NOTES ON BJORN'S TALK

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Let K be a number field, S a finite set of places contaiing all archimedian places,

$$\mathscr{O} := \{x \in K : v(x) \geq 0 \text{ for all } v \notin S\}$$

U a smooth affine geometrically integral curve over K and $\mathscr{U}$ any finite type $\mathscr{O}$ model of U.

Theorem 0.1 (Siegel 1929). *If $\chi(U_{\mathbb{C}}) < 0$ then $\mathscr{U}(\mathscr{O})$ is finite.*

$U_{\mathbb{C}} = X - R$ for X smooth projective of genus g and R finite nonempty. Then, $\chi(U_{\mathbb{C}}) = (2-2g) - \#R$.

Example 0.2. Take $\mathscr{U} = \{x + y = 1\} \subset \mathbb{G}_m \times \mathbb{G}_m$. Then $\mathscr{U} \simeq \mathbb{P}^1_{\mathscr{O}} - \{0, 1, \infty\}$. Then, $\chi(U_{\mathbb{C}}) = 2 - 3 < 0$. So, Siegel says $\mathscr{U}(\mathscr{O}) = \{(x, y) \in \mathscr{O}^\times \times \mathscr{O}^\times : x + y = 1\}$ is finite.

Corollary 0.3. *Take $\mathscr{O} = \mathbb{Z}[1/2, 1/3]$. Then $3^r - 2^s = 1$ has only finitely many solutions.*

Theorem 0.4 (Baker + ϵ theorem, 1966). *Given K a number field, and $|\bullet|$ an absolute value on K, $\alpha_1, \ldots, \alpha_n \in K, \varepsilon > 0$, the set of $(b_1, \ldots, b_n) \in \mathbb{Z}^n$,*

$$0 < \left|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1\right| < e^{-\varepsilon \max |b_i|}.$$

is finite and can be effectively bounded.

Corollary 0.5. *Take $\alpha_1 = 3, \alpha_2 = 2$, then*

$$0 < 3^r 2^{-s} - 1 \leq 2^{-s}$$

has only finitely many solutions with an explicit bound, so we can also bound the number of solutions to $3^r - 2^s = 1$ from the previous Corollary 0.3.

Corollary 0.6 (Baker). *$\mathscr{U}(\mathscr{O})$ is compatible for certain $\mathscr{U}$ such as $\{x + y = 1\}$ in $\mathbb{G}_m \times \mathbb{G}_m$.*

Call U *good* if there exists a nondegenerate morphism $U \to \mathbb{G}_m \times \mathbb{G}_m$, meaning the image is not contained in a coset of a proper algebraic subgroup.

Theorem 0.7 (Bilu, 1995). *If U is good, then $\mathscr{U}(\mathscr{O})$ is finite and computable.*

Lemma 0.8. *Let $U = X - R$ over K. The following are equivalent:*

1. *U is good*
2. $\operatorname{rk}_{\mathbb{Z}} \mathscr{O}(\mathscr{U})^\times / K^\times \geq 2$
3. *There are two indpenedent principal divisors on X supported on R*
4. *There are 2 independent relations between the points of R in $\operatorname{Pic}(X)$.*

If these fail, try

1. enlarging K or S.
2. (Chevalley-Weil descent) Replace U by a finite étale cover $f : U' \to U$. One has $\mathscr{U}(\mathscr{O}) \subset f(\mathscr{U}'(\mathscr{O}'))$ for some computable $\mathscr{O}' \subset K'$.

Example 0.9 (Baker-Coates, 1970)**.** Let $U = E - \{0\}$. Replace U by $U' = E - E[2]$, the preimage of U under multiplication by 2 on E. This has linear equivalences $2P \equiv 2O, 2Q \equiv 2O$ in $\operatorname{Pic}(E)$. Thus $\mathscr{U}(\mathscr{O})$ is computable.

Remark 0.10. This can be viewed directly as a Thue equation.

Question 0.11. Does every smooth affine curve over $\overline{\mathbb{Q}}$ of genus $g \geq 2$ admit a finite étale cover that is good?

Conjecture 0.12. No.

Theorem 0.13. *The answer is no if you replace $\overline{\mathbb{Q}}$ by $\mathbb{C}$.*

Here is a stronger version:

Theorem 0.14. *For any smooth projective curve X of genus $g \geq 2$ over $\mathbb{C}$, there exists $x \in X(\mathbb{C})$ such that $X - \{x\}$ has no finite étale cover with even one non-constant morphism to $\mathbb{G}_m$. Equivalently, for every cover $Y \to X$ ramified at mosst above x, the preimages $\{y_1, \ldots, y_n\} := f^{-1}(x)$ are $\mathbb{Z}$-independent in $\operatorname{Pic}(Y)$.*

1. PROOF OF THEOREM 0.14

It suffics to consider Galois covers $f : Y \to X$, possibly ramified above x. For each finite group G, there exist only finitely many such G covers, (by the Riemann existence theorem). For simplicity, let's suppose there is just one such G cover. Let's also assume $y_1, \ldots, y_n$ are the distinct points of $f^{-1}(x)$.

Now, Bjorn drew a picture of a family over $X \times X$ (really $M \times X$ where M is a finite étale cover of X), where the base X parameterizes points $x \in X$, and the fibers are Y_x. The fibers over z in the second copy of x are divisors swept out by the points $\{y_1, \ldots, y_n\}$. We get disjoint divisors $D_1, \ldots, D_n$. The total space of the family is $\mathscr{Y} \to X \times X$.

The stronger theorem will be implied by the following claim.

Lemma 1.1. *For each finite group G, for all but countably many $x \in X(\mathbb{C})$, $y_1, \ldots, y_n$ are $\mathbb{Z}$-independent in $\operatorname{Pic} Y_x$.*

We can rephrase the lemma as follows, by reducing to the generic version of the above claim:

Lemma 1.2. *For the generic point η, $y_1, \dots, y_n$ are $\mathbb{Z}$-independent in* $\operatorname{Pic} Y_\eta$.

This lemma would be implied by the following:

Lemma 1.3. *The divisors $D_1, \dots, D_n, Y_0$ are $\mathbb{Z}$-independent in* $\mathrm{NS}(\mathscr{Y})$.

One can check this by computing intersection numbers.

Lemma 1.4. *The intersection matrix is*

$$\begin{pmatrix} d & \cdots & 0 & 1 \\ \vdots & \ddots & 0 & 1 \\ 0 & \cdots & d & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$$

for some $d \neq 0$ (using that d is a multiple of $2 - 2g$, so it fails for $g = 1$). This has nonzero determinant.

2. PROSPECTS FOR GETTING THE NO OVER $\overline{\mathbb{Q}}$

Question 2.1. What would you have to do to make this argument work over $\overline{\mathbb{Q}}$?

The problem is that if you puncture in a very general place, there might be none defined over $\overline{\mathbb{Q}}$. That is, suppose X is over $\overline{\mathbb{Q}}$. Can we find $x \in X(\overline{\mathbb{Q}})$ as in Theorem 0.14. For each finite group G, we can define a set $S_G := \{x \in X(\overline{\mathbb{Q}}) : y_1, \dots, y_n\}$ are $\mathbb{Z}$-independent in $\operatorname{Pic} Y_x$.

To prove the stronger theorem, we want

To prove the stronger theorem, we want

$$\cup_G S_G \neq X(\overline{\mathbb{Q}}) \tag{2.1}$$

By Silverman's specialization theorem, S_G is a set of bounded height. This is not good enough, because we have to take a union over countably many Gs. If the bound from Silverman's specialization theorem could be made independent of G, we'd get (2.1).

REFERENCES