1. The Hopf invariant

Pick a representative $f: S^{2n-1} \to S^n$ of an element of $\pi_{2n-1}(S^n)$. We may use this as an attaching map for a CW-complex

$$D(f) = S^n \cup_f D^{2n}.$$ 

If $n \geq 2$, then $H^*(D(f)) \cong \mathbb{Z}$ for $* = 0, n, 2n$ and 0 otherwise. The cells give preferred generators $a_n \in H^n(D(f))$ and $b_{2n} \in H^{2n}(D(f))$.

**Definition 1.1.** The Hopf invariant $H(f) \in \mathbb{Z}$ is defined by $a_n \cup a_n = H(f) \cdot b_{2n}$.

**Problem 1.2.**

(i) Show that $H(0) = 0$.

(ii) Prove that $H: \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism.

**Problem 1.3** (Recommended). Show that $\mathbb{C}P^2$ has a cell structure with a single 0-, 2- and 4-cell. Use this to deduce that $H: \pi_3(S^2) \to \mathbb{Z}$ is surjective.

**Problem 1.4.** Construct a Hurewicz fibration $S^3 \to S^2$ with fiber $S^1$ and use this to prove that $H: \pi_3(S^2) \to \mathbb{Z}$ is an isomorphism.

Combining the two previous problems, the Hopf map $\eta \in \pi_3(S^2)$ with Hopf invariant one is represented by the Hopf fibration and by the attaching map of the 4-cell in $\mathbb{C}P^2$ (and hence any $\mathbb{C}P^n$, $n \geq 2$).

**Problem 1.5** (Hard). Use $F_2J(S^{2n})$ (see Section 4) to show that $\pi_{4n-1}(S^{2n})$ contains a $\mathbb{Z}$-summand.
2. Postnikov towers

Definition 2.1. A Postnikov tower for $X$ path-connected is a sequence of spaces

\[
\vdots \\
| \\
\downarrow \\
X_3 \\
| \\
\uparrow \\
X_2 \\
| \\
\downarrow \\
X_1 \\
\rightarrow X \\
\rightarrow X_1
\]

such that (i) $X \to X_n$ induces an isomorphism on $\pi_i$ for $i \leq n$, and (ii) $\pi_i(X_n) = 0$ for $i > n$.

Problem 2.2 (Recommended). By attaching cells, construct a Postnikov tower for path-connected CW-complexes $X$, such that all $X_n$ are CW-complexes.

Problem 2.3 (Hard). Show that a Postnikov tower such that $X$ and all $X_n$ are CW-complexes is unique up to homotopy equivalence (of such diagrams).

Problem 2.4. Show that up to homotopy equivalence of towers you may replace a Postnikov tower by one where all maps $X_{n+1} \to X_n$ are Hurewicz fibrations.

Problem 2.5. Suppose we are given a Postnikov tower where all maps $X_{n+1} \to X_n$ are Hurewicz fibrations.

(i) Show that the fibers of $X_{n+1} \to X_n$ are Eilenberg-Mac Lane space $K(\pi_{n+1}(X), n+1)$.

(ii) Show that the map $X \to \lim_{n \to \infty} X_n$ is a weak homotopy equivalence.

Definition 2.6. A Postnikov tower is said to be principal if each $X_{n+1} \to X_n$ is homotopy equivalent to the inclusion in $X'_n$ of the fiber of a Hurewicz fibration

\[
X'_n \to K(\pi_{n+1}(X), n + 2).
\]

The following is Theorem 4.69 of [Hat02].

Theorem 2.7. Any 1-connected $X$ has a principal Postnikov tower.

