

A COINCIDENCE BETWEEN GENUS 1 CURVES AND CANONICAL CURVES OF LOW DEGREE

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1. A COINCIDENCE IN DEGREES 3, 4, AND 5

Describing equations for the low degree canonical embeddings are a very classical element of algebraic geometry. Indeed, it is well known that

- (1) a general genus 2 canonical curve is hyperelliptic,
- (2) a general genus 3 canonically embedded curve is a plane quartic,
- (3) a general genus 4 canonically embedded curve is the intersection of a quadric and a cubic,
- (4) a general genus 5 canonically embedded curve is the intersection of three quadrics,
- (5) a general genus 6 canonically embedded curve is the complete intersection of a quadric and quintic del Pezzo surface (where the quintic del Pezzo is itself an intersection of $G(2, 5)$ under the Plücker embedding into \mathbb{P}^9 with a linear subspace $\mathbb{P}^4 \subset \mathbb{P}^9$),
- (6) A general genus 7 curve is a linear section of the orthogonal grassmannian $\mathbb{P}^6 \cap OG(5, 10) \subset \mathbb{P}^{15}$,
- (7) A general genus 8 curve is a linear section of the $G(2, 6) \subset \mathbb{P}^{14}$ under the Plücker embedding with a plane $\mathbb{P}^7 \subset \mathbb{P}^{14}$.

The cases up to genus 5 are quite classical, while descriptions of general canonically embedded curves of genus between 6 and 9 are described in Mukai's series of papers [M⁺92] (genus 6 and 8), [Muk95] (genus 7), and [Muk10] (genus 9). On the other hand, there is a similar classification of genus 1 curves, embedded by divisors of positive degree. In particular, a general

- (1) degree 1 genus 1 curve is contracted to a point,
- (2) degree 2 genus 1 curve is mapped 2 to 1 onto \mathbb{P}^1 ,
- (3) degree 3 genus 1 curve is mapped into \mathbb{P}^2 as a plane cubic,
- (4) degree 4 genus 1 curve is the intersection of two quadrics in \mathbb{P}^3 ,

d	3	4	5
Genus 1 curve of degree d in \mathbb{P}^{d-1}	C	$Q_1 \cap Q_2$	$G(2,5) \cap \mathbb{P}^4$
Canonical curve of genus d + 1 in \mathbb{P}^d	$Q \cap C$	$Q \cap Q_1 \cap Q_2$	$Q \cap G(2,5) \cap \mathbb{P}^5$

TABLE 1. Descriptions of general Genus 1 Curves and Canonical curves in low degree. Here, H_i denote hyperplanes, Q_i denote quadrics, and C denotes a cubic. Also, $G(2,5) \cap \mathbb{P}^j$ refers to the intersection of $G(2,5) \hookrightarrow \mathbb{P}^9$ under the Plücker embedding with a general plane $\mathbb{P}^j \subset \mathbb{P}^9$

d	3	4	5
Genus 1 curve of degree d in \mathbb{P}^d	$H \cap C$	$H \cap Q_1 \cap Q_2$	$H \cap G(2,5) \cap \mathbb{P}^5$
Canonical curve of genus d + 1 in \mathbb{P}^d	$Q \cap C$	$Q \cap Q_1 \cap Q_2$	$Q \cap G(2,5) \cap \mathbb{P}^5$

TABLE 2. A more uniform description of general Genus 1 Curves and Canonical curves in low degree

- (5) degree 5 genus 1 curve is the intersection of $G(2,5) \subset \mathbb{P}^9$ under the Plücker embedding with some $\mathbb{P}^4 \subset \mathbb{P}^9$.

For proofs of the descriptions of lower genus canonical curves and for proofs or references for the descriptions for genus 1 curves of higher degree, see [Vak, Chapter 19].

