MATH21B – LECTURE 23: SYMMETRIC MATRICES
SPRING 2018, HARVARD UNIVERSITY

1. Eigenvalues and eigenvectors of symmetric matrices

Problem 1. Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for $a, b \in \mathbb{R}$.

(i) Give the eigenvalues of A in terms of a and b and verify they are real.
(ii) Give the eigenvectors of A in terms of a and b and very that A is diagonalizable using an orthonormal eigenbasis.

Solution. (i) The characteristic polynomial is $(a-\lambda)^2 - b^2 = (a-\lambda+b)(a-\lambda-b)$, so the roots are $a+b$ and $a-b$. These are indeed real.
(ii) To compute the eigenvectors we find a basis of

$$\ker \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}, \quad \ker \begin{bmatrix} b & b \\ -b & b \end{bmatrix}.$$

These are given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Normalizing them to have length 1, we get

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

which are indeed an orthonormal basis. ∎

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

(i) What is the dimension of the kernel of the matrix $A - 2 \cdot \mathrm{id}_6$?
(ii) Use part (i) and the trace to find the eigenvalues of A.
(iii) Find an eigenbasis. Can you find an orthonormal eigenbasis?

Solution. (i) We have that $A - 2\mathrm{id}_6$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

which has 5-dimensional kernel. Thus we have an eigenvalue 2 with geometric multiplicity 5.

(ii) To find the remaining eigenvalues, we see that there is one which remains to be found. We use that $2+2+2+2+2+\lambda = 6 \cdot 3 = 18$, so that $\lambda = 8$.

(iii) The eigenspace for eigenvalue 2 is spanned by

$$\begin{bmatrix}-1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\0\\0\\1\end{bmatrix}.$$

These are not yet orthonormal, so we can apply Gram-Schmidt to get:

$$\begin{bmatrix}-(1/\sqrt{2})\\1/\sqrt{2}\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-(1/\sqrt{6})\\-(1/\sqrt{6})\\\sqrt{2/3}\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-(1/(2\sqrt{3}))\\-(1/(2\sqrt{3}))\\-(1/(2\sqrt{3}))\\\sqrt{3}/2\\0\\0\end{bmatrix}, \begin{bmatrix}-(1/(2\sqrt{5}))\\-(1/(2\sqrt{5}))\\-(1/(2\sqrt{5}))\\-(1/(2\sqrt{5}))\\2/\sqrt{5}\\0\end{bmatrix}, \begin{bmatrix}-(1/\sqrt{30})\\-(1/\sqrt{30})\\-(1/\sqrt{30})\\-(1/\sqrt{30})\\-(1/\sqrt{30})\\\sqrt{5/6}\end{bmatrix}.$$

For the eigenvalue 8, we get

$$\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$$

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Problem 3. Show that every symmetric matrix A with positive eigenvalues has a "square root", a matrix C such that $C^2 = A$.

Solution. By the spectral theorem, there is an (orthogonal) matrix S such that $S^{-1}AS$ is a diagonal matrix D with positive eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal. Let $\sqrt{D}$ be the diagonal matrix with entries $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ on the diagonal. Then we have that $C := S\sqrt{D}S^{-1}$ satisfies

$$S\sqrt{D}S^{-1}S\sqrt{D}S^{-1} = SDS^{-1} = A.$$

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Summary

- If an $(n \times n)$-matrix A satisfies $A^T = A$ it is called *symmetric*, and if it satisfies $A^T = -A$ it is called *anti-symmetric*.
- Symmetric matrices have real eigenvalues and eigenvectors for different eigenvalues are orthogonal.
- The *spectral theorem* says that a symmetric matrix can always be diagonalized and has an orthonormal eigenbasis. Thus there exists an orthogonal matrix S such that $S^{-1}AS$ is diagonal with diagonal entries the real eigenvalues of A.