Problem 2.8. Show that if $X$ is 1-connected and based, then $\Omega(-)$ of a Postnikov tower for $X$ is a principal Postnikov tower for $\Omega X$. 
3. Localization

In this section we work in the full category $\text{Ho}(\text{CW})$ of $\text{Ho}(\text{Top})$ of spaces having the homotopy type of a CW-complex. That is, you may assume that every space in these problems has the homotopy type of a CW-complex. Let $W_R$ be the class of morphisms in $\text{Ho}(\text{CW})$ consisting of $f: X \to Y$ consisting such that $f_*: H_*(X;R) \to H_*(Y;R)$ is an isomorphism. These are called the $R$-equivalences. We say that a space $Z$ is $R$-local if for all $R$-equivalences $f: X \to Y$ the map $[Y,Z] \to [X,Z]$ on homotopy classes is an isomorphism.

**Problem 3.1.** Show that $K(M,n)$ is $R$-local for any $R$-module $M$ and $n \geq 1$.

**Problem 3.2.**
(i) Let $\mathbb{F}_p$ denote the field with $p$ elements. Show that $K(\mathbb{Z}/p^k\mathbb{Z},n)$ is $\mathbb{F}_p$-local for all $k \geq 1$ and $n \geq 1$.
(ii) Let $\mathbb{Z}_p^\wedge$ denote the $p$-adic integers. Show that $K(\mathbb{Z}_p^\wedge,n)$ is $\mathbb{F}_p$-local for all $n \geq 1$.

**Problem 3.3.** Suppose that $p: E \to B$ is Hurewicz fibration such that $E$ and $B$ are $R$-local and 1-connected. Show that the fiber of $p$ over any $b_0 \in B$ is also $R$-local.

**Problem 3.4.** Suppose that $\ldots X_n \to X_{n-1} \to \ldots \to X_1$ is a tower of Hurewicz fibrations between $R$-local and 1-connected spaces. Show that $\lim_{n \to \infty} X_n$ is also $R$-local and 1-connected.

We will now use some of the results from the section on Postnikov towers. If $S \subset \mathbb{Z}$ is a set of primes, then $S^{-1}\mathbb{Z}$ is the subset of $\mathbb{Q}$ of $n/m$ with denominator $m$ (in lowest terms) divisible only by $p \in S$.

**Problem 3.5** (Recommended). Let $R = S^{-1}\mathbb{Z} \subset \mathbb{Q}$. Use Postnikov towers to show that if $X$ is 1-connected with $\pi_n(X) \otimes R \cong \pi_n(X)$, then $X$ is $R$-local.

**Problem 3.6.** Let $R = S^{-1}\mathbb{Z} \subset \mathbb{Q}$. Show that if $X$ is a 1-connected $R$-local space, then $\pi_n(X) \otimes R \cong \pi_n(X)$.

**Problem 3.7.** Let $R = S^{-1}\mathbb{Z} \subset \mathbb{Q}$.
(i) Use Postnikov towers to prove that for every 1-connected $X$ there exists a $R$-equivalence $X \to X_R$ with target 1-connected and $R$-local. You may use the fact that $f: X \to Y$ between 1-connected spaces is an $R$-equivalence if and only if $\pi_n(X) \otimes R \to \pi_n(Y) \otimes R$ is an isomorphism for all $n$.
(ii) An $R$-equivalence $X \to X_R$ with $R$-local target is called an $R$-localization (so that part (i) says that $R$-localizations of 1-connected $X$ exist). Show that the $R$-localization of $X$ is unique up to homotopy if it exists.
4. THE JAMES CONSTRUCTION

Let \((X,e)\) be a pointed space, where we use \(e\) to denote the basepoint (as it will serve as an identity element soon). The James construction \(J(X)\) is the “free topological monoid” on \(X\). It is defined to be

\[
J(X) := \left( \bigsqcup_{k \geq 0} X^k \right) / \sim
\]

with \(\sim\) the equivalence relation generated by \((x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)\).

**Problem 4.1.** (i) Show that \(J(X)\) is a (unital, associative) topological monoid and state its universal property.

(ii) Let \(\Omega^M X\) (the Moore loop space) denote the subspace of \([0, \infty) \times \text{Map}([0, \infty), X)\) of pairs \((t, \gamma)\) such that \(\gamma(0) = e\) and \(\gamma(s) = e\) for \(s \geq t\). Use concatenation of paths to give this the structure of a (unital, associative) topological monoid, and show that it is homotopy equivalent to \(\Omega X\) as an \(H\)-space.

