\mathbb{G}_m$ acts by weight at least $2k+1$. Many parts of these filtrations may be trivial. To give a vector bundle on SB_d, we need to descent datum for the map $[\mathbb{A}^1/G_m] \to \mathrm{SB}_d$. The double fiber product $[\mathbb{A}^1/\mathbb{G}_m] \times_{\mathrm{SB}_d} [\mathbb{A}^1/\mathbb{G}_m]$ is a copy of $S_{d-2} \times [\mathbb{A}^1/\mathbb{G}_m]$ glued along $S_{d-2} \times \eta$ (for η the generic point of $[\mathbb{A}^1/\mathbb{G}_m]$ which is isomorphic to Spec k) with $S_d \times \eta$, via a fixed inclusion $S_{d-2} \to S_d$, say the subgroup fixing the last 2 elements. This choice of inclusion corresponds to the particular cover we chose, i.e., which two sheets were branched. This data amounts to saying that there should be an S_{d-2} representation preserving the filtration and decomposition, together with a compatible S_d representation on V.

So, to recap, vector bundles on SB_d are described by the following:

1. A vector space V with a decomposition $V = V_{\mathrm{even}} + V_{\mathrm{odd}}$
2. A filtration on V_{even} and V_{odd}
3. An S_{d-2} representation on V preserving the decomposition and filtration
4. A representation $\pi : S_d \to \mathrm{GL}(V)$ satisfying the condition that its restriction to S_{d-2} agrees with the prior S_{d-2} representation on V

10.4.3 Degree 3

We'd like to use equivariant Hilbert series to compute resolutions of degree d covers of a base. Recall we are using SB_d to denote the stack parameterizing simply branched degree d covers of Definition 10.4.1. This has a universal degree d cover which is branched over the closed point of SB_d. We'd like to compute the resolution of the cover, which can be described in terms of certain representation theoretic data, via the description of § 10.4.2. (To repeat, the relevant data is a filtered $k[x]$ module with an even and odd filtration, with an S_{d-2} representation that is the restriction of an S_d representation away from the closed point).

We now explain how this works when $d = 3$ in more detail, concentrating on the situation over the closed point, which is isomorphic to $B\mathbb{G}_m$. In particular, in what follows, we will only describe the $\mathbb{G}_m$ representation. To fully describe the cover over SB_3, we will need a bit more (such as an S_3 representation and a filtration), and we will discuss this additional data in future subsections.

To set up notation, over $B\mathbb{G}_m$, the special fiber of the universal degree 3 cover corresponds the $\mathbb{G}_m$ representation $A := k[\varepsilon]/(\varepsilon^2) \oplus k$ where $\mathbb{G}_m$ acts by weight 1 on ε and weight 0 on the two copies of k. Let $X := \operatorname{Spec} A$. Let $\rho : X \to \operatorname{Spec} k$ denote the structure map. Then, recall that the canonical embedding is induced by the subsystem $V \subset \rho_* \omega_{X/k}$ given by the kernel of the trace map. In our setting, we can describe V quite explicitly. Recall that $\rho_* \omega_{X/k} = (A)^\vee$ as an A-module. Let us notate an element of A as $(a + b\varepsilon, c)$ for a, b, c of weight 0. Explicitly, $b\varepsilon$ is thought of as lying in weight 1. Then, because X is Gorenstein, we know $\rho_* \omega_{X/k}$ can be identified with A (ignoring the $\mathbb{G}_m$ action), and we notate an element as $(\alpha + \beta\varepsilon, \gamma)$ where now α, β have weight -1 and γ has weight 0, so that $\beta\varepsilon$ is thought of as lying in weight 0. Using that $\rho_* \omega_{X/k} = \mathrm{Hom}_k(A, k)$, we are separately identifying $\alpha + \beta\varepsilon$ as dual to $a + b\varepsilon$ and γ as dual to c. The trace map is then dual to the inclusion $k \hookrightarrow A$ sending $1 \mapsto (a, c)$. The resulting trace map is given by $(\alpha + \beta\varepsilon, \gamma) \mapsto \beta - \gamma$, where we think of the latter as an element of k with weight 0.

Therefore, the subspace $V \subset \rho_* \omega_{X/k}$ inducing the canonical embedding, which is the kernel of the trace map, is given by elements of the form $(\alpha + \beta\varepsilon, -\beta)$. In other words it is spanned by the elements $x = (\alpha, 0)$ and $y = (\beta\varepsilon, -\beta)$ where x has weight -1 and y has weight 0.

We see that x and y satisfy the relation $xy^2 = 0$, since $(\alpha, 0) \cdot (\beta\varepsilon, -\beta)^2 = (\alpha, 0)\left(0, (-\beta)^2\right) = 0$. Then, this implies that the resulting embedding $X \hookrightarrow \mathbb{P}^1$ is defined by $\mathrm{Proj}\, k[x, y]/(xy^2)$ where x has weight -1 and y has weight 0. Let us use V_i to denote the 1-dimensional weight i representation of $\mathbb{G}_m$. The equivariant Hilbert series of this ring is then

$$H(n) = V_{-n} \oplus V_{-n+1} \oplus V_0$$

for $n \geq 2$ corresponding to the fact that the degree n monomials not in the span of xy^2 are $x^n, x^{n-1}y$, and y^n.

If we wish to compute the associated resolution, we can do so as follows. Let $p(t) = \sum_{j=0}^{d} (-1)^j W_j t^j$, where W_j denotes the $\mathbb{G}_m$ representation depicting the jth term of the resolution. The hilbert polynomial and $p(t)$ satisfy the relation

$$\frac{p(t)}{1 - Vt + \wedge^2 V t^2} = H(t).$$

Since $V = V_0 + V_{-1}$, we can simplify $1 - Vt + \wedge^2 V t^2 = (1 - t)(1 - V_{-1}t)$. On the other hand,

$$\begin{aligned} H(t) &= 1 + (V_0 + V_{-1})t + (V_0 + V_{-1} + V_{-2})t^2 + (V_0 + V_{-2} + V_{-3})t^3 + \cdots \\ &= \frac{1}{1-t} + \frac{V_{-1}t}{1 - V_{-1}t} + \frac{V_{-1}t^2}{1 - V_{-1}t}. \end{aligned}$$

Therefore, we find

$$\begin{aligned} p(t) &= (1 - V_{-1}t) + (V_{-1}t)(1-t) + V_{-1}t^2(1-t) \\ &= 1 - V_{-1}t + V_{-1}t - V_{-1}t^2 + V_{-1}t^2 - V_{-1}t^3 \\ &= 1 - V_{-1}t^3. \end{aligned}$$

This is in accordance with the coefficient of t^3 being the determinant of V, as predicted in the degree 3 resolution of Casnati-Ekedahl.

10.4.4 Higher degree

We now want to generalize the description of the previous subsection when $d = 3$ to arbitrary values of d. That is, we want to compute the weights of the Casnati-Ekedahl resolution of the degree d cover of the special fiber of SB_d as a $\mathbb{G}_m$ representation.

Let us next describe a trick we can use to compute higher degree Hilbert series: With notation as in the previous section, we find that there is an injection $\ker(\rho_*\omega \to \mathscr{O}_{\mathrm{Spec}\,k}) \to \rho_*\omega$ with cokernel $\mathscr{O}_{\mathrm{Spec}\,k}$. Since our degree d cover embeds into the projectivization of both of these, and induces an isomorphism on the restrictions in degree 2 or more, at least in degrees ≥ 2, we can compute the equivariant Hilbert series by computing it for the embedding into $\rho_*\omega$. This is the difference between the "standard representation" and the "permutation representation." On degree 1 pieces, we can directly see this is the dual of $\mathscr{E}$, and we know $\mathscr{E}$ has 1-dimensional weight 1 part and a $d-1$ dimensional weight 0 part.

