MATH21B – LECTURE 32: PARTIAL DIFFERENTIAL EQUATIONS II
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1. The wave equation

Problem 1. (i) Solve the wave equation

$$f_{tt}(x,t) = c^2 f_{xx}(x,t)$$

for $x \in [0,\pi]$, $t \geq 0$, with initial conditions $f(x,0) = \sin(x)$ and $f_t(x,0) = 0$, and boundary conditions $f(0,t) = 0 = f(\pi,t)$.

(ii) Explain how this can be interpreted as a lowest tone of a string, the fundamental frequency. What does increasing c correspond to?

(iii) What should the initial condition be to get the first overtone?

Solution. (i) The solution is $f(x,t) = \sin(x)\cos(ct)$.

(ii) It is exactly solving the wave equation with the right initial and boundary conditions, as can be seen in the figure.

(iii) It could be $f(x,0) = \sin(2x)$, $f_t(x,0) = 0$, or any other initial condition of the form $f(x,0) = A\sin(2x)$, $f_t(x,0) = B\sin(2x)$.

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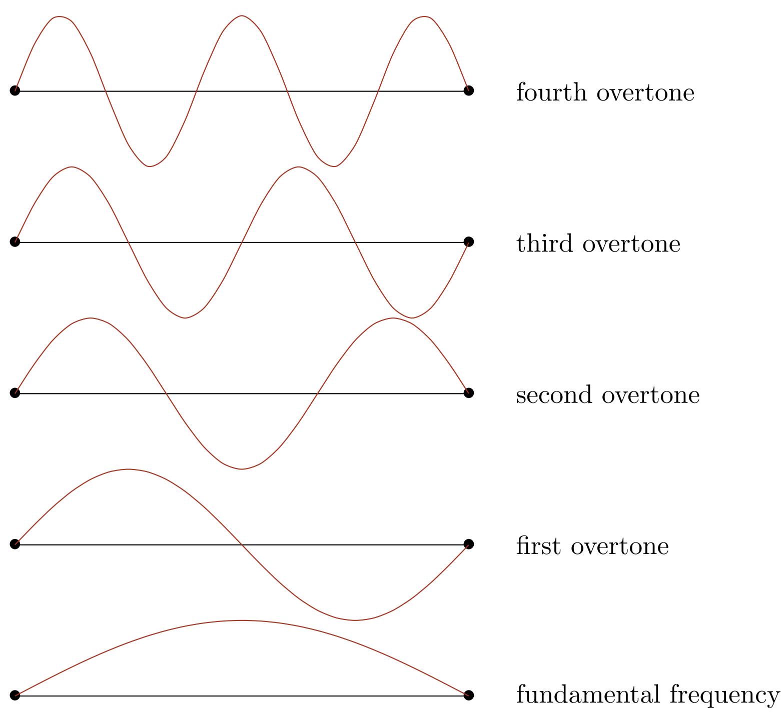

Problem 2. Maxwell's equations for electromagnetism in vacuum without any currents or charges present are given by:

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

where $\vec{E}(x,y,z,t)$ is the *electric field* measured in *volts* $V = A/s$, $\vec{B}(x,y,z,t)$ is the *magnetic field* measured in *tesla* $T = \text{kg}/(A\cdot s^2)$, $\nabla\cdot -$ is the divergence, and $\nabla\times -$ is the curl. The constant $\mu_0 = 4\pi\times 10^{-7}\, kg\cdot m\cdot A^{-2}\cdot s^{-2}$ is the *magnetic permeability* of vacuum and the constant $\epsilon_0 \approx 8.854\times 10^{-12}\, A^2\cdot s^4\cdot kg^{-1}\cdot m^{-3}$ is the *electric permittivity* of vacuum.

Applying $\nabla\times -$ to the second two equations, and using $\nabla\times(\nabla\times -) = \nabla(\nabla\cdot -) + \nabla^2(-)$ together with the first equations, one derives that

$$\frac{\partial E_x}{\partial t} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}\right) \qquad \text{and similarly for } E_y,\ E_z,\ B_x,\ B_y \text{ and } B_z.$$

Thus the components of the electric and magnetic field satisfy a wave equation!

(i) Light is an electromagnetic wave. Compute the speed c of light in vacuum.
(ii) In a medium like water, μ_0 and ϵ_0 change to the magnetic permeability μ and electric permittivity ϵ of the medium. For water at room temperature $\epsilon \approx 1.77\cdot\epsilon_0$ and $\mu\approx\mu_0$. What is the speed of light v in water?
(iii) The *refractive index of a medium* is defined as $n = \frac{c}{v}$. What is the refractive index of water?

Solution. (i) It is given by

$$\begin{aligned} 1/\sqrt{\mu_0\epsilon_0} &= 1/\sqrt{4\pi\times 10^{-7}\, kg\cdot m\cdot A^{-2}\cdot s^{-2}\cdot 8.854\times 10^{-12}\, A^2\cdot s^4\cdot kg^{-1}\cdot m^{-3}} \\ &= 1/\sqrt{4\pi\cdot 8.854\times 10^{-19}\cdot s^2\cdot m^{-2}} \\ &\approx 2.998\cdot 10^8\, m\cdot s^{-1}. \end{aligned}$$

(ii) It is divided by $\sqrt{1.77}\approx 1.33$, so about $2.25\cdot 10^8\, m\cdot s^{-1}$.
(iii) It is 1.33 approximately.

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Summary

- The *wave equation* is
$$f_{tt}(x,t) = c^2 f_{xx}(x,t).$$
with c is the "wave speed."
- To solve the wave equation for $x\in[0,\pi]$, and $t\geq 0$, with initial condition smooth function $f(x,0)$ for $x\in[0,\pi]$ with initial conditions smooth functions $f(x,0) = g(x)$ and $f_t(x,0) = h(x)$ for $x\in[0,\pi]$, and boundary condition $f(0,t) = f(\pi,t) = 0$. To do so, we write the initial conditions as
$$g(x) = \sum_{n\geq 1} b_n\sin(nx) \qquad \text{with } b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)dx,$$
$$h(x) = \sum_{n\geq 1} c_n\sin(nx) \qquad \text{with } c_n = \frac{2}{\pi}\int_0^\pi h(x)\sin(nx)dx,$$
$$f(x,t) = \sum_{n\geq 1} b_n\sin(nx)\cos(nct) + \frac{c_n}{nc}\sin(nx)\sin(nct).$$