BHARGAVA'S CONJECTURE AND MUMFORD'S CONJECTURE

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In this note, we would like to exposit a beautiful relationship between Bhargava's conjecture on the constant for the number of S_d extensions and Mumford's conjecture on the stable cohomology of $\mathscr{M}_g$. This relationship is largely based on work of Andrea Bianchi, who provided a recent new proof of Mumford's conjecture. Although Bianchi was not aware of this relationship when he proved Mumford's conjecture, we will explain how it could have led one to predict his beautiful result, which is described in Theorem 3.2.

To start, let's review Bhargava's conjecture and Mumford's conjecture.

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1. Bhargava's conjecture

Bhargava's conjecture is a refinement of Malle's conjecture. Let's begin by recalling Malle's conjecture.

Conjecture 1.1 (Malle)**.** Fix a finite group G. The number of G extensions K over $\mathbb{Q}$ of discriminant at most X has asymptotic growth of the form $C(G)X^{1/a(G)}(\log X)^{b(G)-1}$, with a specific prediction for the values of $a(G)$ and $b(G)$.

In general, the specific values of $b(G)$ predicted by Malle are fairly cumbersome to write down (so we do not do it here) and moreover are not correct in general. Additionally, Malle does not predict a value for $C(G)$, although there has been much recent work on this. See [AOWW25, §1.4] for a recent summary of work on this topic. However, in the special case $G = S_d$, one can see that $a = 1$ and $b = 1$, and hence one gets that the number of extensions of discriminant at most X should be a multiple of X. It is natural to ask what the constant $C(S_d)$ is in this case. This is the subject of Bhargava's conjecture.

Conjecture 1.2 (Bhargava [Bha07, Conjecture 1.2])**.** Let $N_d(X)$ denote the number of S_d field extensions of degree d having absolute discriminant at most X. Then

$$\lim_{X\to\infty} \frac{N_d(X)}{X} = c_d = \frac{r_2(S_d)}{2} E_d,$$

where $r_2(S_d)$ denotes the number of elements of order at most 2 in S_d and

$$E_d = \prod_{p \text{ prime}} \frac{p-1}{p} \left(\sum_{k=0}^{d} \frac{q(k, d-k)}{p^k} \right),$$

for $q(n,k)$ the number of partitions of n into at most k nonempty parts. In other words, it is the number of ways to write n as a sum of k positive numbers, where two such ways are equivalent if one is obtained from the other by reordering the numbers.

Example 1.3. We use $\zeta(s) = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \cdots)$ to denote the Riemann Zeta function The first few values of c_d turn out to be

$$\begin{aligned} c_2 &= \zeta(2)^{-1}/2 \\ c_3 &= \zeta(3)^{-1}/3 \\ c_4 &= \frac{5}{24} \prod_p \left(1 + p^{-2} - p^{-3} - p^{-4}\right). \end{aligned}$$

Remark 1.4. Although the first couple values of c_i can be expressed in terms of Zeta values, c_4 and later values do not appear to have any nice expression in terms of Zeta values.

However, one can also look at what happens as d grows. In that case, as mentioned in [Bha07, §7], we have the following:

Lemma 1.5.

$$\lim_{d\to\infty} E_d = \prod_{i=2}^{\infty} \zeta(i) = \zeta(2)\zeta(3)\zeta(4)\cdots.$$

Proof. By definition, $E_d = \prod_{p \text{ prime}} \frac{p-1}{p} \left(\sum_{k=0}^{n} \frac{q(k,n-k)}{p^k} \right)$, so the limit will be

$$\prod_{p \text{ prime}} \frac{p-1}{p} \left(\sum_k \frac{\text{the number of partitions of k}}{p^k} \right).$$

We can express

$$\begin{aligned}
&\frac{p-1}{p}\left(\sum_k \frac{\text{the number of partitions of k}}{p^k}\right)\\
&=\frac{p-1}{p}\prod_{j=1}^{\infty}\left(1+p^{-j}+p^{-2j}+p^{-3j}+\cdots\right)\\
&=\frac{p-1}{p}\left(1+p^{-1}+p^{-2}+\cdots\right)\prod_{j=2}^{\infty}\left(1+p^{-j}+p^{-2j}+p^{-3j}+\cdots\right)\\
&=\frac{p-1}{p}\frac{p}{p-1}\prod_{j=2}^{\infty}\left(1+p^{-j}+p^{-2j}+p^{-3j}+\cdots\right)\\
&=\prod_{j=2}^{\infty}\left(1+p^{-j}+p^{-2j}+p^{-3j}+\cdots\right).
\end{aligned}$$

And hence, since $\zeta(r) = \prod_p \left(1+p^{-r}+p^{-2r}+p^{-3r}+\cdots\right)$, we obtain the lemma. (Strictly speaking, one should probably say some more things to justify convergence issues and reordering terms in the way we did. We leave a careful verification of this to the reader.) □

We'd next like to understand what this beautiful infinite product has to do with Mumford's conjecture.