In any case, let's now compute the representation decomposition associated to the degree n part of $\rho_*\omega$. Here, to introduce some variables, we can think of coordinates α and β (slightly different from the previous section) as parameterizing a k-basis for $k[\varepsilon]/(\varepsilon)^2$. Specifically, α corresponds to the element 1 and β to the element ε so that $\beta^2 = 0$ in the two dimensional space $\mathscr{O}_D$. Let z_i correspond to the $d-2$ copies of $\mathrm{Spec}\,k$. We have to be careful here, since β actually corresponds to the coordinate in weight 0 and α corresponds to the coordinate in weight -1. We find the relations $\beta^2 = 0, z_i z_j = 0, z_i\alpha = 0, z_i\beta = 0$. After quotienting by these, we obtain the remaining monomials z_i^n, $\alpha^{n-1}\beta$, and α^n. Recall we are using V_i to denote the weight i representation of $\mathbb{G}_m$. As an $\mathbb{G}_m \times S_{d-2}$ representation, we see this is the standard representation for S_{d-2} acting on the z_i, and the $\mathbb{G}_m$ has $d-2$ copies of V_0, corresponding to the z_i, one copy of V_{-n+1} corresponding to $\alpha^{n-1}\beta$, and one copy of V_{-n} corresponding to α^n. So, for $n \geq 2$, we have

$$H(n) = V_0^{\oplus d-2} \oplus V_{-n+1} \oplus V_{-n},$$

while for $n = 1$ we have the $\mathbb{G}_m$ representation

$$H(1) = V_0^{\oplus d-2} \oplus V_{-1}.$$

Using this Hilbert series, we can now solve for the resolution. Indeed, as in the $d = 3$ case, we find

$$\frac{p(t)}{1 - Vt + \wedge^2 Vt^2 - \wedge^3 Vt^3 + \cdots} = H(t).$$

Using that $V = V_0^{d-2} + V_{-1}$, we see $1 - Vt + \wedge^2 Vt^2 - \wedge^3 Vt^3 + \cdots = (1-t)^{d-2}(1 - V_{-1}t)$. Also, as computed above,

$$H(t) = 1 + (d - 2 + V_{-1})\, t + (d - 2 + V_{-1} + V_{-2})\, t^2 + (d - 2 + V_{-2} + V_{-3})\, t^3 + \cdots .$$

We can see

$$\frac{p(t)}{(1-t)^{d-2}(1 - V_{-1}t)} = \frac{1}{1-t} + \frac{(d-3)t}{1-t} + \frac{V_{-1}t}{1 - V_{-1}t} + \frac{V_{-1}t^2}{1 - V_{-1}t}.$$

Simplifying, we get

$$\begin{aligned} p(t) &= (1 - V_{-1}t)(1-t)^{d-3}(1 + (d-3)t) + (1-t)^{d-2}(V_{-1}t)(1+t) \\ &= (1-t)^{d-3}\left(1 - (d-3)t + V_{-1}(d-3)t^2 - V_{-1}t^3\right). \end{aligned}$$

10.4.5 The resolution on the special fiber

In the previous section, we computed the equivariant Hilbert series for the special fiber of SB_d, when we only kept track of the $\mathbb{G}_m$ action. We'd now like to keep track of the full resolution on the special fiber, which is a $S_{d-2} \times \mathbb{G}_m$ action. As usual, this corresponds to a sequence of $S_{d-2} \times \mathbb{G}_m$ representations. For W an S_{d-2} representation and V_i the 1-dimensional weight i representation of $\mathbb{G}_m$, we notate as $W \otimes V_i$ as the corresponding $S_{d-2} \times \mathbb{G}_m$ representation.

Next, let us compute the equivariant resolution. To start, let us describe the bundle $\mathscr{E}$. If $\rho : X \to Y$ is our given cover for $Y = B(S_{d-2} \times \mathbb{G}_m)$ we see that the canonical vector bundle $\mathscr{E} = \ker\left(\rho_* \omega_{X/Y} \to \mathscr{O}_Y\right)$ into which X embeds is the kernel of the map to $\mathrm{triv} \otimes V_0$ from $\mathrm{triv} \otimes V_0 \oplus \mathrm{triv} \otimes V_{-1} \oplus \mathrm{triv} \otimes V_0 \oplus \mathrm{std} \otimes V_0$. Therefore,

$$\mathscr{E} = \mathrm{triv} \otimes V_0 \oplus \mathrm{triv} \otimes V_{-1} \oplus \mathrm{std} \otimes V_0.$$

In order to compute the equivariant Hilbert series, we note that in degree 1 it is given by $\mathscr{E}$ and in higher degrees, it agrees with the embedding into $\rho_* \omega_{X/Y}$. Using the coordinates α, β, z_i of the previous section, we find the embedding is given by the quotient

$$k[\alpha, \beta, z_1, \ldots, z_{d-2}] \to k[\alpha, \beta, z_1, \ldots, z_{d-2}]/(\{z_i z_j\}_{1 \le i < j \le d-2}, \{z_i \alpha, z_i \beta\}_{1 \le i \le d-2}, \beta^2).$$

The latter in degree n is spanned by $z_1^n, \ldots, z_{d-2}^n, \alpha^n, \alpha^{n-1}\beta$. Recall the representation corresponding to $\rho_* \omega_{X/Y}$ is given by the action by $\mathbb{G}_m$ acting on α with weight -1 and the others with weight 0, while S_{d-2} acts by permuting the z_i. Altogether, this shows that in degree n the representation is

$$\operatorname{triv} \otimes V_{-n} \oplus \operatorname{triv} \otimes V_{-n+1} \oplus \operatorname{triv} \otimes V_0 \oplus \operatorname{std} \otimes V_0.$$

To simplify our notation, for W an S_{d-2} representation, we write W for the $S_{d-2} \times \mathbb{G}_m$ representation $W \otimes V_0$, write V_i for the representation $\operatorname{triv} \otimes V_i$, and write 1 for the representation $\operatorname{triv} \otimes V_0$. Combining these across degrees, we find that the equivariant Hilbert series is

$$\begin{aligned} H(t) &= 1 + (1 + V_{-1} + \operatorname{std})\, t + (V_{-2} + V_{-1} + 1 + \operatorname{std})\, t^2 + (V_{-3} + V_{-2} + 1 + \operatorname{std})\, t^3 + \cdots \\ &= \frac{1}{1-t} + \frac{\operatorname{std} t}{1-t} + \frac{V_{-1} t}{1 - V_{-1} t} + \frac{V_{-1} t^2}{1 - V_{-1} t}. \end{aligned}$$

Note that

$$\begin{aligned} H(t) &= \frac{p(t)}{1 - \mathscr{E} + \wedge^2 \mathscr{E} - \wedge^3 \mathscr{E} + \cdots} \\ &= \frac{p(t)}{(1-t)(1 - V_{-1} t)\left(1 - \operatorname{std} t + \wedge^2 \operatorname{std} t^2 - \wedge^3 \operatorname{std} t^3 + \cdots\right)}. \end{aligned}$$

Therefore, solving for $p(t)$, we find

$$\begin{aligned} p(t) = {} & (1 + \operatorname{std} t)(1 - V_{-1} t)\left(1 - \operatorname{std} t + \wedge^2 \operatorname{std} t^2 - \wedge^3 \operatorname{std} t^3 + \cdots\right) \\ & + V_{-1} t (1-t)(1+t)\left(1 - \operatorname{std} t + \wedge^2 \operatorname{std} t^2 - \wedge^3 \operatorname{std} t^3 + \cdots\right) \\ = {} & \left(1 + \operatorname{std} t - \operatorname{std} \otimes V_{-1} t^2 - V_{-1} t^3\right)\left(1 - \operatorname{std} t + \wedge^2 \operatorname{std} t^2 - \wedge^3 \operatorname{std} t^3 + \cdots\right). \end{aligned}$$

For sufficiently large d and i not too close to 0 or d, we can also compute the coefficient of t^i (we can also compute the values close to 0 and d, where close means within 3 or so, but there are a few more cases). For this, we notate representations of S_{d-2} by partitions of $d-2$. Expanding the above formula for $p(t)$, we see the coefficient of t^i for $i \geq 3$ and $i \leq (d-2)-3$ we find

$$\begin{aligned}[p(t)]_i &= (-1)^{i+1}\left(-\wedge^i \operatorname{std} + \left(\operatorname{std} \otimes \wedge^{i-1} \operatorname{std}\right) + \left(\operatorname{std} \otimes \wedge^{i-2} \operatorname{std} - \wedge^{i-3} \operatorname{std}\right) \otimes V_{-1}\right) \\ &= (-1)^{i+1}\left((d-1-i, 1^{i-1}) \oplus (d-i, 1^{i-2}) \oplus (d-2-i, 2, 1^{i-2}) \oplus (d-1-i, 2, 1^{i-3})\right) \\ &+ (-1)^{i+1}\left(\left((d-i, 1^{i-2}) \oplus (d-i-1, 1^{i-1}) \oplus (d-i, 2, 1^{i-4}) \oplus (d-i-1, 2, 1^{i-3})\right) \otimes V_{-1}\right).\end{aligned}$$

We also find the coefficients of 1 is 1, the coefficient of t is 0, the coefficient of t^d is $(-1)^d \wedge^{d-3} \operatorname{std} \otimes V_{-1}$, and the coefficient of t^{d-1} is 0. Also, for low values of d, and when $i = 3$ (in which case 1^{i-4} is negative) we throw out members above where values in the partition are negative or are not in decreasing order. The coefficient of t^2 is $(-1)(\operatorname{triv} \oplus \operatorname{std} \oplus (d-4, 2) \oplus \operatorname{std} \otimes V_{-1})$, at least when $d-2 \geq 4$. When $d-2 \geq 4$, the coefficient of t^{d-2} is $(-1)^{d-1}\left(\wedge^{d-4} \operatorname{std} + \left((2, 1^{d-4}) \oplus (2, 2, 1^{d-6}) \oplus (1^{d-2})\right) \otimes V_{-1}\right)$.

