NOTES FOR TALK ON GEOMETRIC LOCAL SYSTEMS ON VERY GENERAL CURVES AND ISOMONODROMY

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For today, we'll work over the complex numbers.

Question 0.1. Given $X \to Y$ a map of complex varieties what restrictions are there on the topology of f?

Question 0.2. Fix an integer $g \geq 2$. What is the smallest integer $h \geq 2$ for which the generic genus g curve, i.e., the generic fiber of $\mathscr{M}_{g,1} \to \mathscr{M}_g$, has a non-constant map to $\mathscr{M}_h$?

There is an upper bound of about $2g * g^{2g}$ coming from the Kodaira parshin trick. I'll explain this later.

What is new is the lower bound:

Theorem 0.3 (Theorem A, Landesman-Litt). *Let C be an (analytically) very general genus g curve. Any abelian scheme over C of dimension $r < \sqrt{g+1}$ is isotrivial (all fibers are isomorphic). In particular, there is no non-isotrivial curve over C of genus $< \sqrt{g+1}$*

Remark 0.4. This also holds over $C - \{x_1, \ldots, x_n\}$ for $(C, x_1, \ldots, x_n)$ an analytically very general curve.

0.1. **Riemann-Hilbert correspondence.** Throughout the talk, it will be useful to move interchangably between local systems, representations, and vector bundles with flat connectoin. Recall a complex local system $\mathbb{L}$ on a complex variety X of rank r is a sheaf locally isomorphic to $\mathbb{C}^r$. We can also think of it as a map $\pi_1(X, x) \to \mathrm{GL}_r(\mathbb{C})$ describing the monodromy or a vector with flat connection $\mathrm{id} \otimes d : \mathbb{L} \otimes_{\mathbb{C}} \mathscr{O}_X \to \mathbb{L} \otimes_{\mathbb{C}} \Omega_X$. The inverse map sends $(V, \nabla) \mapsto \ker \nabla$.

The above follows from the following more general result:

Definition 0.5. A local system $\mathbb{L}$ on X is *of geometric origin* if there is an open subset $U \subset X$ and a family of varieties $f : Z \to U$ so that $\mathbb{L}$ is a direct summand of $R^i f_* \mathbb{C}$ for some i.

Theorem 0.6 (Theorem B, Landesman-Litt). *If C is a very general curve, and $\mathbb{V}$ is a local system on C of geometric origin and* $\operatorname{rk} \mathbb{V} < 2\sqrt{g+1}$ *then $\mathbb{V}$ has finite monodromy.*

Theorem 0.6 implies Theorem 0.3 (Theorem B implies Theorem A). Use $R^1 f_* \mathbb{Z}$, which has rank $2r < 2\sqrt{g+1}$. This shows the family has finite monodromy. One can then use the theorem of the fixed part to deduce the corresponding family of abelian varieties factors through a point of $\mathscr{A}_g$. $\square$

To state the more general result, I need to recall the notion of a polarizable complex variation of hodge structure PVHS.

Definition 0.7. A complex PVHS on X is $(V, V^{p,q}, D, \psi)$ where V is a holomorphic vector bundle on X, $V^{p,q}$ are C^∞ subbundles, and $D = \partial + \overline{\partial} + \theta + \overline{\theta}$ is a flat connection sending

$$V^{p,q} \mapsto V^{p,q} \otimes \Omega^{1,0} \oplus V^{p,q} \otimes \Omega^{0,1} \oplus V^{p+1,q-1} \otimes \Omega^{1,0} \oplus V^{p-1,q+1} \otimes \Omega^{0,1}$$

Further, ψ is a flat Hermitian form $\psi : V \times V \to \mathbb{C}$ so that $(-1)^p \psi|_{V^{p,q}}$ is positive definite and $\psi(V^{p,q}, V^{p',q'}) = 0$ for $(p,q) \neq (p',q')$.

