Torsion line bundles on finite covers

AARON LANDESMAN

We outline a new series of moduli spaces whose points are closely related to n-torsion line bundles on degree d covers of a base B. The story is especially clean when $d = 2$, in which case these spaces parameterize n-coverings of generically singular genus 1 curves over B, as described in [1]. For example, when $B = \mathbb{P}^1$, we construct a moduli space whose $\mathbb{P}^1$ points parameterize n-torsion line bundles on hyperelliptic curves.

This investigation is motivated by the Cohen-Lenstra heuristics in number theory, which work in the case that B is the spectrum of the integers, and describe the average number of n-torsion elements in class groups of quadratic number fields. It is a big open question in arithmetic statistics to count the asymptotic number of these n-torsion elements in quadratic fields, and it would be quite interesting if one were able to use these moduli spaces to approach that problem. A simple-to-state consequence of our approach is the following: under the correspondence between quadratic forms and line bundles on spectra of rings of integers of quadratic fields, a quadratic form q corresponds to an n-torsion line bundle if and only if there exists a degree n polynomial whose resultant with q is ± 1.

We now describe our moduli space in the case $d = 2$. Let S denote the secant variety to the rational normal curve in $\mathbb{P}^n$, and let $U \subset S$ denote the open subscheme where one removes the rational normal curve. Then, the relevant moduli space is the quotient of U by PGL_2, acting as automorphisms of the rational normal curve.

Let us explain why the secant variety to the rational normal curve is related to n-torsion line bundles on degree 2 covers. Starting with a degree 2 cover $g : X \to B$ and an n-torsion line bundle L on X, we produce an embedding $X \to \mathbb{P}(g_* L)$. The condition that this is n-torsion amounts to the existence of an isomorphism $L^{\otimes n} \simeq \mathcal{O}_X$. To understand the nth tensor power of L, we compose with the n-Veronese embedding to get a map $X \to \mathbb{P}(g_* L) \to \mathbb{P}(\mathrm{Sym}^n(g_* L))$. We have a surjection $\mathrm{Sym}^n(g_* L) \to g_* L^{\otimes n} \simeq g_* \mathcal{O}_X \to \mathrm{coker}(\mathcal{O}_B \to g_* \mathcal{O}_X)$ which picks out a line M in $\mathbb{P}(\mathrm{Sym}^n g_* L)$ and a point p on M which does not meet X. In fibers, we can think of X as two points on the rational normal curve. These two points span the line M, and the point p on M is then a point on the secant variety to the rational normal curve.

After explaining the above construction, we investigated in the case $d = 2$ and $n = 3$. For this, we reviewed the classical correspondence between 3-torsion line bundles on degree 2 covers and degree 3 covers. We used our construction to understand this classical correspondence as a corollary of the fact that the secant variety to the twisted cubic is all of $\mathbb{P}^3$.

We then proceeded to explain an analogous construction in degree $d = 3$. Here, instead of the moduli space being the 2-secant variety to the rational normal curve in $\mathbb{P}^n$, the relevant space was the space of lines contained in the 3-secant variety to the n-Veronese surface. Following this, we investigated the case $d = 3$ and $n = 2$, which recovers the classical Recillas correspondence. We used our construction to understand this as a corollary of the fact that any line in $\mathbb{P}^5$ not meeting the 2-Veronese surface lies on a unique 3-Secant 2-plane to the 2-Veronese surface.

To conclude, we returned to the case that $d = 2$ and gave two other equivalent descriptions of our moduli space. First, we exhibited it as a moduli space for families of generically singular genus 1 curves with a degree n line bundle and geometrically integral fibers. Second, we exhibited it as a moduli space of families of smooth divisors in the linear system $\mathcal{O}_Y(1)$, for $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ the Hirzebruch surface.

References

[1] A. Landesman, *A geometric approach to the Cohen-Lenstra heuristics*, https://arxiv.org/abs/2106.10357v1, (2021).