

NOTES FOR A TALK ON MALLE'S CONJECTURE

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1. MALLE'S CONJECTURE

Given a finite group G , the inverse Galois problem asks whether there is a G Galois extension of \mathbb{Q} . Malle's conjecture is even more difficult, and asks how many G extensions of \mathbb{Q} there are. One can make a similar conjecture with $\mathbb{F}_q(t)$ in place of \mathbb{Q} .

Conjecture 1.1 (Malle). Given a finite permutation group G , and a global field K , there are specific constants $a(G), b_M(K, G)$ so that, up to a constant, there are $X^{\frac{1}{a(G)}} (\log X)^{b_M(K, G)-1}$ extensions of K with discriminant at most X .

Remark 1.2. Malle moreover conjectures specific values of the constants $a(G)$ and $b_M(G, K)$. Specifically, $a(G) = \min_{g \in G - \text{id}} (\#G - \text{the number of orbits of } g)$. It is a bit trickier to define $b_M(G, K)$.

Remark 1.3. Some things are known about Malle's conjecture (mostly over \mathbb{Q}) but many things remain open. Here are some cases that are known and unknown

- (1) Abelian groups
- (2) S_3, S_4, S_5 due to Bhargava in his thesis and leading up to his fields medal, though S_6 is open.
- (3) D_8 is known but D_{10} is open.
- (4) Many nilpotent groups and some wreath products of cyclic groups
- (5) A_4 is open

Recently, Alberts, Lemke Oliver, Wang, and Wood have developed a fairly comprehensive machine to count the number of G extensions of \mathbb{Q} for many groups G . Their results apply to 1665 of the 4953 transitive permutation groups of degree at most 23 (and many more of higher degree) so in this sample, they are batting around .336.

One interesting aspect of Malle's conjecture is that the valuit is incorrect for general groups G .

Theorem 1.4 (Klüners). *If $G = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$ the value b_M which Malle predicted is wrong.*

In light of this Türkelli made a modified conjecture, using motivation from the function field setting.

Conjecture 1.5 (Türkelli). There is a constant $b_T(K, G)$ so that Malle's conjecture is true if one replaces $b_M(K, G)$ by $b_T(K, G)$.

Nevertheless, Wang shows Türkelli's conjecture is false when $K = \mathbb{Q}$.

Theorem 1.6 (Wang). *If $K = \mathbb{Q}$, there are counterexamples to Türkelli's conjecture, which are of the form $\mathbb{Z}/\ell\mathbb{Z} \times (\mathbb{Z}/m\mathbb{Z})^\ell$ for suitable primes ℓ, m .*

Theorem 1.7 (Landesman-Levy). *For each finite group G , there is a constant C_G so that if $q > C_G$ and $\gcd(q, |G|) = 1$, then Türkelli's conjecture is correct over $K = \mathbb{F}_q(t)$.*

Moreover, Malle's conjecture is correct if one restricts to counting extensions of $\mathbb{F}_q(t)$ which do not contain $\mathbb{F}_{q^r}(t)$ for some value of $r > 1$.

Remark 1.8 (Comparison with Ellenberg–Tran–Westerland). With the goal of proving Malle's conjecture, there has been substantial progress prior to this paper over function fields. Namely, [ETW17] prove a weak upper bound for the number of G extensions with the correct power of X , but with a value of b that is too large in general. Additionally, they do not obtain a lower bound for the number of G extensions. In contrast, our result Theorem 1.7 obtains both upper and lower bounds, as well as the correct power of $\log X$.

Remark 1.9. We can prove substantially more, not just counting by discriminant but any reasonable invariant. We also prove a version where we restrict the ramification. Finally, we are able to say something about what the constant in the asymptotic growth is. The constant doesn't exist in general, but in some cases, as periodic version of the constant exists.

2. REDUCING TO A STATEMENT ABOUT COHOMOLOGY

The key input to our proof is the following homological stability result.