Let's now make the above resolution explicit for $d \leq 6$ by writing it out.

$d = 3$

In degree 3 the resolution series is

$$p(t) = 1 - V_{-1} t^3.$$

$d = 4$

In degree 4 the resolution series is

$$p(t) = 1 - (\operatorname{triv} + \operatorname{std} \otimes V_{-1}) t^2 + \operatorname{std} \otimes V_{-1} t^4.$$

$d = 5$

In degree 5 the resolution series is

$$p(t) = 1 - (\operatorname{std} + \operatorname{triv} + \operatorname{std} \otimes V_{-1}) t^2 + \left(\operatorname{std} + \operatorname{std} \otimes V_{-1} + \wedge^2 \operatorname{std} \otimes V_{-1}\right) t^3 - (\wedge^2 \operatorname{std} \otimes V_{-1}) t^5.$$

$d = 6$

In degree 6 the resolution series is

$$\begin{aligned} p(t) = 1 & \\ & - (\mathrm{triv} + \mathrm{std} + (2,2) + \mathrm{std} \otimes V_{-1})\, t^2 \\ & + \Big((2,1^2) + (2,2) + (3,1) + \big((3,1) + (2,2) + (2,1^2)\big) \otimes V_{-1}\Big)\, t^3 \\ & - \Big(\wedge^2 \mathrm{std} + \big((2,1^2) \oplus (2,2) \oplus (1^4)\big) \otimes V_{-1}\Big)\, t^4 \\ & + \Big(\wedge^3 \mathrm{std} \otimes V_{-1}\Big)\, t^6. \end{aligned}$$

10.4.6 Computing the resolution over the compactification

Up until now, we have computed the resolution only over the closed point of the compactification. Now, we compute the resolution over the whole compactification. Let's start by just examining the $\mathbb{G}_m$ action. Recall that SB_d has a cover by $\mathbb{A}^1 = \operatorname{Spec} k[x]$, factoring through $[\mathbb{A}^1/\mathbb{G}_m]$ with the weight 2 $\mathbb{G}_m$ action, as was described in § 10.4.2.

Pulling back to $\mathbb{A}^1$, the universal cover is a disjoint union of $d-2$ copies of $\mathbb{A}^1$ mapping isomorphically and one degree 2 cover which is ramified over $x = 0$. We'll let y be the coordinate for this degree 2 cover and let $\rho : X \to \mathbb{A}^1$ denote this cover. The vector bundle $\mathscr{E}^\vee = \rho_* \mathscr{O}_X$ is then given by $k[x]^{\oplus d-2} \oplus k[y] = k[x]^{\oplus d-2} \oplus k[x] \oplus y k[x]$ where y lives in weight 1 and x in weight 2. The dual bundle is then $\rho_* \omega_{X/\mathbb{A}^1} = \mathscr{E}$ which can be described in the form

$$(\oplus_{i=1}^{d-2} z_i k[x]) \oplus \alpha k[x] \oplus \beta k[x]$$

where z_i live in weight 0, α lives in weight -1 and β lives in weight 0. The canonical sheaf is the kernel of the map $\mathscr{E}^\vee \to \mathscr{O}$ which sends $\alpha \mapsto 0, \beta \mapsto 1, z_i \mapsto 1$. As in the special fiber resolution of § 10.4.5, β should be thought of as dual to 1 and α should be thought of as dual to y (which was previously called ε). Really, when we write $\alpha f(x) + \beta g(x) \in \beta k[x] \oplus \alpha k[x]$, we can think of such elements as dual to elements of the form $h(x) + y k(x) \in k[y] = k[x] \oplus y k[x]$ with $x = y^2$. Said again, α is dual to 1 and β is dual to y, and this is the natural Serre duality pairing on these rank finite covers of $\mathbb{A}^1$.

As above, we have an canonical embedding into $k[x, z_i, \alpha, \beta]$ defined by equations $z_i z_j, z_i \alpha, z_i \beta$, but now there is a new phenomenon. In this case, we get the homogeneous (in α, β, z_i) degree 2 relation $\beta^2 = x\alpha^2$. In order to calculate the equivariant Hilbert series associated to the canonical embedding, as in § 10.4.5, in degree ≥ 2, it is equivalent to calculate the equivariant Hilbert series in this "permutation" embedding. As before, this is spanned by $\alpha^n, \alpha^{n-1}\beta, z_i^n$. Though here we should be careful that there are relations identifying $\beta^n = x\alpha^2\beta^{n-2}, \alpha\beta^{n-1} = x\alpha^3\beta^{n-3}$ and so on. In any case, this gives a "Hilbert series" over $\mathbb{A}^1$ with degree n piece equal to

$$H(n) = (\oplus_{i=1}^{d-2} z_i^n k[x]) \oplus \alpha^n k[x] \oplus \alpha^{n-1}\beta k[x].$$

Again, this is indeed compatible with reduction $\bmod x$ from § 10.4.5. If we wish, we can now describe this as a sum of two filtered vector spaces (corresponding to even weight and odd weight). Note that the S_{d-2} action is always acting on the z_i, and this should be the restriction of an S_d action on the generic fiber (or even just the open where we remove $x = 0$).

The precise description of the filtration of the above vector space depends on whether n is even or odd, and we write it out below in (10.4.1). The odd part of the filtration is 1-dimensional and the even part has a 1-dimensional subspace given by either α^n or $\alpha^{n-1}\beta$, and a $d-2$ dimensional quotient spanned by the images of the z_i^n.

We let W_i denote the $\mathbb{A}^1$ bundle with lowest degree term living in filtration piece i. So W_0 is just $k[x]$ while $\alpha k[x]$ could be identified with W_{-1}. As in § 10.4.5, we implicitly omit trivial tensor products, and we will use std to denote the standard representation of S_{d-2}. If we want to describe this in terms of filtered vector spaces, we can write

$$\begin{aligned} H(t) &= 1 && (10.4.1)\\ &+ (W_{-1} \oplus W_0 \oplus \mathrm{std})t && (10.4.2)\\ &+ W_{-3} \oplus (W_{-2} \to \bullet \to (W_0 + \mathrm{std}))\, t^3 && (10.4.3)\\ &+ W_{-3} \oplus (W_{-4} \to \bullet \to (W_0 + \mathrm{std}))\, t^4 && (10.4.4)\\ &+ W_{-5} \oplus (W_{-4} \to \bullet \to (W_0 + \mathrm{std}))\, t^5 && (10.4.5)\\ &+ W_{-5} \oplus (W_{-6} \to \bullet \to (W_0 + \mathrm{std}))\, t^6 && (10.4.6)\\ &+ \cdots && (10.4.7) \end{aligned}$$

The notation $\bullet$ here corresponds to the full even subspace V_{even}, and we give the filtration surrounding bullet, which is always a two step filtration.

10.5 The pairing on the Casnati-Ekedahl resolution

In this section, we'd like to concretely construct the canonical pairing on the Casnati-Ekedahl resolution. The main motivation for doing this is to see whether in the middle degree it is symmetric or skew symmetric. We will find that in degree $4k$ it is skew symmetric and in degree $4k + 2$ it is symmetric, see Corollary 10.5.13.