2. Mumford's conjecture

In the past several decades, there has been much interest in the cohomology of the moduli space of curves. It was shown by Harer that dim $H_i(\mathscr{M}_g)$ stabilizes as g grows. Specifically, once g is sufficiently large (roughly more than $3i/2$,) the dimension of this group is independent of g. Mumford conjectured what the stable value of this cohomology group was.

Conjecture 2.1 (Mumford [Mum83, p. 309])**.** For g sufficiently large, the ith rational cohomology of $\mathscr{M}_g$ agrees with the ith graded part of the ring

$$\mathbb{Q}\left[\kappa_2, \kappa_4, \kappa_6, \kappa_8, \cdots\right],$$

the free polynomial ring with a generator in degree $2i$ for each positive integer i.

Mumford's conjecture was originally proved by Madsen and Weiss. There have since been several proofs, including a recent one of a different nature via Hurwitz spaces due to Andrea Bianchi [Bia23, Corollary 6.4].

Let's now massage the formula in Mumford's conjecture a bit. We can write

$$\mathbb{Q}\left[\kappa_2, \kappa_4, \kappa_6, \kappa_8, \cdots\right] = \otimes_{i=1}^{\infty} \mathbb{Q}\left[\kappa_{2i}\right].$$

Now, let's record the Poincaré polynomial of the above. (Note that everything is in even degree, so we won't have to worry about possible sign issues coming from odd cohomology groups.) We get that the Poincaré polynomial is

$$\prod_{i=1}^{\infty} \left(1 + (T^2)^i + (T^2)^{2i} + (T^2)^{3i} + \cdots\right).$$

Now, just for fun, let's look at what happens when we plug in q^{-1} for T^2.

$$\prod_{i=1}^{\infty} \left(1 + q^{-i} + q^{-2i} + q^{-3i} + \cdots\right) = \prod_{i=1}^{\infty} \frac{1}{1 - q^{-i}}.$$

Recall that the zeta function of $\mathbb{A}^1_{\mathbb{F}_q}$ is

$$\zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(r) = \frac{1}{1 - q \cdot q^{-r}}.$$

Therefore, $\frac{1}{1-q^{-i}} = \frac{1}{1-q\cdot q^{-(i+1)}} = \zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(i+1)$. Hence, when we plug in q^{-1} for T^2 in the Poincaré polynomial for the stable cohomology of $\mathscr{M}_g$, we get

$$\zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(2)\zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(3)\zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(4)\cdots.$$

Question 2.2. Is it a coincidence that this is the same as the limiting constant in Bhargava's conjecture?

Of course not. In the rest of this note, we'd like to explain the relation between these two conjectures.

3. Bianchi's result

The key to relating Bhargava's conjecture and Mumford's conjecture is a recent result of Bianchi.

Let $\mathscr{M}_g^1$ denote the moduli stack of genus g curves with one boundary component. Harer's original proof showed that the cohomology of $\mathscr{M}_g^1$ agrees in a stable range with that of $\mathscr{M}_g$.

Definition 3.1. Let $\mathrm{Bra}^d_{\square \,\mathrm{rel}\, \partial}$ denote the space defined in [DP24, §4.1] whose points parameterize degree d branched covers of the square with a trivialization on the boundary.

The next result is due to Bianchi and may be extracted from [Bia23, §6]. A direct proof follows from combining [DP24, Proposition 5.4] with [DP24, Proposition 5.5].

Theorem 3.2 (Bianchi)**.**

$$\operatorname{colim}_{d\to\infty} H_i(\Omega_0 B\,\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial};\mathbb{Z}) = \operatorname{colim}_{g\to\infty} H_i(\mathscr{M}^1_g;\mathbb{Z}).$$

The colimit in g is obtained via the Harer stability maps while the colimit in d is obtained via the inclusion of degree d branched covers into degree $d+1$ branched covers by adding a disjoint covering sheet.

Using a variant of the above as a first step, Bianchi goes on to give a new proof of Mumford's conjecture by computing the stable cohomology of Hurwitz spaces sufficiently well.

4. Bhargava's conjecture over function fields

It turns out that (a version of) Bhargava's conjecture over function fields can be reformulated as predicting the stable cohomology of CHur^d_g. In Lemma 4.2 below, we relate the stable cohomology of CHur^d_g to the cohomology of $\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial}$. Strictly speaking this doesn't really directly connect to Bhargava's conjecture because it counts all degree d covers, not just S_d degree d covers. When d is prime, it should connect to Bhargava's conjecture more directly. The reason understanding stable cohomology is related to understanding the number of covers is via the Grothendieck-Lefschetz trace formula, which will be explained further in the next section.

