A VERIFICATION OF POINCARÉ DUALITY IN AN EXAMPLE

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1. The example

Let k be a field. We use $0 \in \mathbb{A}^n$ to denote the closed point corresponding to the origin. Throughout this document, we consider the following setup

$$\begin{array}{ccc} \mathbb{A}^1 - \{0\} & \xrightarrow{g'} & \mathbb{A}^2 - 0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \{0\} & \xrightarrow{g} & \mathbb{A}^1 \\ & & \downarrow{\scriptstyle h} \\ & & \operatorname{Spec} k \end{array} \tag{1.1}$$

To be precise, the map f is the projection onto the first coordinate, and the map g is the inclusion of the origin in $\mathbb{A}^1$. The square is a fiber square.

Example 1.1. To understand what Poincare duality is saying, recall the statement

$$Rf_* R\mathscr{H}om(\mathscr{F}, \mathbb{Z}/n\mathbb{Z}(1)) \simeq R\mathscr{H}om(Rf_! \mathscr{F}, \mathbb{Z}/n\mathbb{Z}[-2])$$

for $\mathscr{F}$ a sheaf on $\mathbb{A}^2 - 0$. We'll compute the case $\mathscr{F} = \mathbb{Z}/n\mathbb{Z}(1)$

$$Rf_* R\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \simeq R\mathscr{H}om(Rf_! \mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])$$

We now wish to compute the left and right hand sides separately, and hope that they output the same result. At least, we will show the cohomologies of these sheaves agree and give $\mathbb{Z}/n\mathbb{Z}$ in degree 0, $g_*(\mathbb{Z}/n\mathbb{Z}(-2))$ in degree 3 and 0 otherwise. We compute these two sides in Proposition 2.1 and Proposition 3.2. Concretely, we will show both sides

Remark 1.2. One aspect of this example we find particularly interesting is that, in this example, cohomology does not commute with base change along (1.1). We will see this later when we compute that $R^1 f_*(\mathbb{Z}/n\mathbb{Z}) = 0$, even though $R^1 f'_*(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}(-1)$, as both follow from Lemma 2.2 which is essentially an application of the Gysin exact sequence. In terms of Poincare duality, this is explained by the fact that $R^1 f_! \mathbb{Z}/n\mathbb{Z}$ is not locally free; rather it is supported at the origin, see Remark 3.4. There are no maps from a sheaf supported at the origin to $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{A}^1$, but there are maps over $\{0\}$, which explains from the viewpoint of Poincare duality why this cohomology jumps at the origin.

Remark 1.3. Another interesting aspect of this example is that $R^1 f_*(\mathbb{Z}/n\mathbb{Z})$ is nontrivial in degree 3. Hence, this gives an example of a smooth morphism of relative dimension 1 with nontrivial third cohomology. If the morphism were also proper, the third cohomology would necessarily be trivial. This nontriviality essentially follows from the Gysin exact sequence, and one can create similar examples by computing cohomology of the complement of points in $\mathbb{A}^n$.

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2. Computing the left hand side

To start, we observe $Rf_* R\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \simeq Rf_*\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1))$ since $\mathbb{Z}/n\mathbb{Z}(1)$ is locally free, and $\mathscr{H}om$ is exact out of locally free sheaves.

Proposition 2.1.

$$R^i f_* \mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \textit{if } i = 0 \\ g_*\mathbb{Z}/n\mathbb{Z}(-2) & \textit{if } i = 3 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Note that $\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}(1)) \simeq \mathbb{Z}/n\mathbb{Z}$, so we are trying to compute $R^i f_* \mathbb{Z}/n\mathbb{Z}$.

The key to computing cohomology as above will be to compute the cohomology of $\mathbb{A}^2 - 0$.