This section is divided into several subsections as follows. We first review the version of the Casnati-Ekedahl resolution over a base ring due to Behnke. This in turn builds on a resolution due to Scheja and Storch, which we review in § 10.5.1. We construct a pairing on the Scheja-Storch resolution in § 10.5.2. We then review Behnke's resolution in § 10.5.6 and construct an induced pairing on this resolution in § 10.5.7. In § 10.5.12 we deduce this perfect pairing is symmetric or skew symmetric in the middle degree for even degree covers. In § 10.5.15 we also show the middle term for even degree covers satisfies a certain isotropicity condition. Finally, in § 10.5.18 we describe the pairing on Behnke's resolution explicitly for degree 6 covers.

10.5.1 The Scheja and Storch Resolution

Let us now review the construction of the Casnati-Ekedahl resolution over a ring, as in [Beh81]. This was due to Behnke, and was used in the construction of the Casnati-Ekedahl to deal with Step A of the proof of Theorem 10.2.2 when one works over a base which is a point. Note that to show the pairing is symmetric or antisymmetric, it is harmless to restrict to the base being affine.

Behnke's resolution is constructed essentially as a subquotient of the Scheja and Storch resolution, which is in turn a variant of the Koszul resolution. We start by describing the Scheja and Storch resolution, and the pairing there. We will work over the ring R, and consider a resolution of a finitely generated projective module E. In the case we are interested in, E will be an algebra of degree $d-2$, constructed as a certain subalgebra of our degree d Gorenstein algebra. Let $S := \oplus_{k\geq 0} \operatorname{Sym}^k(E)$ be the symmetric algebra. We use x_i for the ith generator of S, thought of as lying in degree 1. We use $\widetilde{M} := M \otimes_R S$, for M an R-module.

The Scheja and Storch resolution is a resolution of a finitely generated projective R module M given as $K_\bullet(N)$ with $K_i(N) = \wedge^i \widetilde{E} \otimes_R N$ for $r \geq i \geq 0$. The boundary maps are given by

$$\begin{aligned}
\delta_N : \wedge^k \widetilde{E} \otimes_R N &\longrightarrow \wedge^{k-1} \widetilde{E} \otimes_R N \\
e_1 \wedge \cdots \wedge e_k \otimes e &\mapsto \left(\sum_{i=1}^{k} (-1)^{i-1} x_i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k \otimes e \right) \\
&\quad + \left(\sum_{i=1}^{k} (-1)^{i} e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k \otimes (e_i \cdot e) \right).
\end{aligned} \tag{10.5.1}$$

Above, the parentheses means the product is taken via multiplication on the module N. This yields a resolution

$$0 \longrightarrow \wedge^r \widetilde{E} \otimes_R N \xrightarrow{\delta_N} \cdots \longrightarrow \wedge^2 \widetilde{E} \otimes_R N \xrightarrow{\delta_N} \widetilde{E} \otimes_R N \longrightarrow \widetilde{N}. \quad (10.5.2)$$

10.5.2 The Pairing on the Scheja and Storch Resolution

We now describe the pairing on the Scheja Storch resolution. To describe this pairing, we will need to have a pairing $\langle , \rangle$ on the module N, and this satisfies the important constraint that $\langle f, gh \rangle = \langle gf, h \rangle$.

Remark 10.5.3. In the case we are interested in, this will be induced from the pairing on the Gorenstein degree d cover and the module N appearing below will be taken to be E, which is a certain degree $d-2$ R-submodule of the pushforward of the structure sheaf of the cover (although it need not be a submodule for the degree d cover of R).

We will overload notation by using $\langle , \rangle$ notation for this pairing on the module N (in addition to using it for the pairing on the complex), but it should be clear from context which pairing we are describing when this notation is used. The pairing on the pieces of the resolution is then given by a map

$$\begin{gathered}\langle , \rangle : \wedge^k \widetilde{E} \otimes_R N \otimes \wedge^{r-k} \widetilde{E} \otimes_R N \to \wedge^r \widetilde{E} \\ (e_1 \wedge \cdots \wedge e_k \otimes n) \otimes (f_1 \wedge \cdots \wedge f_{r-k} \otimes m) \mapsto (-1)^{\binom{k}{2}} \langle n, m \rangle e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge f_{r-k}.\end{gathered} \quad (10.5.3)$$

We'd now like to check that this pairing is compatible with the boundary maps in the Scheja and Storch resolution. This amounts to checking that

Lemma 10.5.4.

$$\langle \delta (e_1 \wedge \cdots \wedge e_k \otimes n), f_1 \wedge \cdots \wedge f_{r-k+1} \otimes m \rangle = \langle e_1 \wedge \cdots \wedge e_k \otimes n, \delta (f_1 \wedge \cdots \wedge f_{r-k+1} \otimes m) \rangle.$$

Proof. We may as well assume that e_i and f_j are all elements of a basis for E. Then, the pairings above both evaluate to 0 unless there is exactly one basis element appears exactly twice in the set $\{e_1, \ldots, e_k f_1, \ldots, f_{r-k+1}\}$. So, assume there is exactly one redundant basis element, say $e_t = f_s$, and both are the jth basis element corresponding to the element x_j. Then,

$$\begin{aligned}&\langle \delta (e_1 \wedge \cdots \wedge e_k \otimes n), f_1 \wedge \cdots \wedge f_{r-k+1} \otimes m \rangle \\ &= (-1)^{\binom{k-1}{2}} \left((-1)^{t-1} x_j \langle n, m \rangle + (-1)^t \langle e_t n, m \rangle \right) e_1 \wedge \cdots \wedge \widehat{e_t} \wedge \cdots \wedge e_k \wedge f_1 \cdots \wedge f_{r-k+1}\end{aligned}$$

On the other hand,

$$\langle e_1 \wedge \cdots \wedge e_k \otimes n, \delta(f_1 \wedge \cdots \wedge f_{r-k+1} \otimes m)\rangle$$
$$= (-1)^{\binom{k}{2}} \left((-1)^{s-1} x_j \langle n, m\rangle + (-1)^s \langle n, f_s m\rangle\right) e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge \widehat{f_s} \wedge \cdots \wedge f_{r-k+1}$$

Now, note that

$$e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge \widehat{f_s} \wedge \cdots \wedge f_{r-k+1} = (-1)^{k-t+s-1} e_1 \wedge \cdots \wedge \widehat{e_t} \wedge \cdots \wedge e_k \wedge f_1 \cdots \wedge f_{r-k+1}$$

Therefore, it suffices to verify

$$(-1)^{\binom{k}{2}} \left((-1)^{s-1} x_j \langle n, m\rangle + (-1)^s \langle n, f_s m\rangle\right)$$
$$= (-1)^{k-t+s-1} (-1)^{\binom{k-1}{2}} \left((-1)^{t-1} x_j \langle n, m\rangle + (-1)^t \langle e_t n, m\rangle\right),$$

which indeed holds using elementary arithmetic and the fact that $\langle e_t n, m\rangle = \langle n, e_t m\rangle = \langle n, f_s m\rangle$. □

Proposition 10.5.5. *The pairing defined in* (10.5.3) *defines a perfect pairing on the Scheja Storch resolution.*

Proof. It defines a perfect pairing on each term of the resolution because for each basis element $e_I := \wedge_{i \in I} e_i$, the element $e_{\{1,\ldots,r\}-I}$ defines an element pairing with e_I to a nonzero value. The compatibility with the maps in the complex is the content of Lemma 10.5.4. □

10.5.6 Behnke's Resolution

We next describe Behnke's Resolution, in terms of Scheja and Storch Resolution. In Behnke's setting, we start with a Gorenstein ring A, we let e_0 be the unit of A and let e^* be an element pairing to e_0 with 1. We can choose some $E \subset A$ so that $Re_0 \oplus E \oplus Re^* = A$. The key to defining Behnke's resolution is that apart from the first and last terms, the kth term $M_k(A; E)$ is given by the homology of

$$\wedge^{k+1} \widetilde{E} \xrightarrow{\Delta} \wedge^k \widetilde{E} \otimes_R E \xrightarrow{P'} \wedge^{k-1} \widetilde{E} \qquad (10.5.4)$$

by [Beh81, Lemma 3.1(ii)] where

$$\Delta : \wedge^{k+1}\widetilde{E} \to \wedge^k \widetilde{E} \otimes_R E$$
$$e_1 \wedge \cdots \wedge e_{k+1} \mapsto \sum_{i=1}^{k+1} (-1)^{i-1}\, e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{k+1} \otimes e_i$$

and

$$P' : \wedge^k \widetilde{E} \otimes_R E \to \wedge^{k-1}\widetilde{E}$$
$$e_1 \wedge \cdots \wedge e_k \otimes e \mapsto \sum_{i=1}^{k} (-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k \langle e, e_i \rangle.$$

For the description of the first and last terms of Behnke's resolution, as well as the induced boundary maps, the reader should consult [Beh81, Theorem 3.3].