Definition 4.1. Let CHur^d_g denote the Hurwitz space parameterizing degree d covers $X \to \mathbb{P}^1$ with X smooth projective and geometrically connected of genus g which are unramified over ∞, together with a marked ordering of the d sheets of the cover over ∞.

Lemma 4.2. *We have $H_i(\Omega_0 B\,\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial};\mathbb{Z}) \simeq \operatorname{colim}_{g\to\infty} H_i(\mathrm{CHur}^d_g;\mathbb{Z})$.*

Proof. This essentially follows from the group completion theorem, as we now explain. The connected components of $\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial}$ are described in [DP24, Remark 4.3], and all the components parameterizing connected covers are given by CHur^d_g for some $g \geq 0$. Moreover, the category of connected components $E\pi_0\,\mathrm{Bra}_{\square\,\mathrm{rel}\,\partial}$ is a filtered category, as was shown in [DP24, Lemma 5.3]. Finally, if we only consider the components parameterizing connected covers, corresponding to some CHur^d_g, these components are cofinal, as the monoid product of any cover degree d cover with a connected cover is again a connected cover. For $s \in \pi_0(\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial})$, let X_s denote the corresponding connected component of $\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial}$. We then have

$$H_i(\Omega_0 B\,\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial};\mathbb{Z}) \simeq \mathrm{colim}_{s\in\pi_0(\mathrm{Bra}^d_{\square\,\mathrm{rel}\,\partial})} H_i(X_s;\mathbb{Z}) \simeq \mathrm{colim}_{g\to\infty} H_i(\mathrm{CHur}^d_g;\mathbb{Z}),$$

where the first equality follows from the group completion theorem and the second follows from cofinality of the components parameterizing connected covers, as shown above. □

Combining Lemma 4.2 with Theorem 3.2, we obtain the following.

Corollary 4.3. $\mathrm{colim}_{d\to\infty}\,\mathrm{colim}_{g\to\infty} H_i(\mathrm{CHur}^d_g;\mathbb{Z}) \simeq \mathrm{colim}_{g\to\infty} H_i(\mathscr{M}^1_g;\mathbb{Z})$.

5. Relating Bhargava's conjecture to Mumford's conjecture using Bianchi's result

Now, let's explain the strange coincidence between Bhargava's conjecture and Mumford's conjecture to answer Question 2.2 using Bianchi's result.

The key point will be that Bhargava's conjecture is roughly about the finite field point counts of Hurwitz spaces, and one can relate that to the Euler characteristic of Hurwitz spaces, which is then related to the Euler characteristic of $\mathscr{M}_g$ by Bianchi's result.

We now define an analog of Bhargava's constant in the function field setting.

Definition 5.1. Define $N_{q,d}(X)$ to be the number of degree d field extensions of $\mathbb{F}_q(t)$ with d ordered marked points over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$, with Galois closure S_d, and with discriminant at most X. Define

$$c_{q,d} := \lim_{n\to\infty} \frac{N_{q,d}(q^{2n})}{q^{2n}},$$

assuming this limit exists.

In order to relate the previous sections which are about all degree d covers, to the value of this constant, which is about S_d degree d covers, for prime d, the following lemma will be key.

Lemma 5.2. *When d is prime, the locus of points of CHur^d_g corresponding to covers whose Galois closure is not S_d has codimension which grows linearly as a function of g.*

Proof. In general, a dimension computation shows that the only types of degree d covers other than S_d covers whose codimension does not tend to ∞ linearly in g are those covers $X \to \mathbb{P}^1$ which factor through an intermediate cover Y. Since the degree of $X \to \mathbb{P}^1$ is the product of the degrees of $X \to Y$ and $Y \to \mathbb{P}^1$, such a factorization cannot occur when d is prime. □

The following lemma is not necessary, but we include it to illustrate the connection to Bhargava's conjecture.

Lemma 5.3. *For d prime and q sufficiently large relative to d, we have*

$$c_{q,d} = \lim_{g\to\infty} \frac{\#\operatorname{CHur}^d_g(\mathbb{F}_q)}{q^{\dim \operatorname{CHur}^d_g}}.$$

Proof. Using Lemma 5.2, the points corresponding to covers which are not S_d Galois have codimension tending to ∞ in g, and hence it follows from [LL25, Theorem 10.1.8] (with a little bit of work) that the number of $\mathbb{F}_q$ points of CHur^d_g corresponding to such covers is also negligible. □

Remark 5.4. In general, $\lim_{g\to\infty} \frac{\#\operatorname{CHur}^d_g(\mathbb{F}_q)}{q^{\dim \operatorname{CHur}^d_g}}$ predicts the constant associated to all degree d covers with some marked ordering of points over ∞, as opposed to only degree d, S_d covers.