Lemma 2.2.

$$H^i(\mathbb{A}^m_{\overline{k}} - 0, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \textit{if } i = 0 \\ \mathbb{Z}/n\mathbb{Z}(-m) & \textit{if } i = 3 \\ 0 & \textit{else} \end{cases}$$

Proof. The point is that we have an Gysin exact sequence (coming from purity of the inclusion of the point into $\mathbb{A}^m$)

(2.1)

$$\cdots \longrightarrow H^{i-2m}(\operatorname{Spec} \overline{k}, \mathbb{Z}/n\mathbb{Z}(-m)) \longrightarrow H^i(\mathbb{A}^m_{\overline{k}}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow$$

$$\longrightarrow H^i(\mathbb{A}^m_{\overline{k}} - 0, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^{i-2m+1}(\operatorname{Spec} \overline{k}, \mathbb{Z}/n\mathbb{Z}(-m)) \longrightarrow \cdots$$

and the fact that the cohomology of Spec $\overline{k}$ and $\mathbb{A}^m_{\overline{k}}$ are concentrated in degree 0, since $\overline{k}$ is algebraically closed, implies the claim. □

First, the 0th cohomology is given by the sheaf $f_*(\mathbb{Z}/n\mathbb{Z})(U) = H^0(U \times_{\mathbb{A}^1} \mathbb{A}^2 - 0, \mathbb{Z}/n\mathbb{Z})$ and since for connected U, $U \times_{\mathbb{A}^1} \mathbb{A}^2 - 0$ is always connected, we obtain $f_*(\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.

To compute the higher cohomology groups, we use the Leray spectral sequence associated to the composition $h \circ g$.

We next claim that the higher cohomology of $R^i f_* \mathbb{Z}/n\mathbb{Z}$ are supported over $\{0\}$. Since the higher cohomology of $\mathbb{A}^1$ vanishes (even in a relative setting; this is essentially the fact that smooth maps are locally acyclic, the key point going into the proof of the smooth base change theorem), and cohomology commutes with smooth base change. We find that the sheaf over the complement of $\{0\}$ has no cohomology, and so it is supported over the origin. Therefore, $R^i f_* \mathbb{Z}/n\mathbb{Z} = g_* \mathscr{G}_j$, for some sheaf $\mathscr{G}_j$ supported on the origin.

The Leray spectral sequence then tells us

$$R^i h_*(g_* \mathscr{G}_j) \simeq R^i h_*(R^j f_*(\mathbb{Z}/n\mathbb{Z})) \implies R^{i+j}(h \circ f)_*(\mathbb{Z}/n\mathbb{Z}).$$

Note that for $i > 0$, the stalk of $R^i h_*(g_* \mathscr{G}_j)$ vanishes because it is identified with $H^i(\text{Spec } \overline{k}, \mathscr{G}_j)$, and Spec $\overline{k}$ is 0-dimensional. Therefore, we find $g_* \mathscr{G}_j$ is given by the sheaf $R^{i+j}(f \circ h)_*(\mathbb{Z}/n\mathbb{Z}) \simeq \underline{H}^{i+j}(\mathbb{A}^2 - 0, \mathbb{Z}/n\mathbb{Z})$ (where the latter is notation for the cohomology sheaf on Spec k). We have computed this already in the above Lemma 2.2, and for $i + j > 0$ it is $\mathbb{Z}/n\mathbb{Z}(-2)$ when $i = 3$ and 0 otherwise. This gives the claim. □

3. Computing the right hand side

We next want to compute $R\mathscr{H}om(Rf_! \mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])$ and show it agrees with the above computation. In particular, it should vanish away from degrees 0 and 3. To start, let us compute $Rf_! \mathbb{Z}/n\mathbb{Z}(1)$. We'll use μ_n for $\mathbb{Z}/n\mathbb{Z}(1)$.

Lemma 3.1.

$$R^i f_! \mu_n = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \textit{if } i = 2 \\ g_* \mu_n & \textit{if } i = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. First, there is a natural map $R^2 f_! \mu_n \to \mathbb{Z}/n\mathbb{Z}$ (the trace map from Poincaré duality) which we know is an isomorphism as f is smooth with geometrically connected fibers, and this yields the claim when $i = 2$. To deal with the other i, we know cohomology commutes with base change, and the compactly supported cohomology for $\mathbb{A}^1$ vanishes in degrees other than 2. Therefore, we only need to show the cohomology of $\mathbb{A}^1 - \{0\}$ (the fiber of f over 0) has degree 1 cohomology equal to μ_n and vanishing degree 0 cohomology, this comes from the long exact sequence on compactly supported cohomology

(3.1)
$$\cdots \longrightarrow H^i_c(\mathbb{A}^1 - \{0\}, \mu_n) \longrightarrow H^i_c(\mathbb{A}^1, \mu_n) \longrightarrow H^i(\{0\}, \mu_n) \longrightarrow \cdots$$

where the right term is concentrated in degree 0, and the middle term is concentrated in degree 2. □

We now wish to compute the cohomology $R\mathscr{H}om(Rf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])$.

