MATH21B – LECTURE 9: DIMENSION
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1. Dimension

Problem 1. What is the dimension of the span of the following collections of vectors?

$$\text{(a)} \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}. \quad \text{(b)} \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}. \quad \text{(c)} \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}. \quad \text{(d)} \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\0\\2\end{bmatrix}.$$

Find within each a maximal linearly independent subset (and hence a basis for its span).

Solution. (a) The dimension is 3, and all of them are already linearly independent.
(b) The dimension is 2, and any two of them form a maximal linearly independent subset.
(c) The dimension is 2, and any two of them form a maximal linearly independent subset.
(d) The dimension is 3, and you can for example take the first three vectors (by looking at where the leading 1's appear in the reduced row echelon form). ∎

Problem 2. For an $(n \times n)$-matrix A, we might want to solve $A\vec{x} = \vec{x}$. That is, we might try to find the collection of vectors that stays fixed under A. (Maybe we are trying to find some equilibrium states in some physical system.)

(i) Show that $\text{Fix}(A) = \{\vec{x} \mid A\vec{x} = \vec{x}\} \subset \mathbb{R}^n$ is a linear subspace.
(ii) If A is reflection in a plane in $\mathbb{R}^3$, what is the dimension of $\text{Fix}(A)$?
(iii) Find the dimension of $\text{Fix}(A)$ and a basis for $\text{Fix}(A)$, when A is the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 4 & 5 \end{bmatrix}.$$

Solution. (i) The equation $A\vec{x} = \vec{x}$ is equivalent to $(A - \text{id}_n)\vec{x} = 0$. Thus $\text{Fix}(A) = \ker(A - \text{id}_n)$ and hence a linear subspace.
(ii) Geometrically, we see that a reflection fixes only those points in the plane. Hence $\text{Fix}(A)$ is equal to the plane and thus 2-dimensional.
(iii) The reduced row echelon form of $\text{rref}(A - \text{id}_4)$ is given by

$$\text{rref}(A - \text{id}_4) = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the dimension is 1, and a basis is given by the vector

$$\begin{bmatrix} -4 \\ 4 \\ -1 \\ 1 \end{bmatrix}.$$

∎

Problem 3 (Challenge)**.** Why is the dimension of linear subspace V well-defined? More precisely, why does every basis of V contain the same number of elements?

Solution. The proof is rather long, so see Theorem 3.3.2 of Bretscher (5th edition). ■

2. Rank-nullity

Problem 4. (i) Given an (4×6)-matrix A, what are the possible ranks and nullities of A

(ii) Given an $(m \times n)$-matrix A, what is the smallest possible nullity of A? What is the largest possible rank of A?

Solution. (i) The rank can be any $0 \leq r \leq 4$ and then the nullity is $6 - r$.

(ii) The smallest possible nullity is n. The largest possible rank is $\min(m, n)$. ■

Problem 5. Suppose that the $(n \times n)$-matrix A satisfies $A^2 = A$.

(i) Give an example of such an A for $n = 3$.
(ii) Show that $(\mathrm{id}_n - A)^2 = \mathrm{id}_n - A$.
(iii) Show that $\dim \mathrm{im}(\mathrm{id}_n - A) = \dim \ker(A)$.
(iv) Show that $\dim \mathrm{im}(A) = \dim \ker(\mathrm{id}_n - A)$.

Solution. (i) Take the matrix for a projection on a plane or a line.

(ii) We have that

$$(\mathrm{id}_n - A)^2 = (\mathrm{id}_n - A)(\mathrm{id}_n - A) = \mathrm{id}_n - \mathrm{id}_n A - A\,\mathrm{id}_n + A^2 = \mathrm{id}_n - A - A + A = \mathrm{id}_n - A.$$

(iii) Firstly, $\mathrm{im}(\mathrm{id}_n - A) \subset \ker(A)$ because $A(\mathrm{id}_n - A) = A - A^2 = A - A = 0$. This means that $\dim \mathrm{im}(\mathrm{id}_n - A) \leq \dim \ker(A)$. Secondly, $\mathrm{im}(\mathrm{id}_n - A) \supset \ker(A)$ because if $A\vec{x} = \vec{0}$, then $(\mathrm{id}_n - A)\vec{x} = \vec{x} - A\vec{x} = \vec{x}$ and we conclude that $\ker(A) \subset \mathrm{im}(\mathrm{id}_n - A)$. Thus $\dim \ker(A) \leq \dim \mathrm{im}(\mathrm{id}_n - A)$.

But if we have two numbers n, m such that $n \leq m$ and $m \leq n$, they have to be equal.

(iv) This is a consequence of rank-nullity and (iii). ■.

Summary

- The dimension of a linear subspace $V \subset \mathbb{R}^n$ is the number of elements in a basis of V.
- The dimension of the image of A is equal to the rank of A. The dimension of the kernel of A is called the *nullity* of A.
- The rank-nullity theorem says that the $\mathrm{rank}(A) + \mathrm{nullity}(A)$ equals the number of columns on A. To see this, recall that $\mathrm{rank}(A)$ equals the number of columns of $\mathrm{rref}(A)$ containing a leading one, and $\mathrm{nullity}(A)$ equals the number of free variables (which correspond to those columns of $\mathrm{rref}(A)$ not containing a leading one).