The next step to relate Bhargava's conjecture to the stable cohomology of $\mathscr{M}_g$ is to use the Grothendieck Lefschetz trace formula. We now recall its statement.

Theorem 5.5 (Grothendieck-Lefschetz trace formula)**.** *For X a smooth Deligne Mumford stack of dimension n over $\mathbb{F}_q$,*

$$\frac{\#X(\mathbb{F}_q)}{q^n} = \sum_{i=0}^{2n} (-1)^i \operatorname{tr}(\operatorname{Frob}_q^{-1} | H^i(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)).$$

Above, Frob_q denotes the geometric Frobenius.

Now, the analog of Bhargava's conjecture [Bha07, Conjecture 1.2] predicts the following:

Conjecture 5.6. For q a prime power relatively prime to $d!$, $c_{q,d}$ exists and is given by

$$c_{q,d} = \prod_{x\in \mathbb{A}^1_{\mathbb{F}_q}} \frac{\#\kappa(x)-1}{\#\kappa(x)} \left(\sum_{k=1}^{d} \frac{q(k,d-k)}{\#\kappa(x)} \right).$$

The analog of Lemma 1.5 predicts that

Conjecture 5.7.

$$\lim_{d\to\infty} c_{q,d} = \prod_{i=2}^{\infty} \zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(i).$$

Remark 5.8. One might wonder where the factor $r_2(d)/2d!$ in Bhargava's conjecture [Bha07, Conjecture 1.2] goes in the function field case. This factor can be viewed as restricting the ramification over the infinite place to be of order 1 or 2. In our setup, we just work with covers unramified over infinity, so Bhargava's heuristic would predict an additional factor of $1/2d!$ in this case. This additional factor is related to the difference in the way Bhargava is counting things from how we are counting things. The factor of $d!$ comes from dividing by the automorphism group of the Galois cover, which has order $d!$. The factor of 2 comes from the fact that any cover unramified over ∞ has discriminant q^n with n even. The fact that n must be even accounts for the factor of 2 in the denominator when translating to the number field case. (A recent paper [LS24] gives a different explanation for this factor of 2.)

Now, if we were to keep pressing the analogy on finite field point counts, we could start to get into some trouble due to not being able to properly deal with the unstable cohomology. However, we next observe that all the stable cohomology of the Hurwitz spaces we are considering (as $d \to \infty$) is pure with geometric Frobenius acting on H^{2i} having eigenvalues equal to q^i. This really uses more of what Bianchi shows, (see [DP24, Corollary 7.14] for part of this statement,) although one could also just assume this as part of a generalized Bhargava's conjecture if one wishes to. Using this, we can translate the above point counting result to a result on the cohomology of Hurwitz spaces. Namely, let $Z_{\mathbb{A}^1}(i) = \frac{1}{1-T^2T^{-2i}}$. Using Lemma 5.2, Bhargava's conjecture in the $d \to \infty$ limit Conjecture 5.7 can be rephrased as saying that $\operatorname{colim}_{d\to\infty} \operatorname{colim}_{g\to\infty} \dim H_{2i}(\mathrm{CHur}_g^d, \mathbb{Q})$, tends to the coefficient of T^i in

$$Z_{\mathbb{A}^1}(2) Z_{\mathbb{A}^1}(3) \cdots . \tag{5.1}$$

as $d \to \infty$.

Finally, using Corollary 4.3, this also agrees with the dimension of the limiting cohomology $\lim_{i\to\infty} H_{2i}(\mathscr{M}_g^1)$ (and all odd stable cohomology vanishes). It then follows from the computation in §2 (by replacing q^{-1} there formally with T^2, which conforms with the fact that the trace of Frob_q^{-1} on H^{2i} is q^{-1} using purity of this cohomology) that this limiting cohomology is precisely $\mathbb{Q}[\kappa_2, \kappa_4, \ldots,]$. In other words, Mumford's conjecture holds true. This finishes our explanation of the relation between Bhargava's conjecture and Mumford's conjecture.

Remark 5.9. One can view the note written above from one perspective as indicating the following: Bhargava's conjecture could have lead to the discovery of Theorem 3.2. As mentioned, in reality, it may well be that no one has realized this before. Still, this is one of my favorite examples of conjectures in disparate fields of math "speaking to each other" across the lines of these disciplines.

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