Proposition 3.2. *We have*

$$H^i(R\mathscr{H}om(Rf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0 \\ g_*\mathbb{Z}/n\mathbb{Z}(-2) & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. To actually pin down the sheaf $R\mathscr{H}om(Rf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])$. in the derived category seems a bit tricky, but at least we can compute its cohomology using the spectral sequence whose (i, j) term is $R^i\mathscr{H}om(R^jf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2])$ which converges to the $i - j + 2$ cohomology of $R\mathscr{H}om(Rf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z})$. (Note here that one might reasonably use alternate notation for $R^i\mathscr{H}om(\mathscr{G}, \mathscr{H})$, such as $\mathscr{E}xt^i(\mathscr{G}, \mathscr{H}) = H^i(R\mathscr{H}om(\mathscr{G}, \mathscr{H}))$, but these all mean the same thing.)

To be in concordance with our computation of the other side of poincare duality, we want to show this is $g_*\mathbb{Z}/n\mathbb{Z}(-2)$ in degree 3 (when $i - j + 2 = 3$) and $\mathbb{Z}/n\mathbb{Z}$ in degree 0, and 0 otherwise. By the above spectral sequence, the below Proposition 3.3 will complete our computation. □

Proposition 3.3.

$$R^i\mathscr{H}om(R^jf_!\mu_n, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0, j = 2 \\ g_*\mathbb{Z}/n\mathbb{Z}(-2) & \text{if } i = 2, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

The proof occupies the remainder of this section. We first check the case $i = 0$ in subsection 3.1 and then the case $i \neq 0$ in subsection 3.2.

3.1. **Proof of Proposition 3.3 in the case** $i = 0$**.** To start, let's deal with the relatively easy computation of the case $i = 0$. In this case, we concretely know $R^jf_!\mu_n$ is $\mathbb{Z}/n\mathbb{Z}$ when $j = 2$ and $g_*\mu_n$ when $j = 1$, by Lemma 3.1. Therefore, when $i = 0, j = 2$, this yields $\mathbb{Z}/n\mathbb{Z}$ as claimed. When $i = 0, j = 1$, we see $\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$ because for any connected étale set U, when we remove the preimage of 0 from U (which we abusively notate $U - \{0\}$, the resulting map will have to vanish, and so commutativity of the diagram

$$\begin{array}{ccc} g_*\mu_n(U) & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\ \downarrow & & \downarrow \text{id} \\ g_*\mu_n(U - \{0\}) & \xrightarrow{0} & \mathbb{Z}/n\mathbb{Z} \end{array} \tag{3.2}$$

forces the top vertical map to be 0 as well. Altogether, this completes the computation when $i = 0$.

Remark 3.4. In terms of Poincaré duality, the above computation explains the reason that the sheaf $\mathbb{Z}/n\mathbb{Z}$ fails to commute with base change along (1.1), as mentioned in Remark 1.2. In this spectral sequence, $R^i\mathscr{H}om(R^jf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}[-2]) \implies H^{i-j+2}(R\mathscr{H}om(Rf_!\mathbb{Z}/n\mathbb{Z}(1), \mathbb{Z}/n\mathbb{Z}))$, we see that the $i = 0, j = 1$ term $\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$. However, although the formation of $Rf_!$ commutes with base change, composition with $\mathscr{H}om$ does not. Indeed, we have seen $\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$ is the trivial sheaf, but when restricted to $\{0\}$ this becomes $\mathscr{H}om(\mu_n, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}(-1)$, which is nontrivial. So this does not commute with base change. Note that this implicitly depends on knowing something about the rest of the spectral sequence, but we will see in what follows that all the relevant sheaves are supported on the point $\{0\}$ and so will automatically commute with base change.