Theorem 0.8 (Theorem C, Landesman-Litt)**.** *Suppose K is a number field with ring of integers $\mathscr{O}_K$ and $\mathbb{V}$ is an $\mathscr{O}_K$ local system on an analyically very general C so that $\mathbb{V} \otimes_\iota \mathbb{C}$ underlies a PVHS for each embedding $\iota : \mathscr{O}_K \to \mathbb{C}$. If $\operatorname{rk} V < 2\sqrt{g+1}$, then $\mathbb{V}$ has finite monodromy.*

Thm C implies Thm B. If $\mathbb{V}$ comes from geometry, it is a sub of $R^1 f_* \mathbb{C}$ for some map f. This has an $\mathscr{O}_K$ structure because $R^1 f_* \mathbb{C}$ has a $\mathbb{Z}$ structure. It underlies a PVHS on the open set U because subs of PVHS acquire a PVHS. And since we can extend the local system to all of C, we can also extend the PVHS, by a theorem of Schmid. $\square$

0.2. **Relevance to number theory.**

There are various reasons these results might be useful in number theory, perhaps more as a method for ruling out potential strategies.

Example 0.9. A very general curve of genus $g \geq 1$ does not map nonconstantly to a Hilbert modular variety associated to K, where $\deg K$ over $\mathbb{Q}$ is d.

If H_K is a Hilbert modular variety for a number field K of degree d over $\mathbb{Q}$, a map from a curve C to H_K is the same as a family of abelian varieties $f : A \to C$ of relative dimension d with an action of $\mathscr{O}_K$. Equivalently, it is a rank $2d$ $\mathbb{Z}$ local system $R^1 f_* \mathbb{Z}$ with an action of $\mathscr{O}_K$, or equivalently a projective $\mathscr{O}_K$ module of rank 2 over $\mathscr{O}_K$. By passing to a further finite extension, we can assume this is a free module and then apply our theorem.

This implies the same for curves over finitely generated fields, and I'd expect the same for curves over number fields, though don't have an idea how to prove it. So our result shows a very general curve of genus at least 1 has no such maps.

(If we ignored the $\mathscr{O}_K$ structure here, and applied our result to the underlying $\mathbb{Z}$ local system, we would only get general curves of genus $g \geq d^2$.) We get similar statements with very general marked curves.

By spreading out, there are many curves over finitely generated fields with no such maps, though I'd expect the same holds over number fields.

Another related idea Sasha relayed to me was that he was hoping that the property of having geometric origin might generize in families. Deep results in the langlands program actually show that over $\mathbb{F}_q$, local systems do all come from geometry. With some work, one can also construct such local systems lifting these with infinite monodromy. But then, morally, our theorem suggests that for sufficiently general such curves, there won't be any with infinite monodromy.

Example 0.10. We don't know how sharp our upper bound $2\sqrt{g+1}$ is in the main result, but some upper bound is necessary, since there are $\mathbb{Z}$ pvhs on every curve of rank roughly $2g6^{2g}$, given by the Kodaira Parshin trick:

Take a curve C of genus g, for any $p \in C$, let $D^p := \cup_i D_i^p$ denote the set of degree 3 covers of C branched only at p. As p varies, we can define PVHS whose fiber over p is $H^1(D^p, \mathbb{Z})$. This has rank roughly $2g6^{2g}g$ since the covers D_i^p are in bijection with maps $\pi_1(D - p) \to S_3$. One can verify these families are not isotrivial using that there are only finitely many maps between two given curves of genus at least 2.

This Kodaira Parshin trick also comes up number theory; we saw it in Alex Betts' talk a couple weeks ago, and it's the starting point of Falting's proof of Mordell's conjecture.

1. PROOF OF THEOREM 0.8

To show finiteness of monodromy, we will show the monodromy is compact and discrete. Discreteness follows from discreteness of $\mathscr{O}_K \to \prod_{\iota:\mathscr{O}_K \to \mathbb{C}} \mathbb{C}$. Compactness will hold if we show the VHS $\mathbb{V}$ has a single nonzero piece of its Hodge filtration, as it will then preserve the associated positive definite hermitian form, so the image of monodromy lies in a unitary group preserving the form. Nonabelian hodge theory connects the Harder-Narasimhan filtration of $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_C$ to the Hodge filtration implies that if $\mathbb{V} \otimes \mathscr{O}_C$ is semistable, $\mathbb{V}$ has only one part of its Hodge filtration. So we reduce to:

Theorem 1.1 (Theorem D, Landesman-Litt)**.** *Fix an irreducible representation $\rho : \pi_1(\Sigma_g) \to \mathrm{GL}_r(\mathbb{C})$. If $r < 2\sqrt{g+1}$, then for a general curve C, the associated vector bundle E is semistable.*

Moreover, if C has any rank, the slopes of the HN filtered parts of E all have slope differing by at most 1.

