MATH21B – LECTURE 7: IMAGES AND KERNELS
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1. Subspaces and spanning sets

Problem 1. (i) Which of the following are subspaces?

(a) The set of vectors in $\mathbb{R}^3$ with third entry 1.
(b) The set of vectors in $\mathbb{R}^3$ with third entry 0.
(c) The set of vectors $\vec{x} = (x_1, x_2)$ in $\mathbb{R}^2$ satisfying $x_1 + x_2 = 0$.
(d) The set of all vectors in $\mathbb{R}^4$ with positive entries.

(ii) Give a spanning set for those sets in (i) that are subspaces.

Solution. The sets (b) and (c) are subspaces; you have to verify that they contain $\vec{0}$, are closed under addition and multiplication by a scalar. The set (a) is not closed under scalar multiplication, e.g. $(0, 0, 1)$ lies in the set but $2 \cdot (0, 0, 1) = (0, 0, 2)$ does not. The set (d) is similarly not closed under scalar multiplication, e.g. $(1, 1, 1, 1)$ lies in the set but $-1 \cdot (1, 1, 1, 1) = (-1, -1, -1, -1)$ does not.

The vectors in (b) are of the form $(a, b, 0) = a \cdot (1, 0, 0) + b \cdot (0, 1, 0)$, so $\{\vec{e}_1, \vec{e}_2\}$ is a spanning set. The vectors in (c) are of the form $(x, -x) = x \cdot (1, -1)$, so $\{(1, -1)\}$ is a spanning set. ■

Problem 2. Suppose that $V, W \subset \mathbb{R}^n$ are subspaces. Show that the intersection $V \cap W \subset \mathbb{R}^n$, the subset given by all $\vec{x}$ which lie in both V and W, is a subspace of $\mathbb{R}^n$.

Solution. We need to show that if $V \cap W$ contains $\vec{0}$ and is closed under addition and scalar multiplication.

Contains $\vec{0}$. If V and W are subspaces, they both contain $\vec{0}$ and hence $\vec{0}$ is in the intersection $V \cap W$.

Closed under addition. Suppose that $\vec{x}, \vec{y} \in V \cap W$, then we must show that $\vec{x} + \vec{y} \in V \cap W$. The assumption tells us that $\vec{x}, \vec{y} \in V$ and $\vec{x}, \vec{y} \in W$. Since V and W are subspaces they are closed under addition and we thus have that $\vec{x} + \vec{y} \in V$ and $\vec{x} + \vec{y} \in W$. This means that $\vec{x} + \vec{y} \in V \cap W$. Since $\vec{x}, \vec{y}$ were arbitrary $V \cap W$ is closed addition.

Closed under scalar multiplication. Suppose that $\vec{x} \in V \cap W$ and $\lambda \in \mathbb{R}$, then we must show that $\lambda \cdot \vec{x} \in V \cap W$. The assumption tell us $\vec{x} \in V$ and $\vec{x} \in W$. Since V and W are subspaces they are closed under scalar multiplication and we thus have that $\lambda \cdot \vec{x} \in V$ and $\lambda \cdot \vec{x} \in W$. This means that $\lambda \vec{x} \in V \cap W$. ■

2. Kernels and images

Problem 3. Give a spanning set for the image and kernel of the matrix A corresponding to a projection on the y-axis:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution. The image is spanned by $\vec{e}_2$. The kernel is spanned by $\vec{e}_1$. ■

Problem 4. Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & 3 \\ 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

(i) Find a spanning set for $\text{im}(A)$.
(ii) Find a spanning set for $\ker(A)$.
(iii) What is the rank of A?

(iv) Is $\text{im}(A^2)$ contained in $\text{im}(A)$?
(v) Is $\ker(A)$ contained in $\ker(A^2)$?
(vi) Compute $\ker(A^2)$.

Solution. (i) It is given by the columns of A.
(ii) We need to compute $\text{rref}(A)$ to read off the solutions to $A\vec{x} = \vec{0}$. This is given by

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and thus the two vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a spanning set.
(iii) The reduced row echelon form has two leading 1's, so we read off that the rank is 2.
(iv) Yes, because if a vector is of the form $A^2\vec{x}$, then it is in particular of the form $A\vec{y}$; take $\vec{y} = A\vec{x}$.
(v) Yes, because if $A\vec{x} = \vec{0}$ then also $A^2\vec{x} = A(A\vec{x}) = A\vec{0} = \vec{0}$.
(vi) We compute that

$$A^2 = \begin{bmatrix} 3 & 2 & 1 & 9 \\ 8 & 8 & 8 & 24 \\ 2 & 4 & 6 & 6 \\ 5 & 6 & 7 & 15 \end{bmatrix}$$

and its reduced row echelon form is equal to that of A. This means its kernel is also equal to that of A. ■

Problem 5. Let $V \subset \mathbb{R}^3$ be the subspace of vectors $\vec{x}$ whose sum $x_1 + x_2 + x_3$ of coordinates is 0.
(i) Find a spanning set for V.
(ii) Find a (3×3)-matrix A whose image is V.
(iii) Find a (3×3)-matrix B whose kernel is V.
(iv) (Challenge) Is it possible to find a (3×3)-matrix whose image and kernel are both given by V?

Solution. (1) If we write V as the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

which is already in reduced row echelon form, we can read of that the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

span $\ker(A)$.
(2) We can use the spanning set as columns:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(3) See (i).
(4) No, because then the number of leading 1's added to the number of free variables needs to be $2+2=4$, which is not equal to 3.

■

Problem 6. Explain why the following are equivalent for an $(n \times n)$-matrix A:

(a) $\ker(A) = \{\vec{0}\}$,
(b) $\operatorname{rank}(A) = n$,
(c) $\operatorname{im}(A) = \mathbb{R}^n$,
(d) A is invertible.

Proof. All may be restated as $\operatorname{rref}(A) = \mathrm{id}_n$. □

Summary

- The kernel of an $(m \times n)$-matrix A is the set of vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{0}$:
$$\ker(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$
This tells you about the uniqueness of solutions.
- The image of an $(m \times n)$-matrix A is the set of vectors $\vec{y}$ in $\mathbb{R}^m$ that can be obtained as $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$:
$$\operatorname{im}(A) = \{A\vec{x} \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\}$$
This tells you about the existence of solutions.
- The kernel and image of an $(m \times n)$-matrix A are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively; this means they are closed under addition of vectors and multiplication of vectors by a scalar.
- A subspace $V \subset \mathbb{R}^n$ is said to be spanned by a collection of $\vec{x}_1, \vec{x}_2, \ldots$ of vectors if every vector in V can be written as a linear combination of $\vec{x}_1, \vec{x}_2, \ldots$.
- The image $\operatorname{im}(A)$ is spanned by the columns of A corresponding to those columns of $\operatorname{rref}(A)$ containing a leading 1. To obtain spanning vectors for the kernel $\ker(A)$, parametrize the solutions of $A\vec{x} = \vec{0}$ using $\operatorname{rref}(A)$.