A close comparison of the above descriptions yields a striking coincidence. When $d \in \{3,4,5\}$, the genus $d + 1$ canonical model is quite similar to the degree d genus 1 curve, except the genus $d + 1$ canonical model (which is of degree $2d$), has an extra quadric in its defining equations. This coincidence is summarized in Table 1. However, it is a bit strange that the degree $d + 1$ canonical model lies in \mathbb{P}^d while the degree d genus 1 curve lies in \mathbb{P}^{d-1} . We can rectify this by simply viewing the genus 1 canonical curve in \mathbb{P}^{d-1} as lying in a hyperplane $\mathbb{P}^{d-1} \subset \mathbb{P}^d$. We then can view both genus 1 curves of degree d and canonical curves of genus $d + 1$ as lying in \mathbb{P}^d . The corresponding models are described in Table 2. Here, the coincidence is even more striking: we can go from the canonical curve to the genus 1 curve, by replacing the first quadric with a hyperplane.

1.1. Goals. In the rest of this note, we will give an explanation why this phenomena occurs. We also show how to partially extend this

coincidence to the more degenerate cases of degrees 1 and 2, and how this appears in higher genera as well.

1.2. Structure and summary. The remainder of this note is structured in a rather long winded fashion to mimic the fashion in which I came to better understand this coincidence. This leads to a rather lengthy discussion of the coincidence.

As a quick summary of the reason for the coincidence, in the case a canonical curve is defined as the complete intersection of a quadric and something else, one can degenerate the quadric to two hyperplanes, to obtain a union of two genus 1 curves meeting at $d - 2$ points in \mathbb{P}^d . Or, as an even better explanation, when a general canonical curve lies on a del Pezzo surface, then the canonical curve will be a complete intersection of that surface with a quadric, while a genus 1 curve will be a hyperplane section of that surface.

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2. A PARTIAL EXPLANATION OF THE COINCIDENCE

We now present an explanation of the coincidence described in section 1. Suppose we know that a canonical curve C of genus g is a complete intersection of a quadric Q and some surfaces S . Note that S is necessarily a surface of degree $g - 1$ inside \mathbb{P}^{g-1} . In other words, it is a surface of almost minimal degree, and the smooth such surfaces are called del Pezzo surfaces. Note that this is the case when the genus of C is 4, 5, or 6.

Then, consider the degeneration of quadric Q to the union of two hyperplanes, H_1 and H_2 . We obtain that $H_i \cap S$ is a curve of degree $g - 1$. By Riemann-Roch $H_i \cap S$ has genus at least 1. Further, we claim $H_i \cap S$ has genus 1. To see this, note that $H_1 \cap H_2 \cap S$ is a collection of $g - 1$ points. So, we have

$$g((H_1 \cup H_2) \cap S) = g(H_1 \cap S) + g(H_2 \cap S) + (g - 1) - 1.$$

Now, by assumption, $Q \cap S$ has genus g , and so $(H_1 \cup H_2) \cap S$, being a flat limit of genus g curves, also has genus g . Since $H_i \cap S$ cannot have genus 0 as it is nondegenerate, if H_1, H_2 are chosen generally, we obtain that $g(H_i \cap S) \geq 1$. Then, the above inequality implies that $g(H_i \cap S) = 1$. So, in the case that a canonical curve is of the form

$Q \cap S$, for S a surface, we will have that a hyperplane section of S is a genus 1 by degenerating Q to a union of two hyperplanes. In this case the canonical curve degenerates to two genus 1 curves meeting at $g - 1$ points.

Conversely, suppose we can express a general genus 1 curve $E \subset \mathbb{P}^{d-1}$ as $E = S \cap H$, with $S \subset \mathbb{P}^d$. Then, if we take two hyperplanes, H_1 and H_2 in \mathbb{P}^d , we have $S \cap (H_1 \cup H_2)$ is a canonical (albeit reducible) genus $d + 1$ curve, which is the union of two genus 1 curves meeting at d points. When we smooth the reducible quadric $H_1 \cup H_2$ into a smooth quadric, we obtain a canonical curve.

