MATH21B – LECTURE 3: THE NUMBER OF SOLUTIONS
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1. The good, the bad and the ugly

Problem 1. The following are matrices in reduced row echelon form. How many solutions do they have?

$$\text{(a)}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(b)}\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(c)}\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{(d)}\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \text{(e)}\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}\right] \quad \text{(f)}\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Solution. (a) Infinitely many.
(b) Infinitely many.
(c) None.
(d) Infinitely many.
(e) A single one.
(f) None. ∎

2. Solution spaces

Problem 2. (i) Solve the system of linear equations

$$\left|\begin{array}{r} x + 2y + z = 0 \\ 2x + 3y + 2z = 0 \\ x + y + z = 0 \end{array}\right|.$$

(ii) Solve the system of linear equations

$$\left|\begin{array}{r} x + 2y + z = 2 \\ 2x + 3y + 2z = 3 \\ x + y + z = 1 \end{array}\right|.$$

(iii) What is the geometric relationship between the answers to (i) and (ii)?
(iv) Can you solve the following without any additional computation?

$$\left|\begin{array}{r} x + 2y + z = 1 \\ 2x + 3y + 2z = 2 \\ x + y + z = 1 \end{array}\right|.$$

Solution. (i) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

and its reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the sets of solutions is given by the equations $x = -z$, $y = 0$, which can be written parametrically as the set of vectors

$$s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $s \in \mathbb{R}$.

(ii) Now the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

and its reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the sets of solutions is given by the equations $x = -z$, $y = 1$, which can be written parametrically as the set of vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $s \in \mathbb{R}$.

(iii) These are parallel lines. This makes sense: suppose that $\vec{p}$ is a solution to (ii) and $\vec{n}$ is a solution to (i), that is, $A\vec{p} = [1, 2, 1]^t$ (this means the column vector with those entries) and $A\vec{n} = \vec{0}$ (this means the vector with all 0 entries). Then $\vec{p} + \vec{n}$ is a solution to (ii) too:

$$A(\vec{p} + \vec{n}) = A\vec{p} + A\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \vec{0} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Taking

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

we see that every

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

for $s \in \mathbb{R}$ is a solution; indeed, here

$$\vec{n} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are exactly to the solutions to (i).

Thus all of the line (1) consists of solutions to (ii). In fact, these are all solutions to (ii): given the particular solution $\vec{p} = [1, 0, 0]^t$, and another solution $\vec{v}$ to (ii), we have that $\vec{v} - \vec{p}$ is a solution to (i):

$$A(\vec{v} - \vec{p}) = A\vec{v} - A\vec{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$$

so $\vec{v} - \vec{p} = s[-1, 0, 1]^t$ for some s (by part (i)) and thus $\vec{v} = \vec{v} - \vec{p} + \vec{p} = [1, 0, 0]^t + s[-1, 0, 1]^t$, which is on the line (1).

(iv) Here we can guess the particular solution $\vec{p} = [1, 0, 0]^t$, so that as in (iii) we see that all solutions are given by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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Conclusion of Problem 2(iii)

If S is the set of solutions to the equation $A\vec{x} = \vec{0}$, then the set of solutions for $A\vec{x} = \vec{b}$ can be obtained by finding a single "particular" solution $\vec{p}$ with $A\vec{p} = \vec{b}$, and taking taking the set of vector $\vec{p} + \vec{v}$ for $\vec{v} \in S$.

3. Self-test

Problem 3. (i) If you perform Gauss-Jordan elimination on an inconsistent system, how will you recognize that the system is inconsistent?

(ii) Suppose A is a 3×3-matrix such that rref(A) has three leading 1's, what can you say about the number of solutions of the system $A\vec{x} = \vec{b}$?

(iii) True or false: There is no matrix A with rank 0.

(iv) True or false: If A is any matrix and $\vec{0}$ a vector with only zero entries, the system $A\vec{x} = \vec{0}$ is consistent.

(v) True or false: If A is a 4×3-matrix and the linear system $A\vec{x} = \vec{b}$ has exactly one solution, then the linear system $A\vec{x} = \vec{c}$ has exactly one solution for all vectors $\vec{c}$ in $\mathbb{R}^4$.

(vi) True or false: If A is an $m \times n$-matrix such that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$, then the rank of A is exactly m.

(vii) Put the following augmented matrix in reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & 1 & -2 & 2 \\ 5 & 5 & 3 & 1 \end{array}\right]$$

and give the unique solution to the corresponding system of linear equations.

(viii) What is the rank of the following matrix?

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Solution. (i) It is inconsistent if rref$(A \mid \vec{b})$ as a leading one in the last column (the one right of the line).

(ii) There is always a unique solution, as the condition on rref(A) says that rank$(A) = 3$ and rank$(A) \leq$ rank$(A \mid \vec{b}) \leq 3$ (since the rank of $[A \mid \vec{b}]$ can not be higher than the number of rows, which is 3) implies rank$(A) = 3$rank$(A \mid \vec{b})$.

(iii) False, take the matrix with only zero entries.

(iv) True, there can never appear a leading 1 on the last column of rref$(A \mid \vec{b})$.

(v) False, e.g. $[A \mid \vec{b}]$ could be

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

but $[A \mid \vec{c}]$ given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is inconsistent.

(vi) True. If $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$, then every row contains a leading 1. Thus the rank of A, the number of leading 1's, is equal to the number of rows, m.

(vii) The unique solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8/7 \\ 16/7 \\ -11/7 \end{bmatrix}$$

(viii) The rank is 6.

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Summary

- We can read off the number of solutions to a system of m linear equations $A\vec{x} = \vec{b}$ in n unknowns from $\text{rref}([A \mid \vec{b}])$:
  - If $\text{rank}(A) = \text{rank}(A \mid \vec{b}) = n$ then there is exactly one solution.
  - If $\text{rank}(A) < \text{rank}(A \mid \vec{b})$ then there are no solutions.
  - If $\text{rank}(A) = \text{rank}(A \mid \vec{b}) < n$ then there are infinitely many solutions.