Definition 2.1. Let $c = c_1 \cup \cdots \cup c_v \subset G$ denote a union of conjugacy classes in a group and $\text{CHur}_{n_1, \dots, n_v}^c$ denote the Hurwitz scheme parameterizing of G covers of \mathbb{A}^1 with connected source, inertia in c , and n_i branch points of inertia type c_i and a trivialization along the boundary.

Also use $\text{CHur}_n^c = \coprod_{n_1 + \dots + n_v = n} \text{CHur}_{n_1, \dots, n_v}^c$ and $[\text{CHur}_{n_1, \dots, n_v}^c / G]$ for the stack of G covers where one doesn't specify a trivialization of the boundary.

Theorem 2.2. *There are constants I and J , depending on c , so that for any $i \geq 0$ and $n_1 > Ii + J$, and any $g \in n_1$, the map $H_i(\text{CHur}_{n_1, \dots, n_v}^c, \mathbb{Z}) \rightarrow H_i(\text{CHur}_{n_1+1, \dots, n_v}^c, \mathbb{Z})$ is an isomorphism.*

Remark 2.3. In the case that c is a single conjugacy class and $H_0(\text{Hur}_{n_1, \dots, n_v}^c, \mathbb{Z})$ has bounded rank, a slightly weaker version of this was shown by Ellenberg-Venkatesh-Westerland.

Theorem 2.4. *Then are constants I and J , depending on c , so that for any $i \geq 0$ and $n_1, \dots, n_v > Ii + J$, and any component $Z \subset \text{CHur}_{n_1, \dots, n_v}^c$, the map $H_i(Z, \mathbb{Z}[1/|G|]) \rightarrow H_i(\text{Conf}_{n_1, \dots, n_v}, \mathbb{Z}[1/|G|])$ is an isomorphism.*

2.5. Stable homology implies Malle. The Hurwitz spaces $\text{CHur}_n^{G-\text{id}}$ exist as $\text{CHur}_{n, \mathbb{Z}[1/|G|]}^{G, G-\text{id}}$ over $\mathbb{Z}[1/|G|]$. A G extension of $\mathbb{F}_q(t)$ roughly corresponds to a \mathbb{F}_q point of $\text{CHur}_{n, \mathbb{Z}[1/|G|]}^{G, G-\text{id}}$. Technically, it corresponds to an \mathbb{F}_q point of $[\text{CHur}_{n, \mathbb{Z}[1/|G|]}^{G, G-\text{id}} / G]$. Therefore, our goal is now to count the \mathbb{F}_q points of this space. To do this, we use the Grothendieck-Lefschetz trace formula, which relates understanding the \mathbb{F}_q points to trace of Frobenius acting on the cohomology.

Theorem 2.6 (Grothendieck-Lefschetz trace formula). *If X is a smooth variety over \mathbb{F}_q of dimension n*

$$\frac{X(\mathbb{F}_q)}{q^n} = \sum_{i=0}^{2n} (-1)^i \text{tr}(\text{Frob}_q^{-1} | H^i(X_{\mathbb{F}_q}, \mathbb{Q}_\ell)).$$

Using this, our new goal is to understand the homology of these Hurwitz spaces, which is the content of Theorem 2.4.

Remark 2.7. Technically, somewhat more is important. It is important to know the homology of these Hurwitz spaces stabilizes in all directions (and not just with all the n_i 's large. It is also important to know the trace of Frobenius stabilizes.

2.8. Difficulties in proving the homological stability result. We now explain the main new idea in the proof of our homological stability results. To explain the proof, we say that a Hurwitz space for c is *non-splitting* if $\dim H_0(\text{Hur}_n^c)$ is bounded as n grows. Our proof builds crucially both on Ellenberg-Venkatesh-Westerland's Annals paper showing that non-splitting Hurwitz spaces have homology which stabilizes [EVW16] (and the alternate proof outline in Oscar Randal-Williams Bourbaki article [RW20]) and our prior work computing the stable value of this homology in the non-splitting case [LL25].

