MATH272X: DIFFEOMORPHISMS OF DISKS – PAPER TOPICS

1. Details about the papers

The goal of the midterm and final papers is to produce nice expository articles about topics in algebra, geometry or topology related to the topic of this course in a broad sense. The intention is that you will learn something from it, while the broader mathematical community benefits from the time you have spent understanding and summarizing some difficult topic in a field where the literature is often lacking.

The midterm paper should be 4–8 pages, and the term paper 8–16 pages. They should be in LaTeX, preferably be handed in electronically, and you should not collaborate with other students. As listed on the syllabus, you will have to hand in a topic and rough draft before handing in the final papers, so I can give comments and suggestions. The main goal is produce understandable and engaging writing, which draws attention to the important ideas and subtleties. You can refer to references for the details, and hence should not feel obliged to give full proofs (though of course it would be nice if you could include them).

1.1. Important dates.

- Midterm paper topic due: Wednesday, October 4th.
- Midterm paper rough draft due: Wednesday, October 18th.
- Midterm paper due: Wednesday, October 25th (2:00pm).
- Final paper topic due: Wednesday, November 8th.
- Final paper rough draft due: Wednesday, November 29th.
- Final paper due: Monday, December 11th (2:00pm).

2. Topic suggestions

Below I give some suggestions for topics, and the one that I think are more suitable for a final paper have been marked with an asterisk. These topics may be too broad, so you might want to select among the different aspects of a topic. Feel free to pick a different topic, and I will comment on your suggestion when you submit it.

2.1. Point-set topology.

- Diffeomorphism groups have the homotopy type of a CW complex, [Pal66].
- The exponential map is not locally surjective, [Mil84].
- Morse homology.

2.2. Low dimensions.

- * Friberg's proof of $\mathrm{Top}(2) \simeq O(2)$ [Fri73].
- * Hamilton's proof the existence and uniqueness of PL-structures on topological 3 manifolds [Ham76]. Can you give a direct proof of the smooth case analogous to Hatcher's note [Hat13]?
- Explain the statement and proof of the Poincaré-Bendixson theorem, [CC00].
- Describe the classification of 3-manifolds, e.g. [Hat76].
- * The "graph model" for diffeomorphism groups of reducible 3-manifolds, which describes the diffeomorphism group of M in terms of a space of graphs and the diffeomorphism groups of its prime summands [HL84]. Hatcher has an incomplete note on this [Hat10].
- A summary of Hatcher's proof of the Smale conjecture [Hat83].
- * Give a proof of Smale's theorem along the lines of [Hat83] (this is not in the literature, but generally believed to be possible).
- * The Smale conjecture for other 3-manifolds than S^3 or D^3, e.g. [Hat76], [Gab01], [HKMR12].

- A collapsible PL manifold is a disk (Corollary 3.28 of [RS82]). This is an interesting characterization of disks in the category of piecewise linear manifolds.
- A counterexample to the Whitney trick in dimension 4 [Lac96].
- Kirby calculus as a tool for understanding smooth 4-manifolds; Morse theory in dimension 4 is possible to understand, but non-trivial enough to prove interesting results. You can describe it and either proof that it works, or give applications. See e.g. [GS99].
- Describe an exotic structure on $\mathbb{R}^4$ (of course you couldn't give a complete proof, but you can explain what it is), see [Sco05].
- * Gabai's recent proof of the lightbulb trick in dimension 4 [Gab17].
- Freedman-Gompf-Morrison-Walker's ideas on the smooth Poincaré conjecture in dimension 4 [FGMW10].
- Describe a Mazur manifold [Maz61].

2.3. High dimensions.

- Stallings' proof of the uniqueness of smooth structures on $\mathbb{R}^n$, $n \geq 5$ [Sta62].
- Milnor's construction of exotic 7-spheres, [Mil56].
- Explicit sphere eversions [Sul99].
- The existence of handlebody structures on topological manifolds of dimension $\neq 4$ [KS77].
- Wall's classification on $(n-1)$-connected $2n$-dimensional manifold (for $n \geq 3$), [Wal65].
- * Haefliger's classification of smooth embeddings $S^n \hookrightarrow S^{n+k}$ [Hae66].
- * The relationship between the Kervaire invariant one problem and manifolds, e.g. [Bro00].
- Fibering manifolds over the circle, [BL66].
- Describe the immersion conjecture, [CT89].
- * Cohen's proof of the immersion conjecture using Brown-Gitler spectra, [Coh85].
- Describe the Gromoll filtration.
- Describe the theory of smoothing PL-manifolds as in [HM74].

2.4. Foliations.

- * The relationship between diffeomorphism groups as discrete groups and foliations [Hae71, Thu74].
- Perfection of diffeomorphism groups as discrete groups, i.e. their abelianization is trivial [Man16].
- * Explain the relationship between perfection of diffeomorphism groups and connectivity of Haefliger space, e.g. [Mat74].

2.5. Surgery theory and algebraic K-theory.

- * A description of how higher Reidemeister torsion detects non-trivial homotopy groups of spheres [Igu02].
- * The computation of π_1 of the concordance diffeomorphisms of a disk [Hat78].
- * An example of a diffeomorphism that is pseudo-isotopic to the identity but not isotopic [Hat78].
- * Variations of the Farell-Hsiang theorem for other manifolds [Bur79].
- Explain how $\mathrm{Wh}^{\mathrm{Diff}}(*)$ and $K(\mathbb{Z})$ are rationally related.
- The Wall finiteness obstruction, e.g. [FR01].
- Computing $K_0(\mathbb{Z}[G])$, e.g. [Rim59].
- * The computation of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ [Rog03].
- A high-level overview of how surgery theory works [LÖ2, Wal99].
- Describe the Farrell-Jones conjecture [KL05].
- Poincaré duality spaces and the Spivak normal fibration.

2.6. Field theories.

- Describe the notion of a framed function and why it's useful, [Igu87].
- * The Eliashberg-Mischachev proof of the framed function theorem, [EM12].
- * Kontsevich's invariants for homology 3-spheres, [Kon94].

- Explain how the Barratt-Priddy-Quillen-Segal theorem is an example of the computation of the homotopy type of the cobordism category, [Hat11].
- Describe Galatius' cobordism category of graphs [Gal11]. How does it relate to other cobordism categories?
- * Discuss the homotopy theory of Madsen-Tillmann spectra.
- * The relationship between the homotopy theory of Madsen-Tillmann spectra and symmetry-protected topological phases [FH16].

References

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