10.5.7 The induced pairing on Behnke's resolution

We claim that the pairing described above on the Scheja and Storch resolution induces a pairing on Behnke's resolution.

Definition 10.5.8. Define the pairing between the first and last terms of Behnke's resolution by

$$\langle e_1 \wedge \cdots \wedge e_r, 1\rangle = (-1)^{\binom{r}{2}} \langle 1, e_1 \wedge \cdots \wedge e_r\rangle \quad = 1. \tag{10.5.5}$$

Define the pairings between the other terms to be induced by the pairing from (10.5.3) on the Scheja and Storch resolution.

It will be fairly involved to check these pairings well defined are compatible with the construction of Behnke's resolution, and we complete the verification in Proposition 10.5.10.

To define the pairing described above, we need to check the following, which will imply that the Scheja and Storch resolution induces a pairing on Behnke's resolution, at least away from the first and last terms.

Lemma 10.5.9. *Under the pairing $\langle , \rangle$ defined above on Behnke's resolution, for $f \in K_{r-k}(E)$ in the Scheja and Storch resolution with $P'(f) = 0$, and $h \in \wedge^{k+1}\widetilde{E}$, we have we have*

$$\langle \Delta(h), f\rangle = 0.$$

Proof. It suffices to check this for h a basis. That is, we will assume h is of the form

$a_1 \wedge \cdots \wedge a_{k+1}$. where the a_i are all elements in a fixed basis of E. We can then write

$$f = \sum_{I=(i_1,\ldots,i_{r-k})} \alpha_I e_{i_1} \wedge \cdots \wedge e_{i_{r-k}} \otimes e.$$

Then, the pairing is given by

$$\langle \Delta(h), f \rangle = \langle \sum_{i=1}^{k+1} (-1)^{i-1} a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_{k+1} \otimes a_i, \sum_{I=(i_1,\ldots,i_{r-k})} \alpha_I e_{i_1} \wedge \cdots \wedge e_{i_{r-k}} \otimes e \rangle.$$

We would like to understand what terms can possibly contribute to this pairing. The only way a term can contribute is when $\{a_1, \ldots, a_{k+1}, e_{i_1}, \cdots, e_{i_{r-k}}\}$ has exactly one repeated index, say $a_t = e_{i_s}$. In this case the sum obtains a contribution of

$$(-1)^{t-1} \alpha_I \langle e, a_t \rangle a_1 \wedge \cdots \wedge \widehat{a_t} \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{r-k}}$$

to the pairing. Therefore, the sum of such terms is

$$\langle \Delta(h), f \rangle = \sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\} \cap \{e_{i_1},\ldots,e_{i_{k-r}}\} = a_t} (-1)^{t-1} \alpha_I \langle e, a_t \rangle a_1 \wedge \cdots \wedge \widehat{a_t} \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{r-k}}. \tag{10.5.6}$$

We wish to show this vanishes. However, we are assuming $P'(f) = 0$, which we claim means that the above expression vanishes, as can be gleaned by looking at the coefficient of $\{e_1, \ldots, e_r\} - \{a_1, \ldots, a_k\}$ in $P'(f)$. Indeed, the coefficient of this term in $P'(f)$ is precisely

$$\sum_{t=1}^{k+1} \sum_{\{e_{i_1},\ldots,e_{i_{r-k}}\} = (I = \{e_1,\ldots,e_r\} - \{a_1,\ldots,a_k\}) \cup \{a_t\}} (-1)^s \alpha_I e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_{r-k}} \langle e, e_{i_s} \rangle,$$

where, by our notation above, we are assuming $e_{i_s} = a_t$. Note that the condition $\{e_{i_1}, \ldots, e_{i_{r-k}}\} = (I = \{e_1, \ldots, e_r\} - \{a_1, \ldots, a_k\}) \cup \{a_t\}$ indicates I is the complement of $\{a_1, \ldots, a_k\}$ in $\{e_1, \ldots, e_r\}$ and $\{e_{i_1}, \ldots, e_{i_{r-k}}\}$ is the union of I and a_t. Observe also that the condition $\{e_{i_1}, \ldots, e_{i_{r-k}}\} = (I = \{e_1, \ldots, e_r\} - \{a_1, \ldots, a_k\}) \cup \{a_t\}$ is precisely the same as requiring $\{a_1, \ldots, a_{k+1}\} \cap \{e_{i_1}, \ldots, e_{i_{k-r}}\} = a_t$.

Therefore, we conclude

$$\sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\} \cap \{e_{i_1},\ldots,e_{i_{k-r}}\} = a_t} (-1)^s \alpha_I \langle e, a_t \rangle e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_{r-k}} = 0.$$

Therefore,

$$\sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\}\cap\{e_{i_1},\ldots,e_{i_{k-r}}\}=a_t} (-1)^s \alpha_I \langle e, a_t\rangle a_1 \wedge \cdots \wedge a_t \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_{r-k}} = 0.$$

Because $a_t = e_{i_s}$, the above implies

$$\begin{aligned}
&0\\
&= \sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\}\cap\{e_{i_1},\ldots,e_{i_{k-r}}\}=a_t} (-1)^s \alpha_I \langle e, a_t\rangle a_1 \wedge \cdots \wedge a_t \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_{r-k}}\\
&= \sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\}\cap\{e_{i_1},\ldots,e_{i_{k-r}}\}=a_t} (-1)^s (-1)^{k+1-t+s-1} \alpha_I \langle e, a_t\rangle a_1 \wedge \cdots \wedge \widehat{a_t} \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge a_t \wedge \cdots \wedge e_{i_{r-k}}\\
&= \sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\}\cap\{e_{i_1},\ldots,e_{i_{k-r}}\}=a_t} (-1)^s (-1)^{k+1-t+s-1} \alpha_I \langle e, a_t\rangle a_1 \wedge \cdots \wedge \widehat{a_t} \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_{r-k}}\\
&= (-1)^k \sum_{t=1}^{k+1} \sum_{\{a_1,\ldots,a_{k+1}\}\cap\{e_{i_1},\ldots,e_{i_{k-r}}\}=a_t} (-1)^{t-1} \alpha_I \langle e, a_t\rangle a_1 \wedge \cdots \wedge \widehat{a_t} \wedge \cdots \wedge a_{k+1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_{r-k}}.
\end{aligned}$$

This implies (10.5.6) vanishes, as desired. □

This previous lemma establishes that the pairing on the Scheja and Storch resolution descends to one on the terms of Behnke's resolution. The rest of this section is concerned with showing this pairing is compatible with the maps between terms in Behnke's resolution. It will be hardest to deal with maps near the ends of the complex. We first deal with maps which are not near the ends of the complex.

We will use notation from [Beh81] and suggest the reader consult that paper. The following lemma is in some sense a straightforward calculation, but one has to be careful because the differentials $\wedge^r \widetilde{E} \to M_{r-1}(A; E)$ and $M_1(A; E) \to S$ (see [Beh81, Theorem 3.3]) are defined in a fairly complicated manner.