Remark 2.9. We note that the non-splitting hypothesis is quite restrictive. First of all, in order for it to be satisfied, we must have that only a single conjugacy class appears in c . Indeed, if there are two conjugacy classes, there will be components of Hur_n^c parameterizing G covers with α branch points in

the first conjugacy class and β elements of the second conjugacy class, with $\alpha + \beta = n$. Thus, the number of components of Hur_n^c will grow linearly in n .

Even if one does restrict to c consisting of a single conjugacy class, we must recall that Hur_n^c includes disconnected G covers as well, and so it is still possible that the number of components is unbounded in n (although it turns out that if one restricts to those components $\text{CHur}_n^c \subset \text{Hur}_n^c$ parameterizing connected covers, there will only be a bounded number of components).

Example 2.10. For example, if we take c to be the set of transpositions in S_3 , c will be non-splitting, but if we take c to be the set of transpositions in S_d for any $d > 3$, it will fail to be non-splitting, because when one restricts to covers with inertia only in $(12) \cup (34)$, we find that $\langle (12), (34) \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the elements (12) and (34) lie in different conjugacy classes, so we again obtain that $\dim H_0(\text{Hur}_n^c)$ is unbounded as n grows. Summarizing the above discussion, it turns out that one can express the condition that c is non-splitting equivalently as the group theoretic condition that c that $c \cap G'$ contains at most a single conjugacy class for each $G' \subset G$.

To explain the importance of the non-splitting condition in prior work, we need to look at the proof of [EVW16]. Their key idea was to show that if $\dim H_0(\text{Hur}_n^c)$ stabilizes, one can also show $\dim H_i(\text{Hur}_n^c)$ stabilizes. In contrast, when one removes the non-splitting condition, this hypothesis disappears. Indeed, it is much less obvious what homological stability should even mean in the non-splitting case, because it is typically false that $\dim H_i(\text{Hur}_n^c)$ is bounded in n once c fails to be non-splitting. Amazingly, it turns out that the failure of stabilization of $\dim H_0(\text{Hur}_n^c)$ is “the only obstruction” to homological stability, in the sense that if one restricts to suitable finite collections of components, then their homologies do stabilize. So the question now becomes if there is some way to show homology of Hurwitz spaces stabilize suitably when the 0th homology not even stabilize.

2.11. Proof idea for the homological stability result. In order to prove a suitable form of homological stability in the non-splitting case, we can think of our idea at a very high level as coming in three steps: First, “differentiate,” second, use the ideas in [EVW16] to prove a suitable version of stability, and, third, “integrate back” by generalizing ideas in [LL25].

Example 2.12. Let’s see what the above means in the concrete case that $G = S_4$ and we consider the Hurwitz space $\text{Hur}^c = \cup_n \text{Hur}_n^c$ associated to the conjugacy class c of transpositions. Let $c = \{x, y, z\}$ be the set of transpositions in S_3 . First, we consider the quotient $\text{Hur}^c / (x^2, y^2, z^2)$, in the sense of higher algebra. We want to show this “space” has homology

which stabilizes. This might seem scary, but if one wishes to understand the homology of Hur^c , this is given as the homology of some the chain complex

$$(2.1) \quad C_{i-1}(\text{Hur}^c) \longrightarrow C_i(\text{Hur}^c) \longrightarrow C_{i+1}(\text{Hur}^c).$$

For $x = (12)$, we can view Hur^c / x^2 as the cofiber $\text{Hur}^c \xrightarrow{x^2} \text{Hur}^c$ and so its homology is given as the homology of the total complex

$$(2.2) \quad \begin{array}{ccccc} C_{i-1}(\text{Hur}^c) & \longrightarrow & C_i(\text{Hur}^c) & \longrightarrow & C_{i+1}(\text{Hur}^c) \\ & & \downarrow x^2 & & \downarrow x^2 \\ C_{i-1}(\text{Hur}^c) & \longrightarrow & C_i(\text{Hur}^c) & \longrightarrow & C_{i+1}(\text{Hur}^c). \end{array}$$

where the map x^2 is given by taking a cover of a disc branched at n points and sending it to a cover of a disc branched at $n + 2$ points with 2 additional branch points labeled x near a specified point on the boundary.