3.2. **Proof of Proposition 3.3 in the case $i \neq 0$.** Next, we have to compute the cases $i \neq 0$. First, when $j = 2$, $\mathbb{Z}/n\mathbb{Z}$ is locally free, and so $R\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and so $R^i\mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$.

Remark 3.5. Note that although $\mathbb{Z}/n\mathbb{Z}$ is injective over a field, it is not injective as a sheaf. This implies we have to deal with computing these higher $R\mathscr{H}om$ and we will see that they are indeed nontrivial (hence giving a very indirect proof that $\mathbb{Z}/n\mathbb{Z}$ is *not* injective as a sheaf in general).

Hence, we just have to deal with the case that $j = 1$ and $i \neq 0$, in which case, using Lemma 3.1, it remains to compute $R^i\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z})$. More specifically, we want to show $R^i\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i \neq 2$ and is $g_*\mathbb{Z}/n\mathbb{Z}(-2)$ when $i = 2$.

For this, consider the exact sequence

$$0 \longrightarrow j_!\mu_n \longrightarrow \mu_n \longrightarrow g_*\mu_n \longrightarrow 0 \tag{3.3}$$

where $j : \mathbb{A}^1 - \{0\} \to \mathbb{A}^1$ is the open immersion which is the complement of the closed immersion g.

Lemma 3.6. *For $j : X \to Y$ an open immersion, and $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y, we have $Rj_* R\mathscr{H}om(\mathscr{F}, j^*\mathscr{G}) \simeq R\mathscr{H}om(j_!\mathscr{F}, \mathscr{G})$.*

Proof. Computing on étale opens, we have

$$\begin{aligned} R\mathscr{H}om(j_!\mathscr{F}, \mathscr{G})(U) &= R\mathscr{H}om((j_!\mathscr{F})|_U, \mathscr{G}|_U) \\ &= R\mathscr{H}om(\mathscr{F}|_{j^{-1}(U)}, j^*\mathscr{G}|_{j^{-1}(U)}) \\ &= Rj_* R\mathscr{H}om(\mathscr{F}, j^*\mathscr{G})(U) \end{aligned}$$

□

Now we want to take the cohomology associated (3.3). This gives the distinguished triangle

$$R\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) \to R\mathscr{H}om(\mu_n, \mathbb{Z}/n\mathbb{Z}) \to R\mathscr{H}om(j_!\mu_n, \mathbb{Z}/n\mathbb{Z})$$

Note that μ_n is locally free so that $R\mathscr{H}om(\mu_n, \bullet) = \mathscr{H}om(\mu_n, \bullet)$, as hom out of a locally free sheaf is exact. Combining the above Lemma 3.6 with the preceding observation, we can replace the last term with

$$R\mathscr{H}om(j_!\mu_n, \mathbb{Z}/n\mathbb{Z}) \simeq Rj_* R\mathscr{H}om(\mu_n, \mathbb{Z}/n\mathbb{Z}) \simeq Rj_* \mathscr{H}om(\mu_n, \mathbb{Z}/n\mathbb{Z}) \simeq Rf_*(\mathbb{Z}/n\mathbb{Z}(-1)).$$

To simplify matters, we also know $R^i\mathscr{H}om(\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$, for the same reason that μ_n is locally free.

Therefore, the associated long exact sequence to the above distinguished triangle is

(3.4)

$$\begin{array}{l} 0 \longrightarrow \mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathbb{Z}/n\mathbb{Z}(-1) \longrightarrow j_*(\mathbb{Z}/n\mathbb{Z}(-1)) \\ \longrightarrow \mathscr{H}^1(R\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z})) \longrightarrow 0 \longrightarrow R^1 j_*(\mathbb{Z}/n\mathbb{Z}(-1)) \\ \longrightarrow \mathscr{H}^2(R\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z})) \longrightarrow 0 \longrightarrow R^2 j_*(\mathbb{Z}/n\mathbb{Z}(-1)) \longrightarrow \cdots \end{array}$$

Now observe that the map $\mathbb{Z}/n\mathbb{Z}(-1) \to j_*(\mathbb{Z}/n\mathbb{Z}(-1))$ is an isomorphism, and so we find $\mathscr{H}^1(R\mathscr{H}om(g_*\mu_n, \mathbb{Z}/n\mathbb{Z})) = 0$. (Said another way, $\mathscr{E}xt^1(g_*\mu_n, \mathbb{Z}/n\mathbb{Z}) = 0$).

