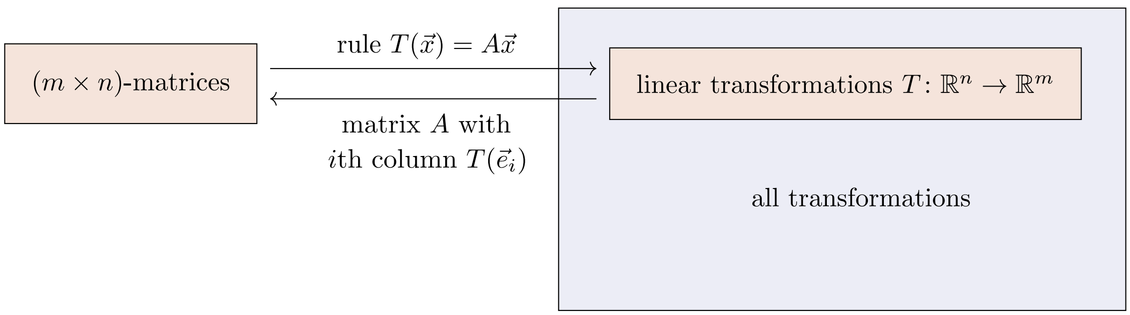
MATH21B – LECTURE 4: LINEAR TRANSFORMATIONS SPRING 2018, HARVARD UNIVERSITY

1. Linear transformations



Problem 1. (i) Which of the following transformations is linear?

(a) $T(x, y, z) = (x, y, 1)$.
(b) $T(x, y, z) = (x, z, 0)$.
(c) $T(x, y, z) = x + y + z$.
(d) $T(x, y) = (2y, 2x)$.
(e) $T(x, y) = x^2 - y$.
(f) $T(x) = (x, -x, 2x)$.
(g) $T(x, y) = (\sin(2)x, 2\sin(y))$.
(h) $T(x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5, x_1)$.

(ii) Find the matrices for those of the above transformation that are linear.

Let me first give a more ridiculous example of a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ which is not linear: T assigns to (x, y, z) the vector $(1, 1)$ unless $(x, y, z) = (0, 0, 0)$ in which case it assigns $(10, 10)$:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^2$$
$$T(x, y, z) = \begin{cases} (1, 1) & \text{if } (x, y, z) \neq (0, 0, 0) \\ (10, 10) & \text{if } (x, y, z) = (0, 0, 0). \end{cases}$$

Solution. We do (i) and (ii) simultaneously.

(a) Not linear, due 1.
(b) Linear, the matrix is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$
(c) Linear, the matrix is
$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$
(d) Linear, the matrix is
$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$
(e) Not linear due to x^2.

(f) Linear, the matrix is

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(g) Not linear due to $\sin(y)$.

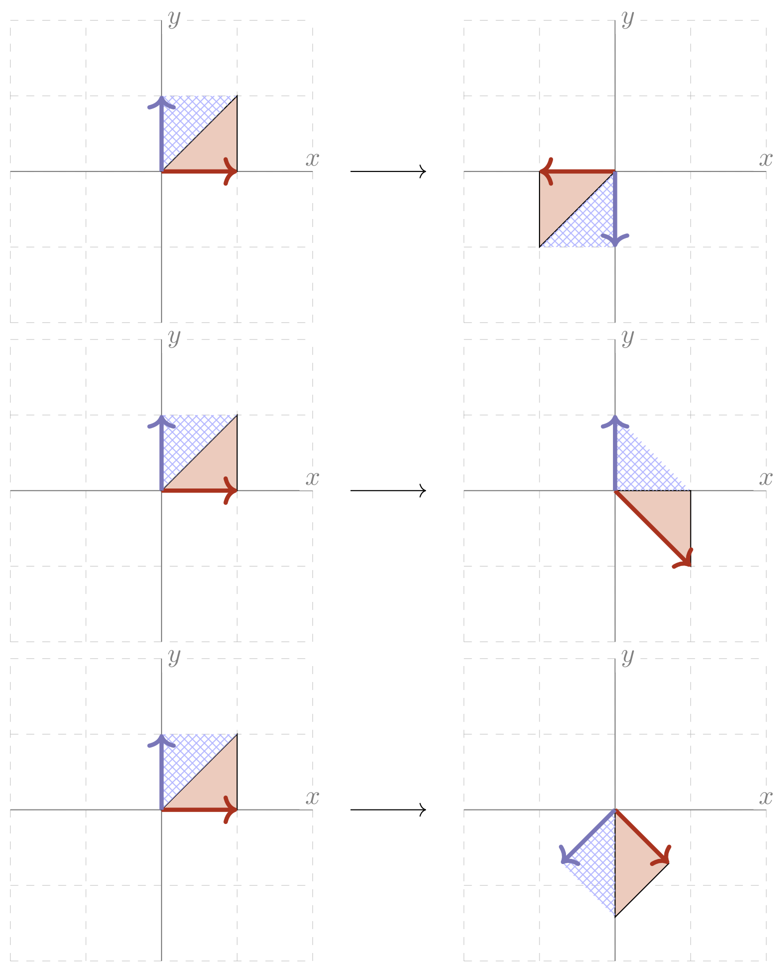
(h) Linear, the matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\blacksquare$

We can interpret linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^n$ geometrically when $n = 2, 3$. Let's fix $n = 2$ for concreteness . Then we think of $(x, y) \in \mathbb{R}^2$ or $(x, y, z) \in \mathbb{R}^3$ as a vector, apply T, and draw the resulting vector as a point in $\mathbb{R}^2$. Thus T is "transforming" the plane. We can find the matrix of T by seeing where the standard basis vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ go, as these will be the two columns of the matrix.