Using ideas from [EVW16], one can show the quotient $\text{Hur}^c / (x^2, y^2, z^2)$ in fact stabilizes to 0. Now, Hur^c can be divided into different parts, depending on the monodromy of the cover. The monodromy can either be all of S_3 , or it can be a subgroup generated by transpositions (so long as there is more than 1 transposition the subgroup will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and there can only be copies of that single transposition, since two transpositions generate S_3). By a filtration argument (or a simple direct argument in this case since $\mathbb{Z}/2\mathbb{Z}$ is so simple) one can also deduce that $\text{Hur}^c / (x^2, y^2, z^2)$ in stabilizes to 0.

Now, recall that our goal is to show the homology of Hur^c stabilizes. Equivalently, we want to show multiplication by x^2 induces an isomorphism from $\text{Hur}_n^c \rightarrow \text{Hur}_{n+2}^c$ and hence we want to show Hur^c / z^2 stabilizes to 0. Therefore we want to “remove” the y^2 and z^2 in the quotient we are taking. The first step is to observe that $\text{Hur}^c / (x^2, y^2, z^2)$ in stabilizing to 0 means that z^2 induces an isomorphism $\text{Hur}^c / (x^2, y^2) \rightarrow \text{Hur}^c / (x^2, y^2)$ which means the homology of $\text{Hur}^c / (x^2, y^2)$ stabilizes. If we knew this stabilizes to 0, we could then deduce that y^2 induces an isomorphism $\text{Hur}^c / x^2 \rightarrow \text{Hur}^c / x^2$ and hence that Hur^c / x^2 stabilizes. And then if we could again show that this stabilizes to 0, we would obtain that the homology of Hur^c stabilizes, as we wished to show.

So, the main remaining step that needs to be explained is why the stable homology of $\text{Hur}^c / (x^2, y^2)$ is 0 (and also the similar statement that the stable value of $\text{Hur}^c / (x^2)$ is 0). When we say stable, we mean that it is the stable value as we iteratively apply z to increase the degree, and hence we are

trying to show the localization $\text{Hur}^c / (x^2, y^2)[z^{-1}] = 0$. The key idea is the following descent statement:

Lemma 2.13. *Suppose $f : A \rightarrow B$ is a map of connective dga's (differential graded algebras) and M a connective A module. Suppose $H_0(f) : H_0(A) \rightarrow H_0(B)$ is surjective with kernel I and I acts nilpotently on $H_i(M)$ for all i . If*

$$(2.3) \quad H_0(B) \otimes_A H_0(B) \simeq H_0(B) \otimes_B H_0(B)$$

then $M \simeq B \otimes_A M$.

Vague idea of the proof. The idea of the proof is that first, the nilpotency condition for I on M allows one to reduce to the case that $M = A/I$. So then in this case one wants to show A/I is equivalent to B . We know they have the same H_0 . One can think of this as a cover. Roughly speaking, a descent argument implies that in order to check A/I is B , it is enough to check the covers agree (which is true as both are $H_0(B)$) and the fiber product of the cover over the base agree, i.e. $H_0(B) \otimes_A H_0(B) \simeq H_0(B) \otimes_{A/I} H_0(B) \simeq H_0(B) \otimes_B H_0(B)$ which was the assumption of the lemma. \square

Now, let's see how to apply this lemma. We will take $A = C_*(\text{Hur}^c)[z^{-1}]$, $B = C_*(\text{Hur}^z)[z^{-1}]$ (where $\text{Hur}^z \subset \text{Hur}^c$ is the space of $\mathbb{Z}/2\mathbb{Z}$ covers where all the branch points are labeled z) and $M = C_*(\text{Hur}^c / (x^2, y^2)[z^{-1}])$. Supposing we can verify (2.3), this will imply $M \simeq B \otimes_A M$. The target is a $C_*(\text{Hur}^z)$ module which implies it has no "connected part." Said another way, one can use this equivalence to deduce $C_*(\text{Hur}^c / (x^2, y^2)[z^{-1}]) = 0$. (I.e. M potentially included S_3 and non- S_3 covers but the S_3 part actually vanished and it actually only included covers with labels in z .) Hence, the key point is to verify (2.3). That is, we want to show

$$H_0(\text{Hur}^z)[z^{-1}] \otimes_{C_*(\text{Hur}^c)[z^{-1}]} H_0(\text{Hur}^z)[z^{-1}] \simeq H_0(\text{Hur}^z)[z^{-1}] \otimes_{C_*(\text{Hur}^z)[z^{-1}]} H_0(\text{Hur}^z)[z^{-1}].$$

