MATH21B – LECTURE 29: FOURIER SERIES I
SPRING 2018, HARVARD UNIVERSITY

1. An inner product on periodic functions

Problem 1. (i) Compute the length of $\cos(x)$.
(ii) Compute the angle between $\sin(x)$ and $\cos(x)$.

Solution. (i) We compute that

$$\langle \cos(x), \cos(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos(2x)) = \frac{2\pi}{2\pi} = 1.$$

So we have $||\cos(x)|| = \sqrt{\langle \cos(x), \cos(x) \rangle} = \sqrt{1} = 1$.
(ii) We compute that

$$\langle \cos(x), \sin(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) \sin(x) dx = 0$$

since $\cos(x)\sin(x)$ is odd. Thus they are orthogonal. ■

2. Fourier series

Problem 2. Compute the Fourier series of the trigonometric polynomial

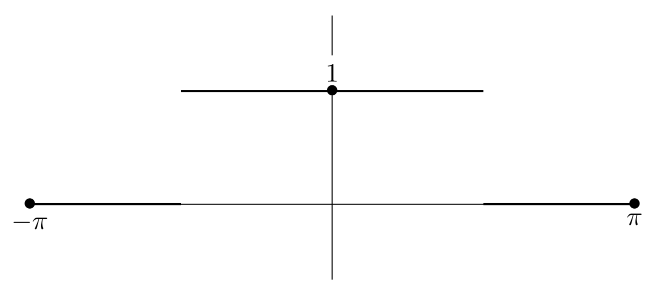
$$\cos(x) + 1/5 \sin(3x).$$

Solution. It is already in the form for a Fourier series. ■

Problem 3. (i) Sketch the periodic function determined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [-\pi/2, \pi/2], \\ 0 & \text{if } x \in [-\pi, -\pi/2) \text{ or } x \in (\pi/2, \pi]. \end{cases}$$

(ii) Is this function even, odd or neither? What does this mean for the Fourier coefficients?
(iii) Compute the Fourier series for f.



Solution. (i)
(ii) It is even. Only a_i's can be non-zero.
(iii) We compute that

$$a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx = \frac{1}{\pi\sqrt{2}} \int_{-\pi/2}^{\pi/2} dx = \frac{1}{\sqrt{2}}.$$

$$a_n = \langle f, \cos(nx) \rangle = \int_{-\pi/2}^{\pi/2} \cos(nx) = \left[\frac{\sin(nx)}{\pi n} \right]_{-\pi/2}^{\pi/2} = \begin{cases} \frac{2(-1)^m}{\pi(2m+1)} & \text{if } n = 2m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get the Fourier series

$$f(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2(-1)^m}{\pi(2m+1)} \cos((2m+1)x).$$

■

Summary

- The linear space C^∞_{per} of 2π-periodic smooth functions has an *inner product*:

$$\langle f, g\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

  This has the same properties as the dot product on $\mathbb{R}^n$, and we call $||f|| = \sqrt{\langle f, f\rangle}$ the *length* of f, and θ determined by $\cos(\theta) = \frac{\langle f,g\rangle}{||f||\cdot||g||}$ the *angle* between f and g.
- The functions $\frac{1}{\sqrt{2}}$, $\cos(nx)$ and $\sin(nx)$ for $n \geq 1$ form an orthonormal set with respect to this inner product.
- Every smooth 2π-periodic function f can be written as an infinite linear combination of these:

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n\geq 1} a_n \cos(nx) + b_n \sin(nx)$$

  with $a_0 = \langle f, \frac{1}{\sqrt{2}}\rangle$, $a_n = \langle f, \cos(nx)\rangle$, $b_n = \langle f, \sin(nx)\rangle$. This is like "additive synthesis" of sounds.
- If f is *even*, $f(-x) = f(x)$, then only $\frac{1}{\sqrt{2}}$ and $\cos(nx)$ for $n \geq 1$ have non-zero coefficients. If f is *odd*, $f(-x) = -f(x)$, then only $\sin(nx)$ for $n \geq 1$ have non-zero coefficients.