Proposition 10.5.10. *For $u \in \wedge^r \widetilde{E}$ and $v \in M_1(A; E)$, we have*

$$\langle u, \delta(v)\rangle = \langle \delta(u), v\rangle.$$

Proof. As a notational convention for the proof of this lemma, we use $e_1, \ldots, e_r$ as a basis for E, and x_i as the corresponding elements $x_i := \Delta(e_i) \in \mathrm{Sym}^1(E) \subset S$, for $\Delta : E \simeq \mathrm{Sym}^1(E)$ the natural map. We also use e_0 to denote the unit element and e^* to denote the final generator of A. We use the maps D, P from [Beh81, p. 223-224], and define $\sum_{i=1}^s p_i \otimes q_i \in E \otimes E$ so that it maps to $\mathrm{id} \in \mathrm{End}(E)$ under the isomorphism

$$\begin{aligned} E \otimes E &\to \mathrm{End}(E) \\ a \otimes b &\mapsto (e \mapsto a\langle e, b\rangle)\,. \end{aligned}$$

We note that for some reason P is only defined in [Beh81, p. 223-224] with $1 \leq k \leq r$, but in fact the same definition applies when $k = 0$. Further, we make the additional assumption that $\langle e^*, E\rangle = 0$ (One may justify this assumption using [Beh81, Proposition 2.1], where we take E to be what is notated as F in [Beh81, Proposition 2.1].) Now, we can check the claim on a basis, and so we are free to assume $u = e_1 \wedge \cdots \wedge e_r$. In general, we can write

$$v = \sum_{1 \leq i,j \leq r} \alpha_{ij} e_i \otimes e_j + \sum_{i=1}^{r} b_i e_i \otimes e_0 + \sum_{i=1}^{r} c_i e_i \otimes e^*.$$

We claim we may assume $c_i = b_i = 0$ for all i. Indeed, because the map D sends $e_i \wedge e_0 \mapsto e_i \otimes e_0$, we may quotient freely add any element in the image of D to v, and hence assume $b_i = 0$. Further, the condition $P(v) = 0$ forces $c_i = 0$, since the coefficient of e_i in the expansion of $P(v)$ is c_i. This uses that $\langle e_0, E\rangle = 0$, by construction of E above.

So, we write $v = \sum_{1 \leq i,j \leq r} \alpha_{ij} e_i \otimes e_j$. For $e = a_* e^* + \sum_{i=0}^{r} a_r e_r$, define $\mathrm{Proj}_E(e) = \sum_{i=1}^{r} a_r e_r, \mathrm{Proj}_{E_0}(e) = \sum_{i=0}^{r} a_r e_r\, \mathrm{Proj}_{e_0}(a) = a_0, \mathrm{Proj}_{e^*}(a) = a_*$. Let $w := \sum_{1 \leq i,j \leq r} \alpha_{ij}(e_i \cdot e_j)$. Here, $e_i \cdot e_j$ denotes multiplication in A.

We next claim

$$\mathrm{Proj}_{e^*}(w) = 0. \tag{10.5.7}$$

Using the assumption that $P(v) = 0$,

$$\begin{aligned} 0 = P(v) &= \sum_{1 \leq i,j \leq r} \left(\alpha_{ij}\langle e_0, e_j\rangle - \alpha_{ij}\langle e_i, e_j\rangle e_0\right) \\ &= -\sum_{1 \leq i,j \leq r} \alpha_{ij}\langle e_i, e_j\rangle e_0 \\ &= -\sum_{1 \leq i,j \leq r} \alpha_{ij}\langle e_0, e_i e_j\rangle e_0. \end{aligned}$$

Observe further that for any $y \in A$, $\langle e_0, y\rangle = \mathrm{Proj}_{e^*}(y)$ since $\langle e_0, E_0\rangle = 0$. It follows that

$$0 = -\sum_{1 \leq i,j \leq r} \alpha_{ij}\langle e_0, e_i e_j\rangle = -\sum_{1 \leq i,j \leq r} \alpha_{ij}\, \mathrm{Proj}_{e^*}(e_i e_j) = \mathrm{Proj}_{e^*}(w),$$

as claimed.

Using that $\mathrm{Proj}_{e^*}(w) = 0$, it follows from the definition of δ in [Beh81, p. 228],

$$\delta(v) = \sum_{1\le i,j\le r} \alpha_{ij} \left(x_i x_j - \Delta(\mathrm{Proj}_E(e_i \cdot e_j)) - \mathrm{Proj}_{e_0}(e_i \cdot e_j) \right).$$

Therefore,

$$\begin{aligned} \langle u, \delta(v)\rangle &= \langle e_1 \wedge \cdots \wedge e_r, \sum_{1\le i,j\le r} \alpha_{ij} \left(x_i x_j - \Delta(\mathrm{Proj}_E(e_i \cdot e_j)) - \mathrm{Proj}_{e_0}(e_i \cdot e_j) \right)\rangle \\ &= (-1)^{\binom{r}{2}} \sum_{1\le i,j\le r} \alpha_{ij} \left(x_i x_j - \Delta(\mathrm{Proj}_E(e_i \cdot e_j)) - \mathrm{Proj}_{e_0}(e_i \cdot e_j) \right). \end{aligned} \tag{10.5.8}$$

We wish to show this agrees with $\langle \delta(u), v\rangle$, so we next calculate $\delta(u) = \delta(e_1 \wedge \cdots \wedge e_r)$. Remember that via the definition from [Beh81, p. 227-228],

$$\delta(e_1 \wedge \cdots \wedge e_r) = \delta_A \left(e_1 \wedge \cdots \wedge e_r \otimes e^* + \sum_{t=1}^{s} \Delta(q_t) e_1 \wedge \cdots \wedge e_r \otimes p_j \right),$$

with δ_A as in (10.5.1). From the definition, we have

$$\begin{aligned} \delta(u) = &\sum_{i=1}^{r} \left((-1)^{i-1} \Delta(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes e^* + (-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes (e^* \cdot e_i) \right) \\ &+ \sum_{i=1}^{r} \sum_{t=1}^{s} \Big((-1)^{i-1} \Delta(q_t) \Delta(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes p_t \\ &\qquad + (-1)^i \Delta(q_t) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes p_t \Big) \Big). \end{aligned} \tag{10.5.9}$$

Calculating, the pairing of the second line of (10.5.9) with v, we find

$$\begin{aligned}
&\langle \sum_{i=1}^{r} \sum_{t=1}^{s} (-1)^{i-1} \Delta(q_t)\Delta(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes p_t \\
&\quad + (-1)^i \Delta(q_t) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes p_t, \sum_{1\le a,b\le r} \alpha_{ab} e_a \otimes e_b \rangle \\
&= (-1)^{\binom{r-1}{2}} \sum_{1\le i,b\le r} \sum_{t=1}^{s} \Big((-1)^{(i-1)+(r-i)} \Delta(q_t)\Delta(e_i)\alpha_{ib}\langle p_t, e_b\rangle + (-1)^{i+(r-i)}\Delta(q_t)\alpha_{ib}\langle p_t \cdot e_i, e_b\rangle \Big) \\
&= (-1)^{\binom{r}{2}} \sum_{1\le i,b\le r} \sum_{t=1}^{s} (\Delta(q_t)\Delta(e_i)\alpha_{ib}\langle p_t, e_b\rangle - \Delta(q_t)\alpha_{ib}\langle p_t \cdot e_i, e_b\rangle) \\
&= (-1)^{\binom{r}{2}} \sum_{1\le i,b\le r} \sum_{t=1}^{s} (\Delta(q_t)\Delta(e_i)\alpha_{ib}\langle p_t, e_b\rangle - \Delta(q_t)\alpha_{ib}\langle p_t, e_i \cdot e_b\rangle) \\
&= (-1)^{\binom{r}{2}} \sum_{1\le i,b\le r} \left(\alpha_{ib}\Delta(e_i)\Delta(e_b) - \alpha_{ib}\Delta(\mathrm{Proj}_E(e_i \cdot e_b)) \right) \\
&= (-1)^{\binom{r}{2}} \sum_{1\le i,b\le r} \left(\alpha_{ib} x_i x_b - \alpha_{ib}\Delta(\mathrm{Proj}_E(e_i \cdot e_b)) \right).
\end{aligned} \tag{10.5.10}$$

Hence, to conclude the proof, comparing (10.5.8) to (10.5.10), it suffices to check that the pairing of v with the right hand side first line of (10.5.9) yields

$$\begin{aligned}
&\left\langle \sum_{i=1}^{r} \Big((-1)^{i-1} \Delta(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes e^* \right. \\
&\qquad\qquad \left. + (-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes (e^* \cdot e_i), \sum_{1\le i,j\le r} \alpha_{ij} e_i \otimes e_j \Big) \right\rangle \\
&= (-1)^{\binom{r}{2}} \sum_{1\le i,j\le r} \alpha_{ij} \left(-\mathrm{Proj}_{e_0}(e_i \cdot e_j) \right).
\end{aligned}$$

For this computation, we first observe

$$\left\langle \sum_{i=1}^{r} \left((-1)^{i-1} \Delta(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes e^*, \sum_{1\le i,j\le r} \alpha_{ij} e_i \otimes e_j \right) \right\rangle = 0$$

because $\langle e^*, e_i\rangle = 0$ for all $1 \le i \le r$ by our assumption that $\langle e^*, E\rangle = 0$.