We can add a disjoint basepoint (notated by a subscript $+$) and take reduced chains to reduce to showing there is an equivalence of spaces

$$\pi_0(\text{Hur}^z)_+[z^{-1}] \otimes_{\text{Hur}^c_+} \pi_0(\text{Hur}^z)_+[z^{-1}] \simeq \pi_0(\text{Hur}^z)_+[z^{-1}] \otimes_{\text{Hur}^z_+} \pi_0(\text{Hur}^z)_+[z^{-1}]$$

Moreover, $\pi_0(\text{Hur}^z)$ is \mathbb{N} so when we invert z it becomes \mathbb{Z} and when we add a basepoint it becomes \mathbb{Z}_+ . So, we want to show there is an equivalence

$$\mathbb{Z}_+ \otimes_{\text{Hur}^c_+} \mathbb{Z}_+ \simeq \mathbb{Z}_+[z^{-1}] \otimes_{\text{Hur}^z_+} \mathbb{Z}_+$$

The first space $\mathbb{Z}_+ \otimes_{\text{Hur}^c_+} \mathbb{Z}_+$ has a model where we put points x, y, z in a configuration space with no two in the same horizontal row. When one point passes above another it gets conjugated by the other, and when z collides into the left or right boundaries in increments the \mathbb{Z} coordinate. When y or x collide, the configuration gets sent to the base point.

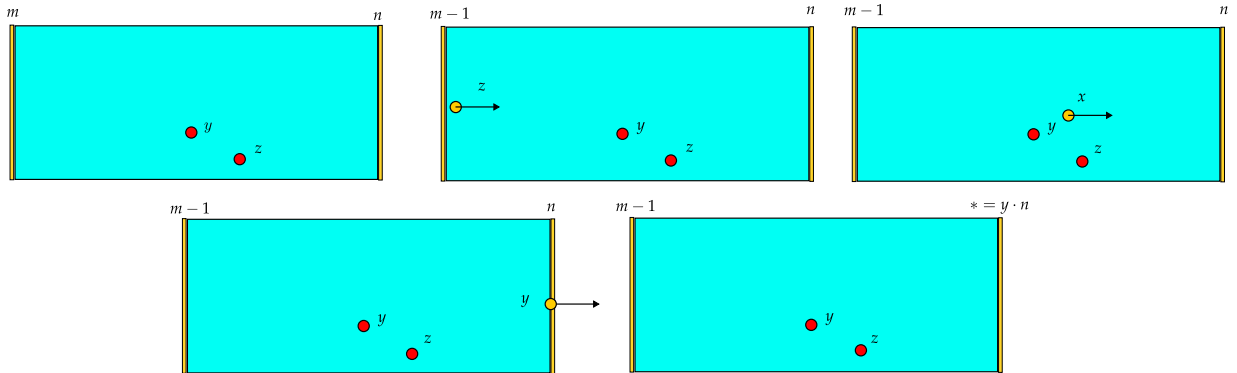


FIGURE 1. This is a pictorial description of the nullhomotopy. We pull a copy of the element z out from the left and let it traverse to the right. When it passes the y it gets replaced with $x = y^{-1}zy$ and then when it passes the z it gets replaced with $y = z^{-1}xz$. When it meets the right boundary, it acts on the boundary by y . Since $y \neq z$, this becomes the base point, and this whole configuration is then identified with the base point.

The space $\mathbb{Z}_+ \otimes_{\text{Hur}_+^z} \mathbb{Z}_+$ is similar but there can only be z 's in the center.

