A BRIESKORN SPHERE

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We discuss the following result, which accounts for a small subset of a much more general class of examples studied systematically by Brieskorn [Bri66].

Theorem 1 (Brieskorn)**.** *For $m > 1$, the manifold*

$$\Sigma = \left\{ z \in \mathbb{C}^{2m+1} \mid z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2m}^2 = 0,\, |z| = 1 \right\}$$

is homeomorphic, but not diffeomorphic, to S^{4m-1}.

The proof will make use of a portion of Kervaire-Milnor's landmark work on exotic spheres [KM63]. Recall that the *signature* $\sigma(M)$ of a $4m$-manifold M is the signature of the quadratic form determined by Poincaré duality on $H_{2m}(M;\mathbb{R})$, which is to say the number of positive eigenvalues minus the number of negative eigenvalues of the associated matrix.

Theorem 2 (Kervaire-Milnor)**.** *Let Σ^{4m-1} be a smooth manifold homeomorphic to S^{4m-1}. If $\Sigma = \partial W$ with W parallelizable, then Σ is diffeomorphic to S^{4m-1} if and only if*

$$\frac{1}{8}\sigma(W) \equiv 0 \mod \frac{3 + (-1)^{m+1}}{2} 2^{2m-2} \left(2^{2m-1} - 1\right) \mathrm{Num}\left(\frac{B_{2m}}{4m}\right),$$

where the Bernoulli numbers B_{2m} are defined by the generating function

$$\frac{\sqrt{z}}{\tanh \sqrt{z}} = 1 + \sum_{m \geq 1} (-1)^{m-1} \frac{2^{2m} B_{2m}}{(2m)!} z^m.$$

Assuming this result, the proof of Theorem 1 will take place in two steps. First, we verify that Σ is a topological sphere, in the process identifying it as the boundary of a parellelizable manifold. Second, we compute that the signature of the bounding manifold is equal to 8, up to sign.

Notation 3. Setting $f(z) = z_0^{a_0} + \cdots + z_n^{a_n}$ with $n > 2$, we write $V(f)$ for the zero set of f in $\mathbb{C}^{n+1}$, and we write $\Sigma = V(f) \cap S^{2n+1}$. We write C_{a_j} for the cyclic group of order a_j with fixed generator ω_j, and we set $G = C_{a_0} \times \cdots \times C_{a_n}$.

Our strategy in understanding the topology of Σ will be to understand that of $S^{2n+1} \setminus \Sigma$ and invoke duality. The idea behind the approach to this complement is that, when leaving the zero locus of f, one must leave in some direction, and so the complement fibers over the space of possible directions. More precisely, defining

$$\tilde{\varphi} : \mathbb{C}^{n+1} \setminus V(f) \to S^1$$
$$z \mapsto \frac{f(z)}{|f(z)|},$$

we have the following result.

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Theorem 4 ([Mil68, 4.8, 5.2, 6.1, 6.4])**.** *In the commuting diagram*

$$\begin{array}{ccc} S^{2n+1}\setminus\Sigma & \hookrightarrow & \mathbb{C}^{n+1}\setminus V(f) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}} \\ S^1 & = & S^1, \end{array}$$

the vertical maps are smooth fiber bundles and the top map is a fiberwise homotopy equivalence. Moreover, the closure of $F_0 := \varphi^{-1}(1)$ is a smooth manifold with boundary Σ, and both Σ and F_0 are connected and simply connected.

The group G acts on F_0 by $\omega_j z = (z_0, \ldots, e^{2\pi i/a_j} z_j, \ldots, z_n)$, and the action of $\pi_1(S^1, 1) =: T$ is via the homomorphism $T \to G$ specified by sending the generator t to $\omega := \omega_0 \cdots \omega_n$.

Proposition 5. *There is a homotopy equivalence*

$$F_0 \simeq \bigvee_{\mu(f)} S^n,$$

where $\mu(f) = \prod_{j=0}^{n}(a_j - 1)$.

Remark 1. The number $\mu(f)$ is exactly the multiplicity of the singularity of f at the origin. This phenomenon is generic—see [Mil68, 7.2].

Recall that the *join* of the spaces A and B is the pushout

$$A * B := C(A) \times B \coprod_{A\times B} A \times C(B).$$

For example, $\Delta^n \cong \mathrm{pt}^{*(n+1)}$.