So, we conclude by verifying

$$\left\langle \sum_{i=1}^{r} (-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes (e^* \cdot e_i), \sum_{1\le i,j\le r} \alpha_{ij} e_i \otimes e_j \right\rangle = (-1)^{\binom{r}{2}} \sum_{1\le i,j\le r} \alpha_{ij} \left(-\mathrm{Proj}_{e_0}(e_i \cdot e_j) \right).$$

To start, observe

$$\begin{aligned}&\left\langle \sum_{i=1}^{r}(-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r \otimes (e^* \cdot e_i), \sum_{1 \le i,j \le r} \alpha_{ij} e_i \otimes e_j \right\rangle \\ &= (-1)^{\binom{r-1}{2}} \left(\sum_{1 \le i,j \le r} \alpha_{ij} (-1)^{i+(r-i)} \langle (e^* \cdot e_i), e_j \rangle \right) \\ &= (-1)^{\binom{r}{2}} \left(- \sum_{1 \le i,j \le r} \alpha_{ij} \langle (e^* \cdot e_i), e_j \rangle \right) \\ &= (-1)^{\binom{r}{2}} \left(- \sum_{1 \le i,j \le r} \alpha_{ij} \langle e^*, e_j \cdot e_j \rangle \right).\end{aligned}$$

Hence, it suffices to show

$$\sum_{1 \le i,j \le r} \alpha_{ij} \langle e^*, e_j \cdot e_j \rangle = \operatorname{Proj}_{e_0} \left(\sum_{1 \le i,j \le r} \alpha_{ij} e_j \cdot e_j \right). \tag{10.5.11}$$

In general, for $y \in A$, if $\operatorname{Proj}_{e^*}(y) = 0$, we claim $\langle e^*, y \rangle = \operatorname{Proj}_{e_0}(y)$. For such a y, writing $y = \sum_{i=0}^{r} a_i y_i$, and using that $\langle e^*, E \rangle = 0$ and $\langle e^*, e_0 \rangle = 1<$ we find $\langle e^*, y \rangle = a_0 = \operatorname{Proj}_{e_0}(y)$. This concludes the proof because we have checked $\operatorname{Proj}_{e^*}\left(\sum_{1 \le i,j \le r} \alpha_{ij} e_j \cdot e_j\right) = 0$ following (10.5.7), and so (10.5.11) holds. □

We now conclude the pairing defined above induces a well defined perfect pairing on the Behnke complex.

Theorem 10.5.11. *The pairing on the Behnke resolution induced by* (10.5.3) *together with* (10.5.5) *in the degrees* 0 *and r is a perfect pairing.*

Proof. First, the pairing is perfect on the Scheja Storch resolution by Proposition 10.5.5. This induces a pairing on the Behnke resolution by Lemma 10.5.9. For $r - 1 \le i \le 2$, the maps $M_i(E; A) \to M_{i-1}(E; A)$ on the Behnke resolution are induced by the maps $K_i(A) \to K_{i-1}(A)$ on the Scheja Storch resolution as in [Beh81, Theorem 3.3(i)]. Therefore, in these degrees $r - 1 \le i \le 2$, the pairings are compatible with the maps, as follows by compatibility with the maps for the Scheja Storch pairing, as is the content of Lemma 10.5.4, and the fact that these descend to Behnke's resolution by Lemma 10.5.9. Compatibility with the maps $\wedge^r \widetilde{E} \to M_{r-1}(E; A)$ and $M_1(E; A) \to S$ is verified in Proposition 10.5.10.

To conclude the proof, we only need to check the pairing is perfect. For this, by [Beh81, Lemma 3.1(i)], we know D is a split monomorphism and P is a split epimorphism.

For $1 \leq k \leq r-1$, it follows from Lemma 10.5.9 and [Beh81, Lemma 3.1(ii)] that $\operatorname{im} D_{r-k} \subset M_{r-k}(E;A)$ vanishes on $\ker P_k \subset K_k(E;A)$. Since $\operatorname{rk}\operatorname{im} D_{r-k} = \binom{r+1}{r+1-k} = \binom{r+1}{k} = \operatorname{rk}\operatorname{im} P_k$, we obtain a perfect pairing between $K_{r-k}(A)/\operatorname{im} D_{r-k}$ and $\ker P_k$. By applying this also with $r-k$ in place of k, we obtain a perfect pairing between $M_{r-k}(E;A) = \ker P_{r-k}/\operatorname{im} D_{r-k}$ and $M_k(E;A) = \ker P_k/\operatorname{im} D_k$. □

10.5.12 Deducing skew symmetry of the pairing

From the construction of the pairing above, we deduce the following:

Corollary 10.5.13. *Suppose* $\deg A = d = 2m$ *is even so that* $\deg E = d-2 = 2m-2$. *The pairing* $\langle , \rangle$ *constructed on the Behnke resolution induces a pairing of the middle term in degree* $e-1$ *of this resolution with itself. This pairing is skew symmetric when* m *is even (so* $d \equiv 0 \bmod 4$) *and symmetric when* m *is odd (so* $d \equiv 2 \bmod 4$).

Proof. Because the pairing on Behnke's resolution is induced by the pairing on the Scheja and Storch resolution, we can verify symmetry and skew symmetry on the Scheja and Storch resolution. Recall that elements in the middle degree of the Scheja and Storch resolution are given by a linear combination of an $(m-1)$st wedge power. Then, the claim amounts to the computation that

$$\begin{aligned}(e_1 \wedge \cdots \wedge e_{m-1}) \wedge (f_1 \wedge \cdots \wedge f_{m-1}) &= (-1)^{(m-1)(m-1)} (f_1 \wedge \cdots \wedge f_{m-1}) \wedge (e_1 \wedge \cdots \wedge e_{m-1}) \\ &= (-1)^{m-1} (f_1 \wedge \cdots \wedge f_{m-1}) \wedge (e_1 \wedge \cdots \wedge e_{m-1}).\end{aligned}$$

Hence, the pairing is skew symmetric when m is even and symmetric when m is odd. □

Remark 10.5.14. A related result describing the parity of the pairing for the case that the degree d is odd is discussed in [Ste19, Theorem 1.5].

10.5.15 Isotropicity of the Casnati-Ekedahl pairing

Recall that via Casnati-Ekedahl's structure theorem, given a finite locally free degree d cover $\rho : X \to Y$, there is a canonical exact sequence of vector bundles

$$0 \to \mathscr{G}_d(-d) \xrightarrow{\alpha_{d-2}} \mathscr{F}_{d-2}(-d+2) \xrightarrow{\alpha_{d-3}} \mathscr{F}_{d-3}(-d+3) \cdots \xrightarrow{\alpha_2} \mathscr{F}_2(-2) \xrightarrow{\alpha_1} \mathscr{O}_{\mathbb{P}}$$

for $\mathbb{P} = \operatorname{Proj}\operatorname{Sym}\mathscr{E}$ with $\mathscr{E} := \ker\left(\rho_*\omega_{X/Y} \to \mathscr{O}_Y\right)$. Note that these vector bundles live on $\mathbb{P}$. We refer to this sequence as $\mathscr{F}_\bullet$, where we include $\mathscr{G}_d(-d)$ as the term in degree $-d+1$ and $\mathscr{O}_{\mathbb{P}}$ as the term in degree 0. As we are using cohomological conventions for degrees, and this is a projective resolution, the complex is concentrated in degrees $[-d+1,0]$. Note that here we have chosen to change our indexing and notation from that in Theorem 10.2.2, as we find this re-indexing easier to think about.