The difference between these spaces is that the component of the basepoint in the first space gets contracted to the base point in the second space. So, to show this is an equivalence, it suffices to show the component of the base point is contractible. (For technical reasons, this isn't strictly correct. Rather, we exhaust the space by a sequence of contractible subspaces, depending on the vertical distance between two points of the configuration.)

We then have an explicit contracting homotopy which is given by pulling a point z out from the right just above the lowest point which is y or x . When z passes past that point, it gets conjugated to y or x . We picture this in Figure 1. This then hits the right boundary and gets sent to the base point. This describes a path from any such point to the base point, and gives a contracting homotopy.

3. COHEN-LENSTRA HEURISTICS

Another fun consequence of this is that we can prove results toward the Cohen-Lenstra heuristics.

Let $IQ_{\leq X}$ denote the set of imaginary quadratic fields (of the form $\mathbb{Q}(\sqrt{-d})$) with discriminant at most X .

Conjecture 3.1. As K ranges over imaginary quadratic fields of discriminant $\leq X$, and ℓ an odd prime number, the average size of ℓ -torsion in the class

group of \mathcal{O}_K is 2. That is,

$$\lim_{X \rightarrow \infty} \mathbb{E}_{K \in IQ_{\leq X}} (\#\text{Cl}(\mathcal{O}_K)[\ell]) = 2.$$

Definition 3.2. Let $\mathcal{MH}_{n,q}$ denote the set of function fields K of monic hyperelliptic curves $y^2 = f(t)$ for $f \in \mathbb{F}_q[t]$ squarefree of degree n . Let \mathcal{O}_K denote $\mathbb{F}_q[t, y]/(y^2 - f(t))$, (the normalization of $\mathbb{F}_q[t]$ in the quadratic extension $K/\mathbb{F}_q(t)$), and let $\text{Cl}(\mathcal{O}_K)$ denote the set of line bundles on $\text{Spec } \mathcal{O}_K$ (or ideal classes in \mathcal{O}_K).

Theorem 3.3 (L-Levy). *For any odd prime ℓ , and q an odd prime power with $\gcd(H, q(q-1)) = 1$, there is an integer C_ℓ depending on H so that for $q > C$,*

$$\lim_{n \rightarrow \infty, n \equiv 1 \pmod{2}} \mathbb{E}_{K \in \mathcal{MH}_{n,q}} (\#\text{Surj}(\text{Cl}(\mathcal{O}_K)[\ell])) = 2.$$

sketch. The rough idea is that an ℓ torsion element in the class group of a quadratic extension $K/\mathbb{F}_q(t)$ corresponds to a $\mathbb{Z}/\ell\mathbb{Z}$ extension L/K . Altogether, this is a $D_{2\ell}$ extension $L/\mathbb{F}_q(t)$, and so we are essentially reduced to counting certain dihedral group extensions, which we can already do using our work on Malle's conjecture. \square

Remark 3.4. More generally, the above works to understand class groups of Γ extensions $K/\mathbb{F}_q(t)$ instead of just quadratic extensions. We do this for Γ extensions split completely over ∞ (meaning there are $|\Gamma|$ \mathbb{F}_q points of the extension over $\infty \in \mathbb{P}_{\mathbb{F}_q}^1$) and show that the average number of surjections of the class group onto an abelian group H is $\frac{1}{[H:H^\Gamma]}$. (And, more generally, for H nonabelian, the average number of surjections from the maximal unramified extension which is totally split over ∞ is $\frac{1}{[H:H^\Gamma]}$.)

4. THE PICARD RANK CONJECTURE

One fundamental question in algebraic geometry is to identify the set of line bundles on moduli spaces of interest. For example, the rational picard group of \mathcal{M}_g , the moduli space of smooth proper genus g curves is generated by the hodge bundle

Another collection of fundamental moduli spaces of importance are Hurwitz spaces.

