MATH21B – LECTURE 28: THE ODE COOKBOOK
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1. Solving linear equations

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

(i) Find the kernel of A.
(ii) Find a single solution $\vec{x}_p$ of

$$A\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}. \tag{1}$$

(iii) Find all solutions of the equation (1) in terms of $\vec{x}_p$ and $\ker(A)$.

Solution. (i) It is given by the span of $[-2, 1]^T$.
(ii) You can take for example $\vec{x}_p = [1, 1]^T$.
(iii) In general the solutions are $\vec{x}_p + \vec{v}$ for $\vec{v} \in \ker(A)$, so they are given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

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2. The operator method

Problem 2. Use the operator method from the cookbook to solve

$$(D^2 - 4D + 4)f = 1.$$

Solution. In this case $\lambda_1 = 2 = \lambda_2$, so we need to first compute

$$(D-2)^{-1}t = Ce^{2t} + e^{2t}\int_0^t e^{-2t}dt = Ce^{2t} + e^{2t}(-\frac{1}{2}e^{-2t})|_0^t = Ce^{2t} - \frac{1}{2}e^{2t}e^{-2t} + \frac{1}{2}e^{2t} = C'e^{2t} - 1/2.$$

Next we need to compute

$$(D-2)^{-1}(C'e^{2t} - 1/2) = Ce^{2t} + e^{2t}\int_0^t C' - 1/2e^{-2t}dt = C''e^{2t} + C'te^{2t} + 1/4.$$

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3. Oscillations

Problem 3. Solve the system

$$f'' + 9f = 0.$$

Solution. This is a harmonic oscillator, given by $C_1\cos(3t) + C_2\sin(3t)$. ■

Problem 4. Solve the system

$$f'' + 9f = \cos(2t).$$

Solution. We first find the particular solution x_p: the heuristic tells us we should try $f(t) = Acos(2t)+B\sin(2t)$: we get $f'' + 9f = -4A\cos(2t) - 4B\sin(2t) + 9A\cos(2t) + 9Bsin(2t)$ so take $A = \frac{1}{5}$ and $B = 0$..

Then all solutions are given by $x_h + x_p$ with $x_p = \frac{1}{5}\cos(2t)$ and x_h a solution of $f'' + 9f = 0$. These are given by $C_1\cos(3t) + C_2\sin(3t) + \frac{1}{5}\cos(2t)$. ■

Problem 5. Solve the system

$$f'' + 9f = \cos(3t).$$

Solution. Again we first find the particular solution x_p Here our heuristic tells us we should try $A\cos(3t) + B\sin(3t)$, but this is among the solutions of the homogeneous system. In that case, we have a modified heuristic: multiply with t first and try $f(t) = At\cos(3t) + Bt\sin(3t)$. We see that $f'' + 9f = -6A\sin(3t) + 6B\cos(3t)$ so we should take $A = 0$ and $B = 1/6$ for the particular solution x_p.

As before all solutions are given by $C_1\cos(3t) + C_2\sin(3t) + \frac{1}{6}t\sin(3t)$. ■

Summary

- When solving $p(D)f = g$ for $p(D)$ a polynomial in the differential operator D, you first find a single *particular solution* x_p satisfying $p(D)x_p = g$. Then you find the *homogeneous solutions* x_h of the differential equation $p(D)x = 0$. All solutions are given by $x_p + x_h$ where x_p is the single particular solution and x_h is a homogenous solution.
- In general, there is an operator method to find both the particular and homogeneous solutions. Because it is a lot of work, one usually follows the "cookbook."
- If p is of degree d, the space of homogenous solutions is d-dimensional. For example, for $p(D) = D^2 + bD + c = (D - \lambda_1)(D - \lambda_2)$ with λ_1, λ_2 real there are two cases: (i) if $\lambda_1 \neq \lambda_2$ we have $C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$, (ii) if $\lambda_1 = \lambda_2$ we have $C_1e^{\lambda_1 t} + C_2te^{\lambda_2 t}$.