2.1. The description is still not completely satisfactory. So, we have just seen how to go between genus 1 curves of degree d and canonical curves of degree $d + 1$. However, this still doesn't quite explain why a general canonical curve of genus 4, 5, and 6 have an analogous model to that of a genus 1 curve. One might argue that this holds because of a dimension count, or because we know in these degrees a general canonical curve is the complete intersection of a quadric and a del Pezzo surface. However, this is not completely satisfactory because this property of the canonical curve was not explained in terms of the genus 1 curve, and we pose the following vague question.

Question 2.1. Why is it the case that in genera 4, 5, and 6 a general genus g canonical curve has a similar description to that of a genus 1 curve, and not that there is just some closed locus of canonical curves which have a similar description to a genus 1 curve? Here, we mean that we replace a hyperplane in the description of the genus 1 curve with a quadric in the description of the canonical curve, as detailed in Table 2.

3. AN EXTENSION OF THE COINCIDENCE TO DEGREE 2

Based on correspondence between genus 1 curves and canonical curves described in section 2, it may seem that one should not be able to extend the description to genus 3. Recall that a general genus 3 curve is a plane quartic. However, there are also hyperelliptic genus 3 curves mapping 2 to 1 onto a plane conic. We will now show how degree 2 hyperelliptic curves correspond, in an analogous sense to section 2, with genus 3 hyperelliptic curves.

Start with a genus 3 hyperelliptic curve. Recall that in section 2, when we had a genus g curve, we degenerated it into two genus 1 curves meeting at $g - 1$ points spanning \mathbb{P}^{g-3} . This degeneration was accomplished by specializing a quadric defining the genus g

curve into a union of two hyperplanes. Here, we are looking to degenerate a hyperelliptic genus 3 canonical curve into two genus 1 degree 2 curves, “meeting at 2 points in \mathbb{P}^0 .” Since \mathbb{P}^0 is only a single point, one might wonder what this means. One’s first guess might be that it should be a double point, but it is actually something different:

If we have a hyperelliptic curve mapping 2 to 1 onto a conic $Q \subset \mathbb{P}^2$, one can degenerate the conic into a union of two lines $L_1 \cup L_2$. The resulting degeneration will map 2 to 1 onto these two lines. In particular, the preimage of $L_1 \cap L_2$ will be two points both mapping to $L_1 \cap L_2 \in \mathbb{P}^2$. These are the “two points in \mathbb{P}^0 ” and they are actually two points mapping onto the same point in \mathbb{P}^0 .

3.1. An alternate view of the degree 2 degeneration with del Pezzo surfaces. Above, we haven’t yet precisely described the degeneration of a genus 3 hyperelliptic curve into two genus 1 hyperelliptic curves meeting at two points. To do this, we bring into play the degree 2 del Pezzo surface. This will be useful both in precisely describing the degeneration for genus 3, and for giving insight into the degeneration for genus 2.

Abstractly, this del Pezzo surface S is \mathbb{P}^2 blown up at 7 points. Its anticanonical divisor maps it 2 to 1 onto \mathbb{P}^2 , and is branched along a plane quartic.

Now, consider the anticanonical map for a degree 2 del Pezzo surface S mapping $\pi : S \rightarrow \mathbb{P}^2$. For $Q \subset \mathbb{P}^2$ a conic, we have $C := \pi^{-1}(Q)$ is a hyperelliptic genus 3 curve, and $\pi_C : C \rightarrow \mathbb{P}^2$ is the canonical map. One can see this is genus 3 because π is branched along a plane quartic T , and so $Q \cap T$ has 8 points by Bezout’s theorem. This implies π_C is branched at 8 points, and so by Riemann Hurwitz, C has genus 3. Now, degenerate Q to a union of two lines, $L_1 \cup L_2$. Let $E_i = \pi^{-1}(L_i)$ for $i \in \{1, 2\}$. Then, if $L_1 \cap L_2$ does not lie on the branching plane quartic T , we have that $E_i \cap E_j \subset S$ consists of two points. Further, each E_i has genus 1 because it maps 2 to 1 onto L_i branched along the four points $L_i \cap T$.