Definition 4.1. The *Hurwitz space* \mathcal{H}_g^d denotes the moduli stack of simply branched covers $X \rightarrow \mathbb{P}^1$ of degree d and genus g with geometrically connected source. For example, if $d = 2$, this parameterizes hyperelliptic curves (with a specified rigidification of the target \mathbb{P}^1).

Property	M_g	Hur $_n$
Dimension	$3g - 3$	n
Irreducible	Yes	Yes
Rational Picard group	\mathbb{Q}	?

TABLE 1. Chart of basic facts about M_g and Hurwitz spaces.

It turns out that there are no “obvious” non-torsion line bundles on Hurwitz spaces, which leads to the following conjecture.

Conjecture 4.2 (Picard rank conjecture). For all $d > 0$, we have $\text{Pic}(\mathcal{H}_g^d) \otimes \mathbb{Q} \simeq 0$.

Remark 4.3. This was stated in [HM06, Conjecture 2.49(1)]. See also the closely related [DE96, Conjecture 3], although there they work with Hurwitz spaces where they do not quotient by the PGL_2 action. We note that the roots of this conjecture extend further back, and a version of it appears in work of Ciliberto from 1986 [Cil86, Conjecture 3.2] and even in work of Enriques from 1919 [Enr19, p. 371].

Remark 4.4. Conjecture 4.2 has been proven for $d \leq 5$ by Deopurkar-Patel [DP15], but this proof relies on explicit parameterizations of low degree Hurwitz spaces quite similar to those Bhargava used to count number fields of degree at most 5. We also note that Mullane has proven Conjecture 4.2 whenever $d \leq g$ (so $n \leq 4d - 2$ for $n = 2g - 2 + 2d$) [Mul23]. However, the case that $d \geq 6$ and $g > d$ remains open.

Theorem 4.5 (Landesman-Levy, Asymptotic Picard rank conjecture). *If $g \gg d$, the Picard rank conjecture holds true.*

To state our more general result, we introduce some notation.

Definition 4.6. Let G be a finite group and $c \subset G - \text{id}$ be a conjugacy class generating G . We use $\mathcal{H}_n^{G,c}$ to denote the Hurwitz stack over \mathbb{C} parameterizing connected G covers of genus 0 curves with n branch points, each of which have inertia in c .

Example 4.7. If $G = S_d$, $c \subset S_d$ is the conjugacy class of transpositions, and $n = 2g + 2n - 2$ then $\mathcal{H}_n^{G,c}$ is \mathcal{H}_g^d the moduli stack of simply branched covers.

We can prove a more general version of the asymptotic Picard rank conjecture, which works for arbitrary G, c .

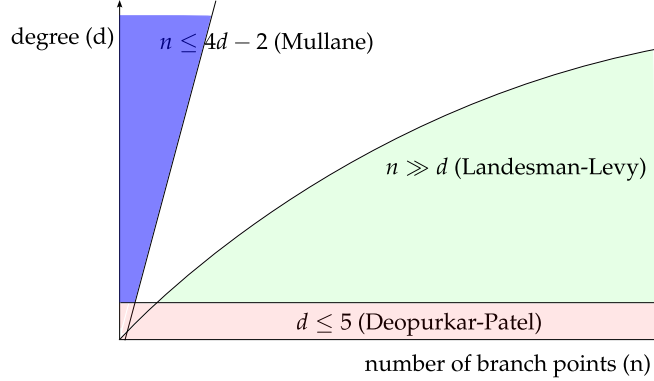


FIGURE 2. This figure depicts known cases of the picard rank conjecture, where $n = 2g + 2d - 2$ is the number of branch points. The $n \leq 4d - 2$ can be reexpressed as $d \leq g$. We computed the constant for our result where it holds roughly for $n > 2^{\binom{d}{2}}$, though it is a bit worse than that.