Hence, we will complete the verification of Proposition 3.3 once we prove the following.

Lemma 3.7. *We have $R^j j_*(\mathbb{Z}/n\mathbb{Z}(-1)) = 0$ for $j \geq 2$ and $R^1 j_*(\mathbb{Z}/n\mathbb{Z}(-1)) = g_*(\mathbb{Z}/n\mathbb{Z}(-2))$.*

Proof. First, these cohomologies are concentrated at the origin because j is an isomorphism (hence j_* is exact) away from the origin. So, we only need compute the stalk at the origin.

To compute this, we may base change to the étale local ring at the origin in $\mathbb{A}^1$, which is $S = \operatorname{Spec} R$ for R the strict henselization of $k[t]$. Let $s \in S$ be the closed point. We then aim to show $H^1(S - s, \mathbb{Z}/n\mathbb{Z}(-1)) = \mathbb{Z}/n\mathbb{Z}(-2)$ while $H^j(S - s, \mathbb{Z}/n\mathbb{Z}(-1)) = 0$ for $j \geq 2$. Indeed, this follows from the Gysin exact sequence (coming from purity of the inclusion of the point into $\mathbb{A}^1$)

(3.5)

$$\cdots \longrightarrow H^{i-2}(s, \mathbb{Z}/n\mathbb{Z}(-2)) \longrightarrow H^i(S, \mathbb{Z}/n\mathbb{Z}(-1)) \longrightarrow$$

$$\longrightarrow H^i(S - s, \mathbb{Z}/n\mathbb{Z}(-1)) \longrightarrow H^{i-1}(s, \mathbb{Z}/n\mathbb{Z}(-2)) \longrightarrow \cdots$$

Because S is a strictly Henselian DVR, its higher cohomology vanishes. We therefore obtain an isomorphism $H^i(S - s, \mathbb{Z}/n\mathbb{Z}(-1)) \simeq H^{i-1}(s, \mathbb{Z}/n\mathbb{Z}(-2))$ when $i \geq 1$. Because s is the spectrum of an algebraically closed field, this is 0 when $i \geq 2$ and $\mathbb{Z}/n\mathbb{Z}(-2)$ when $i = 1$, as claimed. □

4. The setup of Grothendieck duality

Since it's quite similar to Poincaré duality, let's also recall the general setup of Serre duality.

Here, we have a proper map $f : X \to Y$ of smooth schemes (probably we just need it is Cohen Macaulay or Gorenstein) and have a complexes of sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. Serre duality says

$$Hom_{\mathscr{O}_Y}(f_*\mathscr{F}, \mathscr{G}) = f_* Hom_{\mathscr{O}_X}(\mathscr{F}, f^!\mathscr{G}).$$

Plugging in $\mathscr{G} = \mathscr{O}_Y$, we find $f^!\mathscr{O}_Y = \omega_{X/Y}[n]$ and the above becomes

$$Hom_{\mathscr{O}_Y}(f_*\mathscr{F}, \mathscr{O}_Y) = f_* Hom_{\mathscr{O}_X}(\mathscr{F}, \omega_{X/Y}[n]).$$

In the case $\mathscr{F}$ is a locally free sheaf and Y is a point, taking ith cohomology yields the usual statement

$$H^i(Y, Hom_{\mathscr{O}_Y}(f_*\mathscr{F}, \mathscr{O}_Y)) = H^i(Y, f_* Hom_{\mathscr{O}_X}(\mathscr{F}, \omega_{X/Y}[n])).$$

which we can rephrase as

$$H^{-i}(X, \mathscr{F}^\vee)^\vee \simeq H^{i+n}(X, \mathscr{F}^\vee \otimes \omega_{X/Y}).$$

The reason we see the $-i$ on the left is that $Hom_{\mathscr{O}_Y}(f_*\mathscr{F}, \mathscr{O}_Y)$ is the dual of the complex $f_*\mathscr{F}$. So for example, if $\mathscr{F}$ were supported in degree i the dual would be supported in degree $-i$. In particular, they should pair to give something in degree 0.