MATH21B – LECTURE 6: MATRIX ALGEBRA
SPRING 2018, HARVARD UNIVERSITY

1. Matrix multiplication

Problem 1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix}.$$

Compute AB, BA, AC, CA, BC and CB.

Solution. The answers are:

$$AB = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$
$$AC = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} \qquad CA = \begin{bmatrix} -1 & -1 \\ -2 & -4 \end{bmatrix}$$
$$BC = \begin{bmatrix} 2 & -7 \\ 0 & -2 \end{bmatrix} \qquad CB = \begin{bmatrix} -2 & 0 \\ -5 & 2 \end{bmatrix}$$

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Problem 2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}.$$

Compute AB and explain why we can not take the matrix product BA.

Solution. We have that

$$AB = \begin{bmatrix} 6 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

but the other multiplication is not possible because B has three columns and A only row rows. ■

Problem 3. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}.$$

Compute A^{-1} and check that $A^{-1}A = \mathrm{id}_2$ by matrix multiplication.

Solution. To compute A^{-1}, we take the reduced row echelon form of $\mathrm{rref}([A \mid \mathrm{id}_2])$ and read off the last two columns:

$$\mathrm{rref}([A \mid \mathrm{id}_2]) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & -1/4 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & -1/4 \end{bmatrix}$$

We leave the verification of $A^{-1}A = \mathrm{id}_2$ to the reader. ■

Problem 4. Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Compute A^2, A^3, A^4, etc. (Hint: you'll be done after a finite number of steps).

Solution. We have that

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus $A^n = 0$ for $n \geq 3$. ∎

2. Composition of linear transformations

Problem 5. (i) Explain in words the linear transformation of the plane $\mathbb{R}^2$ obtained by first reflecting in the line $y = -x$ and then in the x-axis. (Hint: it is a rotation.)

(ii) Write down the matrix A for reflection in the line $y = -x$ and the matrix B for reflection in the x-axis.

(iii) Compute BA and interpret the corresponding linear transformation to verify your answer to (i).

(iv) (Challenge) Can every rotation of the plane be obtained as a composition of two reflections?

Solution. (i) The vector $\vec{e}_1$ gets first mapped to $-\vec{e}_2$ and then to $\vec{e}_2$. The vector $\vec{e}_2$ gets first mapped to $-\vec{e}_1$ and stays $-\vec{e}_1$ under the second transformation. This is exactly what counterclockwise rotation by $90°$ does.

(ii) The matrices are

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(iii) The product BA is given by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is the matrix corresponding to the composition of linear transformations in part (i). Indeed, we recognize it as the matrix for counterclockwise rotation by $90°$.

(iv) Yes, reflecting in lines that are 2θ apart is rotation by θ. ∎

Problem 6 (Challenge)**.** Derive the addition laws for sine and cosine from matrix multiplication:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha),$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

Solution. The linear transformation given by rotation by α has corresponding matrix

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Since rotation by α followed by rotation by β is rotation by $\alpha + \beta$, we must have $R_\beta R_\alpha = R_{\alpha+\beta}$. Reading of the top-left and top-right entries of both sides, which must be equal, gives the equations. ∎

Summary

- You can add matrices and multiply them with a number:

$$\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}.$$

- You can multiply an $(m \times n)$-matrix A with an $(n \times k)$-matrix B:

$$AB = \begin{bmatrix} | & \cdots & | \\ A(\vec{b}_1) & \cdots & A(\vec{b}_k) \\ | & \cdots & | \end{bmatrix}$$

  where $\vec{b}_i$ denotes the ith column of B. Matrix multiplication is associative, $A(BC) = (AB)C$. It is not commutative, in general $AB \neq BA$ (though this can occasionally happen for particular A and B).
- If A is the $(m \times n)$-matrix for $T\colon \mathbb{R}^n \to \mathbb{R}^m$, and B is the $(n \times k)$-matrix for $S\colon \mathbb{R}^k \to \mathbb{R}^n$, then AB is the $(m \times k)$-matrix for $T \circ S\colon \mathbb{R}^k \to \mathbb{R}^m$.