Corollary 1.2. *The Tschirnhausen bundle of a generic degree d cover of $\mathbb{P}^1$ has consecutive summands. (The same statement holds for the associated syzygy bundles, and for other irreducible representations associated to finite covers of $\mathbb{P}^1$.)*

Proof. Choose a representation of $\pi_1(\mathbb{P}^1 - \{x_1, \ldots, x_t\})$ with finite image. This corresponds to a Hurwitz space $\mathscr{H}$ parameterizing certain covers $f : C \to \mathbb{P}^1$ branched over at most t points. Over $\mathbb{P}^1$, we can construct a vector bundle E in families (say, the Tschirnhausen bundle $f_*\mathscr{O}_C/\mathscr{O}_{\mathbb{P}^1}$) and the HN pieces of E have slopes differing by degree at most 1. This means that we we write $E = \oplus_i \mathscr{O}(a_i)^{b_i}$ the a_i with all $b_i \neq 0$, all a_i are consecutive. $\square$

Recall a vector bundle is semistable if there are no subbundles $U \subset V$ with $\deg U/\operatorname{rk} U > \deg V/\operatorname{rk} V$.

Remark 1.3. In fact the above was "proven" by Biswas Heu and Hurtubeise without the hypothesis on the rank of E. We were able to use their result to prove a longstanding conjecture in geometric topology, Ivanov's conjecture. Unfortunately, the proof had an error, a counterexample is given by the Kodaira Parshin trick. The above theorem was an attempt to salvage some part of their result.

2. PROOF OF THEOREM 1.1

2.1. **Proving the HN part.** Let's explain the proof of the final statement of Theorem 1.1. Let's focus on the case that E_ρ has rank 2. For simplicity, let's say $E_\rho = \mathscr{O}(a) \oplus \mathscr{O}(a+2)$. We want to show it's possible to deform the cover of curves so that E_ρ becomes $\mathscr{O}(a+1) \oplus \mathscr{O}(a+1)$. One can interpret this deformation theoretically. Suppose f is branched along $D \subset \mathbb{P}^1$. Suppose that any way of deforming f preserves the splitting type. It turns out one can interpret this condition deformation theoretically as a nonzero map $T_{\mathbb{P}^1}(-D) \to \operatorname{Hom}(\mathscr{O}(a+2), \mathscr{O}(a)) \simeq \mathscr{O}(-2)$ which vanishes on H^1. That is, it induces the 0 map $H^1(T(-D)) \to H^1(\mathscr{O}(-2))$.

Applying Serre duality, we get a nonzero map $\alpha : \mathscr{O} \to \omega_{\mathbb{P}^1}^{\otimes 2}(D)$ which induces the 0 map on $H^0(\mathscr{O}) \to H^0(\omega^{\otimes 2}(D)$. However, any nonzero map as above also induces a nonzero map on H^0, because it sends any nonzero section s of $\mathscr{O}$ to the nonzero section $\alpha \circ s$. This is a contradiction, which implies it was possible to deform f to make the bundle balanced.

2.2. **Proving the semistability part.** Let's consider the case that E is not semistable with rank $< 2\sqrt{g+1}$, but has a positive degree subbundle $F \subset E$ with quotient $Q := E/F$. The idea is to show it is possible to take a deformation of our curve C so that F deforms away. If it is possible to deform F along with E, deformation theory tells us the map $H^1(C, T_C) \to$

$H^1(C, \mathrm{Hom}(F, Q))$ is 0. Serre dualizing gives $H^0(C, \mathrm{Hom}(F, Q)^\vee \otimes \omega_C) \to H^0(\omega_C^{\otimes 2})$. In other words, all global sections of $\mathrm{Hom}(F, Q)^\vee \otimes \omega_C$ factor vanish in $\omega_C^{\otimes 2}$ and hence factor through a subbundle of $\mathrm{Hom}(F, Q)^\vee \otimes \omega_C$ (by irreducibility of the connection). Note that the $\deg F > 0, \deg Q < 0$ so $V := \mathrm{Hom}(F, Q)^\vee \otimes \omega_C$ has $\deg V / \operatorname{rk} V > 2g - 2$. Using AM-GM, we also see $\operatorname{rk} V < g + 1$.

