MATH21B – LECTURE 24: DIFFERENTIAL EQUATIONS I
SPRING 2018, HARVARD UNIVERSITY

1. Differential and linear differential equations

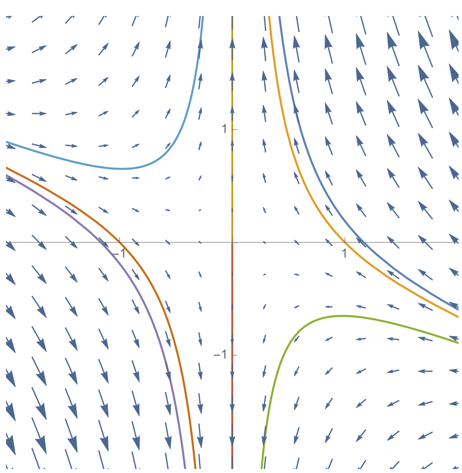
Problem 1. (i) Solve $\frac{dx(t)}{dt} = t$, $x(0) = 1$.
(ii) Solve $\frac{dx(t)}{dt} = t/x(t)$, $x(0) = 1$.
(iii) Solve $\frac{dx(t)}{dt} = x(t)$, $x(0) = 1$.

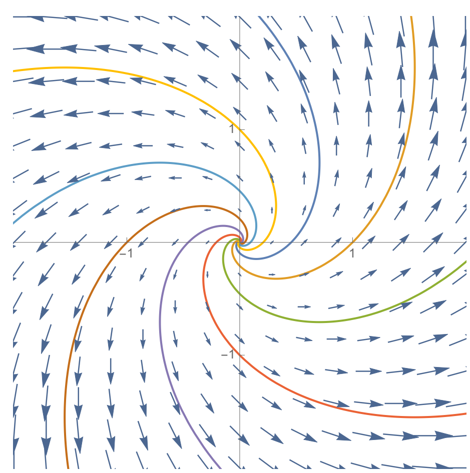
Solution. (1) The solution is $x(t) = 1/2t^2 + 1$.
(2) We get $1/2x^2 = 1/2t^2 + C$, which gives $x(t) = \sqrt{t^2 + 2C}$ and to match the initial condition we need $2C = 1$. The solution is $x(t) = \sqrt{t^2 + 1}$.
(3) We get $\log(x) = t + C$, so $x(t) = \exp(t + C)$ and to match the initial condition we take $C = 0$ and get $x(t) = \exp(t)$. ■

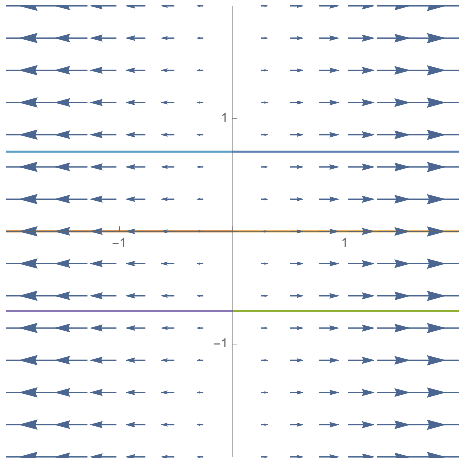
2. Systems of linear differential equations

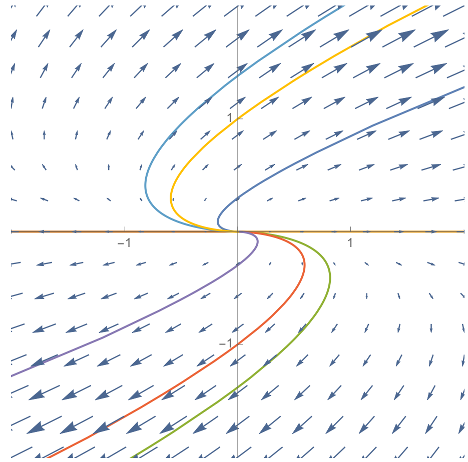
Problem 2. (i) Which phase portrait corresponds to which matrix?

$$(a) \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (d) \begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix}.$$









(ii) Which of the matrices have asymptotically stable systems of linear differential equations?

(iii) Solve the system of (c) with initial condition $\vec{x}(0) = [1,0]^T$.

Solution. (i) The solutions are

$$\begin{matrix} (d) & (c) \\ \\ (a) & (b). \end{matrix}$$

To see this, note that the vectors are horizontal in the third picture, so the matrix must have image in $\operatorname{span}(\vec{e}_1)$. Next note that (b) has eigenvector $\vec{e}_1$ and hence an initial conditions on the x-axis gives a solution on the x-axis. Similar (d) has eigenvector $\vec{e}_2$ and hence an initial conditions on the y-axis gives a solution on the y-axis.

(ii) None of them are: from the arrows we see that solutions grow. Alternatively, you can compute the eigenvalues.

(iii) The characteristic polynomial

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

is $(1-\lambda)^2 + 1$, which has roots $1 \pm i$. These are the eigenvalues, with eigenvectors for $1+i$ and $1-i$ given respectively by

$$\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

We have that

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{i}{2}\begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Thus we have that

$$\begin{aligned} \vec{x}(t) &= -\frac{i}{2}e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{i}{2}e^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= e^t\left(-\frac{i}{2}(\cos(t) + i\sin(t))\begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{i}{2}(\cos(t) - i\sin(t)\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) \\ &= e^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}. \end{aligned}$$

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Summary

- An equation of the form $\frac{dx(t)}{dt} = f(x(t))$ is called a *(ordinary) differential equation*. You can solve them when given an *initial condition* $x(0)$. They can often be solved by *separation of variables*.
- A linear differential equation is one of the form $\frac{dx(t)}{dt} = cx(t)$ for some constant $c \in \mathbb{R}$ (or $c \in \mathbb{C}$). The solution with initial condition $x(0)$ is given by $x(t) = e^{ct}x(0)$.
- A system of linear differential equation is one of the form $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$, with A an $(n \times n)$-matrix and $\vec{x}(t) = [x_1(t), \ldots, x_n(t)]^T$. To solve $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ with initial condition $\vec{x}(0)$, diagonalize A and write $\vec{x}(0)$ as a linear combination $\vec{x}(0) = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ of eigenvectors $\vec{v}_i$ with eigenvalues λ_i. Then the solution is $\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + \cdots + c_ne^{\lambda_n t}\vec{v}_n$.
- The system of linear differential equation $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ is *asymptotically stable* if $\vec{x}(t) \to \vec{0}$ as $t \to \infty$. This holds if and only if $\mathrm{Re}(\lambda_j) < 0$ for all eigenvalues λ_j.
- A *phase portrait* of a system of linear differential equations $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ for $n = 2$ is given by drawing a vector $A\vec{x}$ at $\vec{x} \in \mathbb{R}^2$. The solutions are curves tangent to these vectors.