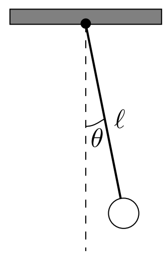
MATH21B – LECTURE 25: DIFFERENTIAL EQUATIONS II
SPRING 2018, HARVARD UNIVERSITY

1. The harmonic oscillator



Problem 1. Classical mechanics tells us that a pendulum satisfies

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{\ell}\sin(\theta(t)) = 0$$

with ℓ the length of the pendulum and g the gravitational constant.

(i) Using the approximation $\sin(\theta) \approx \theta$ for small angles, show that for small angles we can instead solve

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{g}{\ell}x_1(t).$$

(ii) If we want to write this as $\frac{d\vec{x}}{dt} = A\vec{x}$, what is A?

(iii) Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ for $\vec{x}(t) = [1, 0]^T$.

(iv) What is the period of the pendulum?

Solution. (i) Substituting θ for $\sin(\theta)$ gives

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{\ell}\theta(t) = 0,$$

use $x_1(t) = \theta(t)$ and add a dummy variable $x_2(t)$ to get the two equations.

(ii) A will be given by

$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix}$$

(iii) The eigenvalues are $\pm i\sqrt{g/\ell}$, and an eigenvector for $i\sqrt{g/\ell}$ is $\vec{v}_1 = [-i, \sqrt{g/\ell}]^T$ and an eigenvector for $-i\sqrt{g/\ell}$ is $\vec{v}_2 = [i, \sqrt{g/\ell}]^T$. We have that $\vec{x}(0) = i/2\vec{v}_1 - i/2\vec{v}_2 = [1, 0]^T$, and that $\vec{x}(t) = i/2e^{i\sqrt{g/\ell}t}\vec{v}_1 - i/2e^{i\sqrt{g/\ell}t}\vec{v}_2$. Writing this out, we get $\vec{x}(t) = [\cos(\sqrt{g/\ell}t), \sqrt{g/\ell}\sin(\sqrt{g/\ell}t)]$.

(iv) Whenever $\sqrt{g/\ell}$ is a multiple of 2π the motion repeats, thus the period is $2\pi\sqrt{\ell/g}$. ■

2. Stability revisited

Problem 2. True or false?

(i) If A is horizontal shear, then $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is not stable.

(ii) For $A = \begin{bmatrix} -1 & 2 \\ 1/4 & -1 \end{bmatrix}$, $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is stable.

(iii) If $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is stable, then $\frac{d\vec{x}(t)}{dt} = A^{-1}\vec{x}(t)$ is also stable.
(iv) If $A = B^T B$, then $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is stable.

Solution. (i) True, as A is given by

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

and $\text{tr}(A) > 0$.
(ii) True, its trace is -2, its determinant is $3/2$.
(iii) True, if $\text{Re}(\lambda_i) < 0$, thne $\text{Re}(1/\lambda_i) < 0$, as $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$.
(iv) False, e.g. $B = \text{id}_2$, then $A = \text{id}_2$ too and the trace is 2.

∎

Summary

- The *harmonic oscillator* is the name for the differential equation $\frac{d^2x(t)}{dt^2} = -k^2x(t)$. It has solutions $x(t) = a\cos(kt) + b\cos(kt)$, with a, b depending on the initial condition, and is solved by adding a dummy variable to make it a system of two linear differential equations.
- Recall the system of linear differential equation $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is *asymptotically stable* if $\vec{x}(t) \to \vec{0}$ as $t \to \infty$. This holds if and only if $\text{Re}(\lambda_j) < 0$ for all eigenvalues λ_j. For $n = 2$, this can be rephrased as $\det(A) > 0$ and $\text{tr}(A) < 0$.