Theorem 4.8. *Let G be a finite group and $c \subset G$ a conjugacy class generating G . We have For n sufficiently large, $\text{Pic}(\mathcal{H}_n^{G,c}) \otimes \mathbf{Q} \simeq 0$.*

This also works semi-integrally. To explain the precise statement, we need some notation. We also recall that for G a group and c a conjugacy class $H_2(G, c)$ as defined in [Woo21, Definition p. 3] was defined as the quotient of the group cohomology $H_2(G; \mathbf{Z})$ by the image of all maps $H_2(\mathbf{Z}^2, \mathbf{Z}) \rightarrow H_2(G; \mathbf{Z})$ induced by the maps $\mathbf{Z}^2 \rightarrow G, (i, j) \mapsto x^i y^j$ for all pairs of commuting $x, y \in c$. Here is our main result toward the Picard rank conjecture.

Theorem 4.9. *Let G be a finite group and $c \subset G$ a conjugacy class generating G . For n sufficiently large and each component $Z \subset \mathcal{H}_n^{G,c}$ we have $\text{Pic}(Z) \otimes \mathbf{Z}[\frac{1}{2|G|}] \simeq \left((\mathbf{Z}/(2n-2)\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{2|G|}] \right)$.*

Let $\text{ord}_{G^{\text{ab}}}(c)$ denote the order of the image of any element of c in G^{ab} . For n large enough depending on c and divisible by $\text{ord}_{G^{\text{ab}}}(c)$, there is a finite number $H_2(G, c)$ so that $\text{Pic}(\mathcal{H}_n^{G,c}) \otimes \mathbf{Z}[\frac{1}{2|G|}] \simeq \left((\mathbf{Z}/(2n-2)\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{2|G|}] \right)^{|H_2(G,c)|}$. If n is sufficiently large and not a multiple of $\text{ord}_{G^{\text{ab}}}(c)$, $\mathcal{H}_n^{G,c}$ is empty.

Remark 4.10. The presence of the factor $|H_2(G, c)|$ merely reflects the fact that there are $|H_2(G, c)|$ many components of the Hurwitz space.

There is a branch locus map sending a cover to its branch locus. Since $\text{Pic}(\text{Conf}_{n, \mathbb{P}^1} \otimes \mathbf{Z}[\frac{1}{2|G|}]) \simeq \left((\mathbf{Z}/(2n-2)\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{2|G|}] \right)$ this is expressing the

statement that all line bundles on Hurwitz space are pulled back from the corresponding line bundles on configuration space.

4.11. Stable homology implies stable picard groups. The key input is Theorem 2.4, which roughly says that the first and second rational cohomology groups of Hurwitz space vanish, once the number n of branch points is large enough. If Hurwitz space were a smooth proper variety, the exponential exact sequence on cohomology gives

$$H^1(X, \mathcal{O}_X) \otimes \mathbb{Q} \rightarrow H^1(X, \mathbb{G}_m) \otimes \mathbb{Q} \rightarrow H^2(X, \mathbb{Q}).$$

The vanishing of $H^1(X, \mathbb{Q})$ implies $H^1(X, \mathcal{O}_X)$ vanishes as well, by Hodge theory. So the latter map is an injection, and we can identify $H^1(X, \mathbb{G}_m) \simeq \text{Pic}(X)$. So the Picard group would inject into the H^2 , and vanishing of H^2 would imply vanishing of the Picard group.

Of course, Hurwitz space is not proper and it is also not a variety (but rather a Deligne-Mumford stack) we can use its proper compactification, along with some technical lemmas to make the above idea rigorous. Basically, there is a compactification $\overline{\mathcal{H}}$ of \mathcal{H} , and by proving the vanishing of $H^1(\mathcal{H})$ one can deduce that $H^1(\overline{\mathcal{H}})$ also vanishes. And then one gets an injection from $\text{Pic } \overline{\mathcal{H}}$ into $H^2(\overline{\mathcal{H}})$ using the above argument. Finally, the vanishing of $H^2(\mathcal{H}, \mathbb{Q})$ implies that all of $H^2(\overline{\mathcal{H}}, \mathbb{Q})$ is generated by boundary divisors, and so one can deduce that $\text{Pic}(\overline{\mathcal{H}}) \otimes \mathbb{Q}$ is generated by boundary divisors and hence $\text{Pic } \mathcal{H} \otimes \mathbb{Q}$ vanishes.

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