Remark 3.1. Before continuing to the degree 1 case, let us observe one further, orthogonal coincidence. Note that the del Pezzo surface we constructed is actually branched along a plane quartic, which itself is a genus 3 canonical curve!

Is there any precise sense in which this plane quartic relates to the hyperelliptic genus 3 curves, and the genus 1 curves meeting at two points?

4. AN EXTENSION OF THE COINCIDENCE TO DEGREE 1

In the previous section, we interpreted how to degenerate hyperelliptic genus 3 curves into a union of two genus 1 curves meeting at "2 points in \mathbb{P}^0 ." Although it may seem nonsensical, we now go one step further, describing how to find certain genus 2 curves meeting at "1 points in \mathbb{P}^{-1} ."

Since the degeneration in genus 3 was gotten by considering del Pezzo surfaces, we use del Pezzo surfaces to understand the genus 2 case. Recall that a degree 1 del Pezzo surface S is abstractly \mathbb{P}^2 blown up at 8 points, p_1, \dots, p_8 . We obtain a blow up map $\eta : S \rightarrow \mathbb{P}^2$. Its anticanonical embedding defines a rational map (not defined everywhere) $\pi : S \rightarrow \mathbb{P}^1$. Note that π is defined everywhere except at a single point q , where $p_9 := \eta(q)$ is the ninth point of the base locus of the linear system of cubics passing through p_1, \dots, p_8 .

Further, the map π is extremely concrete, and we now describe it. Note that there is a pencil of cubics in \mathbb{P}^2 passing through p_1, \dots, p_8 . Every member of this pencil passing through p_1, \dots, p_8 also passes through p_9 . We can choose a basis for our pencil C_1 and C_2 , so that every member of the pencil can be written as $aC_1 + bC_2$ for $[a, b] \in \mathbb{P}^1$. Then, the map π sends a point on $\eta^{-1}(aC_1 + bC_2)$ to $[a, b] \in \mathbb{P}^1$. This is well defined away from $q = \eta^{-1}(p_9)$.

Using the above description, let us now examine how to construct genus 2 curves as the union of two genus 1 curves meeting at 1 point in \mathbb{P}^0 . As usual, we will need to start with a quadric $Q \subset \mathbb{P}^1$. This is just two points in \mathbb{P}^1 . Then, $\pi^{-1}(Q)$ is a union of two curves $D_1 \cup D_2 \subset S$, so that $\eta(D_i) \subset \mathbb{P}^2$ is a plane cubic, and $\eta_{D_i} : D_i \rightarrow \mathbb{P}^2$ is an isomorphism onto its image. In particular, D_i is a genus 1 curve. But, $\pi_{D_i} : D_i \rightarrow \mathbb{P}^1$ realizes D_i as a degree 1 curve, because it maps D_i onto a single point. So, inside S , the two curves D_1 and D_2 meet at a single point q . However, the birational map π is not defined at q , and so the image is the empty set in \mathbb{P}^1 . It makes sense to interpret this empty set as $\mathbb{P}^{-1} \subset \mathbb{P}^1$. So, we obtain two genus 1 curves meeting at one point, whose image in \mathbb{P}^1 is \mathbb{P}^{-1} . This is what we mean by two genus 1 curves meeting at two points in \mathbb{P}^{-1} .

Remark 4.1. So, the above produces a description of a special locus of genus 2 curves as the union of two genus 1 curves meeting at 1 point in \mathbb{P}^0 . In this case, that special locus only contained the genus 2 curves which were the union of two genus 1 curves meeting a point, and did not contain any more general genus 2 hyperelliptic curves. In the case of degree 2 and genus 3, it contained all hyperelliptic

genus 3 curves, and in the cases of degrees 3, 4, and 5 described in section 2, it contained all genus 4, 5, and 6 canonical curves.

Question 4.2. Is there any way to make sense of the above description of two genus 1 curves meeting at a point in \mathbb{P}^{-1} more precisely, perhaps using stacks, since \mathbb{P}^{-1} can actually be realized as a stack?