Proof of Proposition 5. Set $J = C_{a_0} * \cdots * C_{a_n}$. We have an embedding

$$J \to \tilde{\varphi}^{-1}(1)$$
$$(t_j\omega_j^{r_j})_{j=0}^n \mapsto (e^{2\pi i r_j/a_j} t_j^{1/a_j})_{j=0}^n,$$

which, by a sequence of straight-line homotopies, is a G-equivariant deformation retract. General properties of the join imply that J is $(n-1)$-connected, and in degree n we have

$$H_*(F_0) \cong H_*(\tilde{\varphi}^{-1}(1)) \cong H_*(J) \cong H_*\left(\mathbb{Z}[G]\sigma \xrightarrow{\sum(-1)^j\partial_j} \mathbb{Z}[G]\{\partial_j\sigma\}/(\partial_j\sigma - \omega_j\partial_j\sigma) \to \cdots\right),$$

where σ is the n-simplex given by the join of the respective units in the C_{a_j}. Thus, setting $\eta = \prod_{j=0}^{n}(1 - \omega_j)$, we also have that

$$H_n(F_0) \cong \mathbb{Z}[G]\eta \cong \frac{\mathbb{Z}[G]}{\mathrm{Ann}(\eta)} \cong \bigotimes_{j=0}^{n} \frac{\mathbb{Z}[C_{a_j}]}{\mathrm{Ann}(1-\omega_j)} \cong \bigotimes_{j=0}^{n} \frac{\mathbb{Z}[C_{a_j}]}{1 + \omega_j + \cdots + \omega_j^{a_j-1}}.$$

This group is free Abelian of rank $\mu(f)$. Since F_0 is simply connected, the claim follows. □

Corollary 6. *The manifold F_0 is parallelizable.*

Proof. Since the top homology of F_0 vanishes, F_0 is non-compact by Poincaré duality, so it suffices to show that TF_0 is stably trivial. Since φ is locally trivial, the normal bundle of F_0 in $S^{2n+1} \setminus \Sigma$, and hence in S^{2n+1}, is trivial. Since the normal bundle of S^{2n+1} in $\mathbb{C}^{n+1}$ trivial, the claim follows. □

Corollary 7. *There are isomorphisms*

$$\widetilde{H}_i(S^{2n+1} \setminus \Sigma) \cong \begin{cases} \operatorname{coker}(1-\omega) & i = n \\ \ker(1-\omega) & i = n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 5, the E^2-page of the Serre spectral sequence for the fibration φ vanishes outside of bidegrees $(0, n)$ and $(1, n)$. For degree reasons, there can be no nonzero differentials, so the spectral sequence collapses at this page. Using the identification $S^1 \simeq K(T, 1)$, we have $E^2_{*,n} \cong H_*(T; H_n(F_0))$, and the claim follows from consideration of the exact sequence of $\mathbb{Z}[T]$-modules

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{t \mapsto (1-t)} \mathbb{Z}[T] \xrightarrow{t \mapsto 0} \mathbb{Z} \longrightarrow 0.$$

□

We write Δ for the characteristic polynomial of ω, thought of as an operator on $H_n(F_0)$. In other words,

$$\Delta(x) = \det(x - \omega)$$

Proposition 8. *The manifold Σ is homeomorphic to S^{2n-1} if and only if $|\Delta(1)| = 1$.*

Proof. Multiplication by ω is an isomorphism if and only if $|\Delta(1)| = 1$ if and only if $S^{2n+1} \setminus \Sigma$ has the homology of a point. By Alexander and Poincaré duality,

$$H_i(S^{2n+1} \setminus \Sigma) \cong H^{2n-i}(\Sigma) \cong H^{i-1}(\Sigma),$$

so this statement is equivalent to the statement that Σ is a homology sphere, again by Poincaré duality. Since Σ is simply connected, the generalized Poincaré conjecture applies. □

Using the splitting of $H_n(F_0)$ exhibited in the proof of Proposition 5 and passing to a convenient eigenbasis for multiplication by ω_j on $\mathbb{C}[C_{a_j}]$, one finds that

$$\Delta(x) = \prod_{0<r_j<a_j} \left(x - \prod_{j=0}^{k} e^{2\pi i r_j / a_j} \right).$$

Example 1. Taking $n = 2m$ and $f(z) = z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2m}^2$ as in the example of interest, $\Delta(x)$ is the cyclotomic polynomial $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$. Thus, in this case, $\Delta(1) = 1$, and Σ is a topological sphere.

In order to determine the smooth structure of Σ, we will compute the signature of F_0. For this, we make use of the following result—see [HM68, 12.4]. Recall that $H_n(F_0) \cong \mathbb{Z}[G]/\operatorname{Ann}(\eta)$.

Theorem 9 (Pham)**.** *The intersection form on $H_n(F_0)$ is given by*

$$\langle g, h \rangle = \epsilon(h^{-1} g \eta),$$

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is defined by stipulating that

$$\epsilon(1) = -\epsilon(\omega) = (-1)^{\frac{n(n+1)}{2}}$$

and extending linearly by zero.

Notation 10. Given a linear map $\lambda : \mathbb{C}[G] \to \mathbb{C}$, we write $\hat{\lambda} = \sum_{g \in G} \lambda(g) g \in \mathbb{C}[G]$. We further define a linear automorphism $x \mapsto \bar{x}$ of $\mathbb{C}[G]$ by the requirement that $\bar{g} = g^{-1}$.