(iii) Construct a map \(J(X) \to \Omega^M \Sigma X\).

We can filter \(J(X)\) by taking \(F_i J(X)\) to be the image of \(\bigsqcup_{0 \leq k \leq i} X^k\).

**Problem 4.2.** Let \(R\) be a commutative ring. Compute \(H_*(F_i J(S^n); R)\) and \(H_*(J(S^n); R)\) (without using Theorem 4.3).

The following is proven in Section 4.1 of \([Hat02]\). You will do parts of the proof, after using it to give an alternative proof of the Freudenthal suspension theorem.

**Theorem 4.3 (James).** If \(X\) is path-connected, the map \(J(X) \to \Omega^M \Sigma X\) is a weak homotopy equivalence.

**Problem 4.4** (Recommended). Suppose that \(X\) is a CW-complex with \(e\) a 0-cell.

(i) Show that \(J(X)\) inherits a CW-structure from \(\bigsqcup_{k \geq 0} X^k\) such that \(F_i J(X) \to F_{i+1} J(X)\) is the inclusion of a subcomplex.

(ii) If \(X\) is \((n-1)\)-connected, show that \(F_i J(X) \to J(X)\) is \((2n-1)\)-connected.

(iii) Use this and Theorem 4.3 to give another proof of the Freudenthal suspension theorem.

Let us commence the proof, and let \(R\) be a commutative ring. For a graded \(R\)-module \(A\), let \(TA\) denote the free tensor algebra (the “free associative algebra” on \(A\)):

\[
TA = \bigoplus_{k \geq 0} A^\otimes \alpha^k
\]

here \(A^\otimes \alpha^0 = \mathbb{Z}\) generated by the unit, the element \(a_1 \otimes \cdots \otimes a_k\) is in degree \(|a_1| + \cdots + |a_k|\), and the product is bilinear and on generators given by \((a_1 \otimes \cdots), (a_1 \otimes \cdots) \cdot (b_1 \otimes \cdots) = a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l\). Note that \(TA\) also has an increasing filtration, with \(F_i TA\) defined as the image of the first \(i + 1\) summands (so \(0 \leq k \leq i\)).

**Problem 4.5.** Let \(R\) be a commutative ring.

(i) State the universal property of \(TA\).

(ii) Construct a homomorphism \(T \hat{H}_*(X; R) \to H_*(J(X); R)\) and show that its restriction to \(F_i T \hat{H}_*(X; R)\) factors through \(H_*(F_i J(X); R)\).

(iii) Suppose that \(X\) is path-connected and \(H_*(X; R)\) is a free \(R\)-module in each degree. Prove that there is a map of short exact sequences

\[
0 \longrightarrow F_i T \hat{H}_*(X; R) \longrightarrow F_{i+1} T \hat{H}_*(X; R) \longrightarrow \hat{H}_*(X; R)^\otimes \alpha^{i+1} \longrightarrow 0
\]

and use this to prove that \(T \hat{H}_*(X; R) \to H_*(J(X); R)\) is an isomorphism.
The proof of James’ theorem then starts with the computation that there is an isomorphism $H_\ast(\Omega \Sigma X; R) \cong T\tilde{H}_\ast(X)$ if $X$ is path-connected and $H_\ast(X; R)$ is a free $R$-module in each degree, which fits into a commutative diagram

$$
\begin{array}{ccc}
T\tilde{H}_\ast(X) & & \\
\downarrow & & \downarrow \\
H_\ast(J(X); R) & \rightarrow & H_\ast(\Omega \Sigma X; R).
\end{array}
$$

We now finish the proof of James’ theorem in the case that $X$ is 1-connected.

**Problem 4.6.**

(i) Let $f: X \rightarrow Y$ be a map of spaces. Show that $f$ induces an isomorphism on homology with all field coefficients if and only if it induces an isomorphism on homology with coefficients in $\mathbb{Z}$.

