MATH21B – LECTURE 31: PARTIAL DIFFERENTIAL EQUATIONS I
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1. The heat equation

Problem 1. (i) Solve the heat equation

$$f_t(x,t) = f_{xx}(x,t)$$

with initial condition $f(x,0) = \sin(x)$ and boundary conditions $f(0,t) = f(\pi,t) = 0$.

(ii) Explain that a (badly insulated) oven which was turned on for a while to 400° F and then turned off, might reasonably be modeled by a modified heat equation

$$f_t(x,t) = f_{xx}(x,t)$$

with initial condition $f(x,0) = 70 + 330\sin(x)$ and boundary conditions $f(0,t) = f(\pi,t) = 70$.

(iii) Solve the equation in (ii). Does the solution make sense?

Solution. (i) $f(x,0)$ is already given as a (rather easy) Fourier series, and $\mu = 1$, so we have

$$f(x,t) = \sin(x)e^{-t}.$$

(ii) In the initial condition at the sides the temperature is 70, while in the center it is 400.

(iii) It is $70 + 330\sin(x)e^{-t}$. This is a much faster cooling than expected. We should have chosen μ to be smaller. ■

Summary

- Partial differential equations are equations involving the partial derivatives of functions of multiple variables (in contrast with the ordinary differential equations for functions of a single variable studied before). We explain how to solve two important ones: the *heat equation* (with "thermal diffusivity" μ)

$$f_t(x,t) = \mu f_{xx}(x,t)$$

  and next lecture, the *wave equation*.
- To solve the heat equation for $x \in [0,\pi]$, and $t \geq 0$, with initial condition smooth function $f(x,0) = g(x)$ for $x \in [0,\pi]$ and boundary condition $f(0,t) = f(\pi,t) = 0$. To do so, we write the initial condition as a Fourier series in sines and diagonalize:

$$g(x) = \sum_{n\geq 1} b_n \sin(nx) \qquad \text{with } b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)dx,$$

$$f(x,t) = \sum_{n\geq 1} b_n \sin(nx) e^{-n^2\mu t}.$$