Lemma 11. *The signature of F_0 is given by*

$$\sigma(F_0) = \sigma_+(F_0) - \sigma_-(F_0),$$

where $\sigma_+(F_0)$ is the number of characters $\chi : G \to \mathbb{C}^\times$ such that $\chi(\hat{\epsilon}\overline{\eta})$ is positive (resp. $\sigma_-(F_0)$, negative).

Proof. The set $\{\hat{\chi} : \chi : G \to \mathbb{C}^\times\}$ forms a basis for $\mathbb{C}[G]$, and it is easily checked that

$$\hat{\chi}\eta = \chi(\overline{\eta})\hat{\chi},$$

so this basis is an eigenbasis for multiplication by η, whence $H_n(F_0;\mathbb{C}) \cong \mathbb{C}\langle \hat{\chi}\eta \mid \chi(\overline{\eta}) \neq 0\rangle$. Moreover, the intersection form is diagonal in this basis, since

$$\langle \hat{\chi}\eta, \hat{\chi}'\eta\rangle = \epsilon(\overline{\hat{\chi}'}\hat{\chi}\eta) = \begin{cases} |G|\chi(\hat{\epsilon}\overline{\eta}) & \chi = \chi' \\ 0 & \text{otherwise.}\end{cases}$$

by orthogonality of characters. □

Now, every character of G is a product of characters of the C_{a_j}, which are all of the form $\chi_j(\omega_j) = e^{2\pi i r_j/a_j}$, and it is easy to see that $\chi(\overline{\eta}) \neq 0$ if and only if $0 < r_j < a_j$ for each j.

Corollary 12. *If n is even, then $\sigma(F_0) = N_+ - N_-$, where*

$$N_+ = \#\left\{(r_0, \dots, r_n) \mid 0 < r_j < a_j, 0 < \sum_{j=0}^{n} \frac{r_j}{a_j} \bmod 2 < 1\right\}$$

$$N_- = \#\left\{(r_0, \dots, r_n) \mid 0 < r_j < a_j, 1 < \sum_{j=0}^{n} \frac{r_j}{a_j} \bmod 2 < 2\right\}.$$

Proof. We compute that

$$\begin{aligned}(-1)^{\frac{n(n+1)}{2}}\hat{\epsilon}\overline{\eta} &= (1-\omega)\overline{\eta} \\ &= \overline{\eta} - \omega\overline{\eta} \\ &= \overline{\eta} - \prod_{j=0}^{n} \omega_j(1-\omega_j^{-1}) \\ &= \overline{\eta} - (-1)^{n+1}\eta,\end{aligned}$$

so that, using the fact that n is even twice, we have

$$\begin{aligned}\chi(\hat{\epsilon}\overline{\eta}) &= (-1)^{n/2}\Re(\chi(\eta)) \\ &= (-1)^{n/2}\|\chi(\eta)\| \cos\arg(\chi(\eta)) \\ &= (-1)^{n/2}\|\chi(\eta)\| \cos\arg\left(\prod_{j=0}^{n}(1-e^{2\pi i r_j/a_j})\right) \\ &= (-1)^{n/2}\|\chi(\eta)\| \cos\left(\sum_{j=0}^{n}\arg(1-e^{2\pi i r_j/a_j})\right) \\ &= \|\chi(\eta)\| \cos\left(\frac{n\pi}{2} + \frac{3(n+1)\pi}{2} + \pi\sum_{j=0}^{n}\frac{r_j}{a_j}\right) \\ &= \|\chi(\eta)\| \sin\left(\pi\sum_{j=0}^{n}\frac{r_j}{a_j}\right),\end{aligned}$$

which implies the claim. □

Proof of Theorem 1. We set $n = 2m$, $a_0 = 3$, $a_1 = 5$, and $a_j = 2$ for $2 \leq j \leq 2m$. The sum in question is now

$$\frac{r_0}{3} + \frac{r_1}{5} + \frac{2m-1}{2} = \frac{10r_0 + 6r_1 + 15(2m-1)}{30},$$

where $r_0 \in \{1, 2\}$ and $r_1 \in \{1, 2, 3, 4\}$. We find that

$$10r_0 + 6r_1 \in \{16, 26, 22, 32, 28, 38, 34, 44\};$$

moreover, $2m - 1$ is odd, so $\frac{(2m-1)}{2} \equiv \frac{(-1)^{m+1}}{2} \mod 2$. There are two cases, then: if m is odd, then the possible values modulo 2 are

$$\left\{\frac{31}{30}, \frac{41}{30}, \frac{37}{30}, \frac{47}{30}, \frac{43}{30}, \frac{53}{30}, \frac{49}{30}, \frac{59}{30}\right\} \subset (1, 2);$$

if m is even, then the possible values modulo 2 are

$$\left\{\frac{1}{30}, \frac{11}{30}, \frac{7}{30}, \frac{17}{30}, \frac{13}{30}, \frac{23}{30}, \frac{19}{30}, \frac{29}{30}\right\} \subset (0, 1).$$

From this we conclude that $\sigma(\Sigma) = (-1)^m \cdot 8$ □

References

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