Similarly, we wonder whether there is any sensible extension of this to the case of degree 0 genus 1 curves, to the surface which is \mathbb{P}^2 blown up at 9 points, whose anticanonical map sends it to \mathbb{P}^0 . This sounds quite plausible, since if we take a genus 1 curve on that surface, we can realize it as “one genus 1 curve” (of course, the condition that it meets at 0 points in \mathbb{P}^{-2} is vacuous).” Going beyond the realm of sensibility, we can ask whether there is some (probably stack theoretic) interpretation of genus 0 curves corresponding to -1 genus 1 curves meeting at -1 points in \mathbb{P}^{-3} .

5. A PARTIAL EXTENSION TO HIGHER GENERA

Using the insight of examining del Pezzo surfaces in degrees 1 and 2, we can now extend the description of the coincidence to higher genera.

Let us start with the case of degree 6 and genus 7. We will want to look at a degree 6 del Pezzo surfaces S , which are abstractly \mathbb{P}^2 blown up at 3 points, embedded in \mathbb{P}^6 . Then, there is a locus inside the space of genus 7 canonical curves which are of the form $S \cap Q$, for $Q \subset \mathbb{P}^6$ a quadric. Note that unlike the cases of degree 3, 4, and 5, this degree 6 case does not correspond to a general canonical curve, because a general genus 7 curve is an intersection of $\mathbb{P}^6 \cap \text{OG}(5, 10) \subset \mathbb{P}^{15}$, where $\text{OG}(5, 10)$ is the orthogonal grassmannian of 5 dimensional Lagrangian subspaces of a 10 dimensional vector space with respect to a quadratic form, as shown in [Muk95]. When we degenerate Q to a union of two hyperplanes, we get two genus 1 curves meeting at 6 points in \mathbb{P}^4 . Note that any genus 1 curve E of degree 6 lies on a del Pezzo surface. One way to see this is to take two nonisomorphic degree three sheaves $\mathcal{L}_1, \mathcal{L}_2$ which determine maps $\pi_i : E \rightarrow \mathbb{P}^2$. Their product determines a map $\pi : E \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$, and the image of E under this map corresponds to the global sections of $\mathcal{L}_1 \otimes \mathcal{L}_2$, and hence lies in \mathbb{P}^5 . Intersecting some \mathbb{P}^6 containing the copy of \mathbb{P}^5 spanned by the image of E , we obtain a del Pezzo surface S containing E which is $S := \mathbb{P}^6 \cap (\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$.

Next, we can examine the case of degree 7 and genus 8. Once again, we consider the locus of canonical curves which are of the

form $Q \cap S$, for S a degree 7 del Pezzo surface. Degenerating Q into a union of two hyperplanes degenerates the canonical curve into two genus 1 curves meeting at 7 points in \mathbb{P}^5 . Once again, $Q \cap S$ does not realize a general genus 8 curve because a general such curve is an intersection $\mathbb{P}^7 \cap G(2, 6) \subset \mathbb{P}^{14}$ where $G(2, 6) \rightarrow \mathbb{P}^{14}$ is the Plücker embedding.

One can continue this description in higher degrees as well. One thing to note is that there are no del Pezzo surfaces in degrees more than 9. However, one can always take the cone over a genus 1 degree d curve, and obtain a degree d surface. Then, one can intersect that cone with a quadric to obtain a canonical curve. When one degenerates that quadric to two hyperplanes, and intersects with the cone, one obtains two genus 1 curves meeting at d points in \mathbb{P}^{d-2} .

So, for degrees up to 5, we have concretely described the locus of canonical curves corresponding to genus 1 curves. In [Muk95, p. 17], Mukai describes the locus of genus 7 canonical curves corresponding to degree 6 canonical curves as the closure of those with two g_6^2 's but finitely many g_4^1 's. Similarly, in the flowchart on page 2 of [IM03], we see that the locus of canonical degree 7 canonical curves is the closure of those with a g_6^2 that are not bielliptic. So, we ask the question for higher genera as well.

Question 5.1. What is the locus inside the moduli space of genus g curves which can be realized as a complete intersection of a quadric and a degree d surface?

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