Duality gives a canonical isomorphism

$$\mathscr{F}_{d-i}(-d+i) \xrightarrow{\phi_i} (\mathscr{F}_i(-i))^\vee \otimes \mathscr{G}_d(-d), \tag{10.5.12}$$

which is equivalent to the pairing

$$\mathscr{F}_{d-i}(-d+i) \otimes \mathscr{F}_i(-i) \to \mathscr{G}_d(-d) \tag{10.5.13}$$

$$(x,y) \mapsto \langle x,y\rangle_{d-i}. \tag{10.5.14}$$

Now, we have a canonical isomorphism $\mathscr{H}om_{\mathscr{O}_{\mathbb{P}}}(\mathscr{F}_\bullet, \mathscr{G}_d(-d))[d-1] \simeq \mathscr{F}_\bullet$ coming from duality. Again, we are using cohomological conventions, so the complex is always concentrated in degrees $[-d+1,0]$. The above isomorphism also induces the isomorphisms of (10.5.12). From this, we have commutativity of the diagram

$$\begin{array}{ccc} \mathscr{F}_{d-i-1}(-d+i+1) & \xrightarrow{\alpha_{d-i}} & \mathscr{F}_{d-i}(-d+i) \\ \downarrow{\phi_{i+1}} & & \downarrow{\phi_i} \\ \mathscr{F}_{i+1}^\vee(-i-1)\otimes\mathscr{G}_d(-d) & \xrightarrow{\alpha_{i+1}^\vee\otimes \mathrm{id}} & \mathscr{F}_i^\vee(-i)\otimes\mathscr{G}_d(-d). \end{array} \tag{10.5.15}$$

This is a restatement of the last aligned equation on [CE96, p. 447]. [There, the complex $\mathscr{N}_\bullet$ is used for what we are calling $\mathscr{F}_\bullet$, with a slightly different indexing convention.]

Our goal is the prove the following:

Proposition 10.5.16. *Suppose $d = 2e$ is even with $e \geq 3$. Then, the image of $\alpha_e : \mathscr{F}_{e+1}(-e-1) \to \mathscr{F}_e(-e)$ is isotropic with respect to the pairing $\mathscr{F}_e(-e)\times\mathscr{F}_e(-e) \to \mathscr{G}_d(-d)$ of* (10.5.13)*.*

This follows fairly formally from the following:

Lemma 10.5.17. *For v a section of $\mathscr{F}_{d-i}(-d+i)(U)$ and w a section of $\mathscr{F}_{i+1}(-i-1)(U)$ defined on the same open set U,*

$$\langle v, \alpha_i(w)\rangle_{d-i} = \langle \alpha_{d-i-1}(v), w\rangle_{d-i-1}.$$

Proof of Proposition 10.5.16. Let's see why the Lemma 10.5.17 implies Proposition 10.5.16. We apply the lemma in the case $i = e$. Then, if x and w are arbitrary sections of

$\mathscr{F}_{e+1}(-e-1)(U)$, then

$$\langle \alpha_e(x), \alpha_e(w) \rangle_e = \langle \alpha_{e-1}\alpha_e(x), w \rangle_{e-1} = \langle 0, w \rangle = 0,$$

using that $\mathscr{F}_\bullet$ is a complex so $\alpha_{e-1}\alpha_e = 0$. This shows the pairing is isotropic on the image of $\mathscr{F}_{e+1}(-e-1)$ in $\mathscr{F}_e(-e)$. □

It remains to prove Lemma 10.5.17.

Proof of Lemma 10.5.17. This essentially follows from (10.5.15) and a standard duality between vector spaces, as we now explain. In terms of diagrams it amounts to "commutativity" of

$$\begin{array}{ccccc} \mathscr{F}_{i+1}(-i-1) \otimes \mathscr{F}_{d-i-1}(-d+i+1) & \xrightarrow{\mathrm{id} \otimes \phi_{i+1}} & \mathscr{F}_{i+1}(-i-1) \otimes \mathscr{F}_{i+1}^\vee(i+1) \otimes \mathscr{G}_d(-d) & \longrightarrow & \mathscr{G}_d(-d) \\ \alpha_i \big\downarrow \quad \big\uparrow \alpha_{d-i-1} & & \alpha_i \big\downarrow \quad \big\uparrow \alpha_i^\vee & & \big\downarrow \\ \mathscr{F}_i(-i) \otimes \mathscr{F}_{d-i}(-d+i) & \xrightarrow{\mathrm{id} \otimes \phi_i} & \mathscr{F}_i(-i) \otimes \mathscr{F}_i^\vee(i) \otimes \mathscr{G}_d(-d) & \longrightarrow & \mathscr{G}_d(-d). \end{array} \tag{10.5.16}$$

The above notation is perhaps a bit nonstandard, so let us spell it out in symbols. Suppose v is a local section of $\mathscr{F}_{i+1}(-i-1)(U)$ and w is a local section of $\mathscr{F}_{d-i}(-d+i)(U)$. Then, we wish to show

$$\langle \alpha_i(v), w \rangle_i = \langle v, \alpha_{d-i-1}(w) \rangle_{i+1}.$$

Let $z = \phi_i(w)$. By commutativity of (10.5.15), it is equivalent to prove

$$\langle \alpha_i(v), z \rangle = \langle v, \alpha_i^\vee(z) \rangle.$$

Note that here the pairings are the natural ones: on the left we are using the pairing $\mathscr{F}_{i+1}(-i-1) \otimes (\mathscr{F}_{i+1}(-i-1))^\vee \to \mathscr{O}_\mathbb{P}$ and on the right we are using the pairing $\mathscr{F}_i(-i) \otimes (\mathscr{F}_i(-i))^\vee \to \mathscr{O}_\mathbb{P}$. The claim then follows from the definition of $\alpha_i^\vee$ in terms of α_i.

In the above analysis, we note it is crucial that $\mathrm{id} \otimes \phi_j$ carries the pairing $\langle , \rangle_j$ to the duality pairing $\langle , \rangle$. This follows as it was in fact implicit in the definition of $\langle , \rangle_j$ from (10.5.12) and (10.5.13). □

10.5.18 The degree 6 pairing

Let's calculate the pairing of Definition 10.5.8 explicitly in the degree 6 case.

Proposition 10.5.19. *The explicit symmetric pairing in the resolution of a degree* 6 *cover is given by a sum of* 8 *hyperbolic spaces.*

Proof. By [Beh81, Lemma 3.1(ii)] we wish to compute the pairing on the middle homology of

$$\wedge^3 E \xrightarrow{\Delta} \wedge^2 E \otimes E \xrightarrow{P'} E \tag{10.5.17}$$

where Δ sends $x \wedge y \wedge z \mapsto y \wedge z \otimes x - x \wedge z \otimes y + x \wedge y \otimes z$ and P' sends $x \wedge y \otimes z \mapsto -\langle x, z\rangle y + \langle y, z\rangle x$. For F as in [Beh81, Proposition 2.1], we may choose $E = F$, meaning E is orthogonal to e_0 and e^*. We can also assume a, b, c, d is an orthonormal basis for E under the pairing $\langle , \rangle$. In this basis, we find that a basis for $\wedge^2 E \otimes E$ is given by elements of the form $x \wedge y \otimes z$ with $\{x, y, z\} \subset \{a, b, c, d\}$ and $x \neq y$. The kernel of P' is spanned by

$$\{x \wedge y \otimes z\}_{\#\{x,y,z\}=3}$$

together with the elements

$$a \wedge b \otimes b - a \wedge c \otimes c,$$
$$a \wedge b \otimes b - a \wedge d \otimes d,$$
$$a \wedge b \otimes a + b \wedge c \otimes c,$$
$$a \wedge b \otimes a + b \wedge d \otimes d,$$
$$a \wedge c \otimes a - b \wedge c \otimes b,$$
$$a \wedge c \otimes a + c \wedge d \otimes c,$$
$$c \wedge d \otimes c - b \wedge d \otimes b,$$
$$c \wedge d \otimes c - a \wedge d \otimes a.$$

Now, when quotienting by the image of the map D, we can choose as a basis, these above 8 elements, together with the 8 elements pairing to 1 with the second terms of each of the above 8 elements. That is, up to signs, we can take the above 8 elements together with the following 8 elements as a basis:

$$\begin{aligned} &b \wedge d \otimes c, \\ &b \wedge c \otimes d, \\ &a \wedge d \otimes c, \\ &a \wedge c \otimes d, \\ &a \wedge d \otimes b, \\ &a \wedge b \otimes c, \\ &a \wedge c \otimes b, \\ &b \wedge c \otimes a. \end{aligned}$$

We see that the resulting quadratic form is 8 copies of the hyperbolic plane, corresponding to these 8 terms in the above two aligned equations. □

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