MATH21B – LECTURE 27: DIFFERENTIAL OPERATORS
SPRING 2018, HARVARD UNIVERSITY

1. Linear spaces, a recollection

Problem 1. Which of the following are linear subspaces? Which are finite-dimensional?

(i) The set $\mathcal{P}_{\text{even}}$ of even polynomials $a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n}$ $(n \geq 0)$ in C^∞.
(ii) The set $\mathcal{P}_{\geq 0}$ of polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ $(n \geq 0)$ with non-negative coefficients $a_i \geq 0$ in C^∞.
(iii) The span of $1, \cos(x), \sin(x)$ in C^∞_{per}.

Solution. The sets (i) and (iii) are linear subspaces: they contain the 0 function and are closed under scaling and addition. The set (ii) is not closed under scaling, in particular multiplication by negative numbers.

Only (iii) is finite-dimensional, in particular it is 3-dimensional. ■

2. Differential operators as linear transformations

Problem 2. Let $\mathcal{P}_n$ be the linear subspace of C^∞ of polynomials with real coefficients of degree $\leq n$.

(i) What is the dimension of $\mathcal{P}_n$.
(ii) Show that differentiation $D\colon \mathcal{P}_n \to \mathcal{P}_n$ is a linear transformation.
(iii) What is the matrix A of D with respect to the basis $\{1, x, x^2, \ldots, x^n\}$?
(iv) What are the eigenvalues of A?
(v) Is there a non-zero polynomial p of degree $\leq n$ which satisfies $Dp(x) = p(x)$?
(vi) Is there any non-zero function f in C^∞ which satisfies $\frac{d}{dx}f(x) = f(x)$?

Solution. (1) It is $n+1$-dimensional, with basis $1, x, x^2, \ldots, x^n$.
(2) We know that D is a linear transformation on C^∞ and since it sends $\mathcal{P}_n$ into $\mathcal{P}_n$, it restricts to a linear transformation on $\mathcal{P}_n$.
(3) It is given by

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & n & 0 \end{bmatrix}$$

(4) They are all 0.
(5) No, as this would be an eigenvalue with eigenvector 1.
(6) Yes, e^x. ■

Problem 3. (i) Fix a smooth function $g(x)$. Show that multiplication by $g(x)$, the map sending $f(x)$ to $g(x)f(x)$, is a linear transformation $C^\infty \to C^\infty$.
(ii) Check that $Af = (D - x)f(x)$ is a linear transformation $C^\infty_{\text{per}} \to C^\infty$.
(iii) Check that $f(t) = e^{\frac{1}{2}x^2}$ is in the kernel of A.
(iv) Check that $f(x)$ is in the kernel of A if and only if $f(x)/e^{1/2x^2}$ is in the kernel of D. Use this to compute the kernel of A.
(v) Solve the differential equation $\frac{df(x)}{dx} = xf(x)$ by separation of variables.

Solution. (i) We have that $g(x) \cdot 0 = 0$, $g(x) \cdot (\lambda f(x)) = \lambda g(x) \cdot f(x)$, and $g(x) \cdot (f(x) + h(x)) = g(x) \cdot f(x) + g(x) \cdot h(x)$.

(ii) It is the sum of the linear transformations D^2 and multiplication by $-x$.

(iii) We compute that $D(e^{1/2x^2}) = te^{1/2x^2}$, so that $(D - x)e^{1/2x^2} = 0$.

(iv) We compute that $D(f(x)/e^{1/2x^2}) = f'(x)/e^{1/2x^2} - f(x)/(e^{1/2x^2})^2 xe^{1/2x^2} = (f'(x) - xf(x))/e^{1/2x^2}$ which is 0 if $f'(x) - xf(x) = 0$. This means that to find the kernel of A, we can find all functions in the kernel of D and multiply these with $e^{1/2x^2}$. The kernel of D are the constant functions, so we get that $\ker(A) = Ce^{1/2x^2}$.

(v) When you separate variables and integrate, you get $\log(f(x)) = \frac{1}{2}x^2 + C'$, and taking exponents on both sides we see the solutions $Ce^{1/2x^2}$. ■

Summary

- We are interested in the linear spaces C^∞ of smooth real-valued functions on $\mathbb{R}$, and C^∞_{per} of smooth periodic real-valued functions on $\mathbb{R}$ with period 2π. These contain interesting subspaces: $\mathcal{P} \subset C^\infty$ given by the polynomials (this is the span of $1, x, x^2, \ldots$) and $\mathcal{T} \subset C^\infty_{\text{per}}$ of trigonometric polynomials (this is the span of $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$).
- On both C^∞ and C^∞_{per}, the map D sending f to $\frac{df}{dt}$ is a linear transformation: $D0 = 0$, $D(\lambda f) = \lambda D(f)$ and $D(f + g) = D(f) + D(g)$ follow directly from the rules of differentiation. A combination of powers of D, such as $D^5 - D + 1$, is called *differential operator*.
- The eigenvalues of the differential operator D on C^∞ are all $\lambda \in \mathbb{R}$ with eigenvector given by $e^{\lambda x}$. The differential operator D on C^∞_{per} only has complex eigenvalues in for $n \in \mathbb{Z}$ with eigenvector given by e^{inx}.