(ii) Prove that $J(X) \simeq \Omega \Sigma X$ if $X$ is 1-connected.
5. Some homotopy theory of simplicial sets

Let’s practice with some of the homotopy theory of simplicial sets that is most relevant to the study of topological spaces. A general reference for simplicial sets is [GJ09].

**Definition 5.1.** Let \( \Lambda^k_i \subset \Delta^k \) be the union of those faces \( \Delta^{k-1}_j \) opposite the \( j \)th vertex for \( j \neq i \). This is called the \( j \)th horn of \( \Delta^k \).

**Definition 5.2.** A map \( f : X \to Y \) of simplicial sets is said to be a Kan fibration if for all \( k \geq 0 \) and all \( 0 \leq i \leq k \), in each commutative diagram

\[
\begin{array}{ccc}
\Lambda^k_i & \to & X \\
\downarrow & & \downarrow \\
\Delta^k & \to & Y
\end{array}
\]

there exists a dashed lift making the diagram commute.

**Definition 5.3.** A simplicial set \( X \) is said to be a Kan complex if the map \( X \to \ast \) is a Kan fibration. This lifting condition is often rephrased as “\( X \) has horn fillers.”

**Problem 5.4** (Recommended). Show that for a topological space \( Z \), the singular simplicial set \( S(Z) \) is a Kan complex.

**Problem 5.5.** Show that if \( G \) is a simplicial group (i.e. a functor \( \Delta^{op} \to \text{Grp} \)), then its underlying simplicial set is a Kan complex.

**Problem 5.6.**

(i) Show that \( G \) is a groupoid, then its nerve \( N_\bullet G \) is a Kan complex and in fact has unique horn fillers.

(ii) Show that a simplicial set with unique horn fillers is the nerve of some groupoid.\(^1\)

(iii) Show that a simplicial set with unique inner horn fillers (i.e. only for \( \Lambda^k_i \subset \Delta^k \) with \( 0 < i < k \)) is the nerve of some category.

**Problem 5.7.** We define a mapping simplicial set \( \text{Map}(K, X)_p := \text{Hom}_{sSet}(K \times \Delta^p, X) \).

(i) Show that \( \text{Map}(K, X) \) is a Kan complex if \( X \) is a Kan complex.

(ii) Show that there is an adjunction \( - \times K \dashv \text{Map}(K, -) \), both functors \( sSet \to sSet \).

**Problem 5.8** (Hard). Show that if \( i : K \to L \) is a levelwise injective map and \( f : X \to Y \) is a Kan fibration, then \( \text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y) \) is a Kan fibration.

Two maps \( f, g : K \to X \) of simplicial set are said to be simplicially homotopic if there is a map \( H : K \times \Delta^1 \to X \) such that the restrictions along the two inclusions \( \Delta^0 \hookrightarrow \Delta^1 \) give \( f \) and \( g \).

**Problem 5.9.**

(i) Show that simplicial homotopy between maps \( \Delta^0 \to X \) is an equivalence relation when \( X \) is a Kan complex.

(ii) Use part (i) and the Problem 5.7 to show that simplicial homotopy is an equivalence relation on maps \( K \to X \) when \( X \) is a Kan complex.

(iii) For \( K \subset L \), define a notion of homotopy of maps \( L \to X \text{ rel } K \) and use Problem 5.8 to show that this gives an equivalence relation.

**Problem 5.10.** Suppose \( X \) is a Kan complex and \( x_0 \) a 0-simplex. Let \( \pi_n(X, x_0) \) be the set of homotopy classes of maps of pairs of simplicial sets \( (\Delta^n, \partial \Delta^n) \to (X, x_0) \). Show that this is a group for \( n \geq 1 \).

Note that if \( X \) were not Kan in the previous problem, we could not have even defined homotopy classes as homotopy might not have been an equivalence relation.

---

\(^1\)A Kan complex is also called an \( \infty \)-groupoid. Feel free to read about them in the first chapter to [Lur09].
References