Problem 2. Find the matrix for the following linear transformations:



Solution. There are three pictures, describing how one of three linear transformations transforms a square with edges given by $\vec{e}_1$ (the red[1] arrow) and $\vec{e}_2$ (the purple[2] arrow).

To get the matrix for T, we use $T(\vec{e}_1)$ and $T(\vec{e}_2)$ as the columns. We can easily read off $T(\vec{e}_1)$ and $T(\vec{e}_2)$ from the pictures.

(a) The matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) The matrix is

$$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

(c) The matrix is[3]

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

■

2. Cracking the code

If A has the property that $A\vec{y} = \vec{b}$ has a unique solution for all $\vec{b}$, then it has to be an $(n \times n)$-matrix of rank n (think through what the reduced row echelon form of A must be). We can write down a transformation $S\colon \mathbb{R}^n \to \mathbb{R}^n$ by the rule

$$S(\vec{b}) = \Big(\text{unique solution } \vec{y} \text{ of } A\vec{y} = \vec{b}\Big).$$

Fact: S is linear, that is, comes from an $(n \times n)$-matrix B. We call S the *inverse of the linear transformation* T and write $S = T^{-1}$. Similarly, we call B the *inverse of the matrix* A and write $B = A^{-1}$.

Remark 3. We have that $T^{-1}(T(\vec{x})) = \vec{x}$ because $T^{-1}(T(\vec{x}))$ is the unique solution $\vec{y}$ to $T\vec{y} = T\vec{x}$, but clearly we can take $\vec{y} = \vec{x}$. It also true that $T(T^{-1}(\vec{x})) = \vec{x}$: to see this, we first remark that since $A\vec{y} = \vec{b}$ always has a solution, taking $\vec{b} = \vec{x}$ we see every $\vec{x} \in \mathbb{R}^n$ is to equal $T\vec{y}$ for some $\vec{y}$. So let us fill this in:

$$T(T^{-1}(\vec{x})) = T(T^{-1}(T(\vec{y}))) = T(\vec{y}) = \vec{x}.$$

By the remark, finding T^{-1} amounts to finding a way of undoing the transformation T. If we happened to think of T has encrypting, then T^{-1} is decrypting:

[1] The computer program mathematicians write in calls this color "Mahogany."

[2] The computer program mathematicians write in calls this color "Periwinkle."

[3] I know the precise entries since the picture was actually produced by applying a linear transformation to the square. It's fine if you guessed a nearby value, such at 0.7 instead of $1/\sqrt{2}$.

Problem 4. We "encrypt" a message with the linear transformation $T(x, y) = (2x + 8y, 3x + 11y)$. This is given by a matrix

$$\begin{bmatrix} 2 & 8 \\ 3 & 11 \end{bmatrix}.$$

Find the matrix for the inverse transformation (if there is one).

Solution. Let A denote the matrix in the problem. To find the columns of T^{-1}, we need to solve $A\vec{x} = \vec{e}_1$ and $A\vec{x} = \vec{e}_2$. This can be done simultaneously by applying Gauss-Jordan elimination to the super-augmented matrx

$$[A \mid \mathrm{id}_2] = \left[\begin{array}{cc|cc} 2 & 8 & 1 & 0 \\ 3 & 11 & 0 & 1 \end{array}\right]$$

Its reduced row echelon form is

$$\mathrm{rref}([A \mid \mathrm{id}_2]) = \left[\begin{array}{cc|cc} 1 & 0 & -11/2 & 4 \\ 0 & 1 & 3/2 & -1 \end{array}\right]$$

so we conclude that

$$A^{-1} = \begin{bmatrix} -11/2 & 4 \\ 3/2 & -1 \end{bmatrix}.$$

(You can check that you didn't make a mistake by computing $A^{-1}A\vec{x}$ for some choice of $\vec{x}$. The answer should be $\vec{x}$.) ■

Summary

- A transformation T is *linear* if $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ (which implies $T(\vec{0}) = \vec{0}$, a helpful check to see whether a transformation is linear). It is then implemented by a matrix A, that is $T(\vec{x}) = A\vec{x}$.
- To find the matrix of a linear transformation, take the ith column to be the image of the ith standard vector.
- An $n \times n$-matrix A is invertible if and only if its rank is n. Then we can find the matrix for the inverse of the linear transformation $T(\vec{x}) = A\vec{x}$ by putting the (super-)augmented matrix $[A \mid \mathrm{id}_n]$ in reduced row echelon form.