Proposition 2.1. *If V is a semistable vector bundle with* $\deg V / \operatorname{rk} V \geq 2g - 2$ *so that all its global sections factor through a subbundle, then* $\operatorname{rk} V \geq g$ *(with strict inequality on the slope implying a strict inequality on the rank).*

Proof. Suppose $U \subset V$ induces an isomorphism on H^0. This follows from clifford's theorem for vector bundles, which says $h^0(C, U)$ is not too big.

More precisely, $h^0(C, U) \leq \deg U/2 + \operatorname{rk} U$. The proof is easier to digest in the case $\deg V / \operatorname{rk} V = 2g - 2$ and $\operatorname{rk} U = \operatorname{rk} V - 1$. In this case,

$$\deg V + (1 - g) \operatorname{rk} V \leq h^0(V) = h^0(U) \leq \operatorname{rk} U + \deg U/2 \leq (\operatorname{rk} V - 1) + (g - 1)(\operatorname{rk} V - 1).$$

Solving for $\operatorname{rk} V$ gives $\operatorname{rk} V \geq g$. □

3. A CONJECTURE OF ESNAULT-KERZ

Definition 3.1. The group $\mathrm{Hom}(\pi_1(X, x), \mathrm{GL}_r(\mathbb{C}))$ can be identified with a complex variety, the *framed character variety of X, $M_r(X)$* .

Example 3.2. Suppose $X = \mathbb{P}^1 - \{0, 1, \infty\}$. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$. Hence, $M_r(X) = \mathrm{Hom}(\mathbb{Z} * \mathbb{Z}, \mathrm{GL}_r(\mathbb{C})) = \mathrm{GL}_r \times \mathrm{GL}_r$.

Conjecture 3.3 (Esnault-Kerz, Budur-Wang)**.** Let X be a normal complex variety. For any $r \geq 1$, the the set of local systems coming from geometry is dense in the framed character variety $M_r(X)$.

Remark 3.4. This was originally stated for the character variety, which is the quotient of the framed character variety by the conjugation action of $\mathrm{GL}_r(\mathbb{C})$, in place of the framed character variety. This version for the framed character variety is potentially slightly stronger, but is equivalent in the case X is a curve. We state it this way in these notes to slightly simplify the presentation.

3.1. **Motivation.**

(1) One of the main sources of motivation for this conjecture is that the analog for curves over finite fields is true. That is, if we assume $\overline{\rho} : \pi_1(X) \to \mathrm{GL}_r(\mathbb{F}_q)$ is a fixed geometrically irreducible representations, then we can consider the set of semisimple $\rho : \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ where ρ reduces to $\overline{\rho}$, which has a natural Zariski topology on it. The locus of such representations/local systems of geometric origin are

dense. The proof of this is very difficult, and rests on Gaitsgory's resolution of de Jong's conjecture, [which says an arithmetic representation of the fundamental group of a variety over a finite field has finite geometric monodromy] as well as results of Lafforgue in the langlands program to say that representations with finite orbit under Frobenius actually come from geometry (they live in the cohomology of a certain space of shtukas).

(2) This is known in the easy case of 1-dimensional representations. (More generally, it is known to hold if one restricts to representations whose image has Zariski closure an abelian group.)

(3) Every irreducible local system has quasi-unipotent monodromy and Esnault-Kerz showed the locus of reps with quasi-unipotent monodromy is dense.

Theorem 3.5 (Landesman-Litt)**.** *The above conjecture is false for X an analytically very general genus g curve and* $1 < r < 2\sqrt{g+1}$*.*

Idea using Theorem 0.6 (Theorem B). By Theorem 0.6, and its corollary, we know any local system coming from geometry has finite monodromy. So we just want to show a local system with finite monodromy lies in a proper Zariski closed subset of the character variety.

The key input is the following theorem of Jordan: Given any fixed integer r, there exists an integer s so that for any finite subgroup $G \subset \mathrm{GL}_r(\mathbb{C})$ G has an index $s(r)$ abelian subgroup.

We then have that for any two elements $g, h \in \pi_1(X)$, and a finite image representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$, g^s commutes with h^s, i.e., $g^s h^s (g^s)^{-1}(h^s)^{-1}$. This defines a hypersurface in the framed character variety. On the other hand, it is easy to make representations with infinite image not satisfying this relation when $r > 1$, so this relation defines a proper Zariski closed (i.e., non-dense) subset of the character variety. □

REFERENCES