MATH21B – LECTURE 8: BASES
SPRING 2018, HARVARD UNIVERSITY

1. Linear combinations, dependence and independence

Problem 1. Describe geometrically the sets of linear combinations of the following vectors:

$$\text{(a)}\ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \in \mathbb{R}^3 \qquad \text{(b)}\ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \in \mathbb{R}^3, \qquad \text{(c)}\ \begin{bmatrix}2\\3\end{bmatrix} \in \mathbb{R}^2.$$

Solution. (a) This is the xz-plane in $\mathbb{R}^3$.
(b) This is just $\mathbb{R}^3$.
(c) This is the line of slope $3/2$ through the origin in $\mathbb{R}^2$. ■

Problem 2. Which of the following sets of vectors are linearly independent?

$$\text{(a)}\ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}. \qquad \text{(b)}\ \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}. \qquad \text{(c)}\ \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

Solution. Only (a) is. This may be shown by taking them to be the columns of a matrix and applying Gauss-Jordan elimination; a maximal linear independent set is given by those vectors corresponding to columns containing a leading 1. ■

2. Bases

Problem 3. Find a basis for the kernel and image of the following matrices:

$$\text{(a)}\ A = \begin{bmatrix}0&1&2&0\\1&2&3&3\\1&0&-1&3\\1&1&1&3\end{bmatrix} \qquad \text{(b)}\ B = \begin{bmatrix}0&1&2&0\\1&3&2&3\\1&0&-1&3\\1&1&1&3\end{bmatrix}.$$

Solution. The reduced row echelon form are

$$\operatorname{rref}(A) = \begin{bmatrix}1&0&-1&3\\0&1&2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \qquad \operatorname{rref}(B) = \begin{bmatrix}1&0&0&3\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}.$$

To get the bases for the images, we need to take the columns of A, resp. B, corresponding to those of the reduced row echelon forms containing a leading 1:

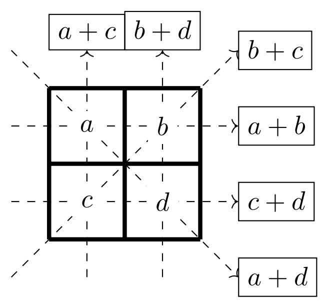
$$\text{basis for } \operatorname{im}(A)\text{:}\ \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \qquad \text{basis for } \operatorname{im}(B)\text{:}\ \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\-1\\1\end{bmatrix}.$$

To get the bases for the kernels, we need to read of the parametrizations of the solutions of $A\vec{x} = \vec{0}$, resp. $B\vec{x} = \vec{0}$. the reduced row echelon forms containing a leading 1:

$$\text{basis for } \ker(A)\text{:} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{basis for } \ker(B)\text{:} \quad \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

■

Problem 4 (Challenge)**.** Consider the "MRI" scanner of the first lecture, which uses six beams and records the amount of material encountered in an attempt to recover the densities a, b, c, and d of different parts of the patient's body:



(i) Give the matrix A for the system of linear equations that one should solve to recover a, b, c, d from the six outputs.
(ii) Compute $\ker(A)$. Interpret what this means for the MRI scanner.
(iii) Give a basis for $\operatorname{im}(A)$. Interpret what this means for the MRI scanner.
(iv) Could we get away with using fewer beams? If so, how few?

Solution. (i) Let us call the measurements $m_{ac}, m_{bd}, m_{bc}, m_{ab}, m_{cd}, m_{ad}$. The equations are then $a+c = m_{ac}$, etc. So the matrix describing m_{ac}, etc. in terms of the a, b, c, d is given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence the kernel is $\{0\}$. This means we can recover a, b, c, d uniquely from the measurements.
(iii) The basis is just the four columns of the matrix given in (i). The fact that the basis has four vectors in $\mathbb{R}^6$ means that not every collection of measurements can occur.
(iv) Yes, you can use any four of them and still uniquely recover a, b, c, d. Any less and you can no longer do this. ■

3. Self-test

Problem 5. (i) True or false: If A is an invertible $(n \times n)$-matrix, then $\{A\vec{e}_1, \ldots, A\vec{e}_n\}$ is a basis of $\mathbb{R}^n$.

(ii) True or false: any collection of $m > n$ vectors in $\mathbb{R}^n$ is linearly dependent.

(iii) True or false: any collection of $m < n$ vectors in $\mathbb{R}^n$ is linearly independent.

(iv) Give a collection of three linearly independent vectors in $\mathbb{R}^3$, all of whose entries are non-zero.

Solution. (1) True, e.g. by noting that $\text{rref}(A) = \text{id}_n$.

(2) True, as the reduced row echelon form of the matrices containing the vectors as columns must have free variables.

(3) False, e.g. $(1, 0, 0)$ and $(2, 0, 0)$ are linearly dependent.

(4) $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$. ∎

Summary

- A collection $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ of vectors in $\mathbb{R}^n$ is *linearly independent* if $a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = 0$ implies that $a_1 = \ldots = a_k$; this means that there are no non-zero relations between them. If it is not linearly independent it said to be *linearly dependent*; in that case we can express at least one of the $\vec{v}_i$'s in terms of the others.
- To find a linear independent subset among a collection of vectors, put them as the columns of a matrix A and solve $A\vec{x} = \vec{0}$ by Gauss-Jordan elimination. If there is a unique solution (i.e. there is no free variable), then they are linearly independent. In there is not a unique solution, the vectors corresponding to the columns containing a leading one will be linearly independent.
- A linearly independent spanning set for a subspace $V \subset \mathbb{R}^n$ is called a *basis*. Every vector in V is a unique linear combination of vectors in the spanning set.
- The spanning sets of $\ker(A)$ and $\text{im}(A)$ read off from the $\text{rref}(A)$ will be linearly independent, and thus give a basis for $\ker(A)$ and $\text{im}(A)$ respectively.